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Pre-functions and Extended pre-functions of Complex Variables

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Abstract

Pre-functions are functions that possess a sequence $\{f_n(z,\beta)\}$ which tends to one of the elementary functions as n tends to infinity and β tends to 0. The main objective of this paper is to broaden the scope of pre-functions from functions of a real variable to functions of a complex variable by introducing pre-functions of a complex variable. We have analyzed the pre-functions of a complex variable for their properties. The pre-Laguerre, pre-Bessel and pre-Legendre polynomials of a complex variable have been obtained as special cases. Graphs have been used to visualize complex pre-functions.

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1. Introduction

Exponential and logarithmic functions have wide variety of applications in science, medicine, business and many other fields. Exponential functions are used to describe growth or decay of a quantity whose rate of change has a relation to its present value. Logarithms have the ability to measure quantities that are vastly different but need an easy way to be talked about and compared to. On the other hand, trigonometry can be used in music, roofing a house, cartography, satellite system in naval, aviation industries and in many other fields.

Deo and Howell [1] introduced and studied trigonometry and trigonometry like functions. Khandeparkar *et al.* [2] have introduced and studied pre-functions of a real variable. Motivated by their work, we have defined pre-functions of a complex variable. Functions which possess a sequence $\{f_n(z,\beta), z \in C, \beta \geq 0\}$ are called pre-functions of a complex variable z, if they tend to one of the elementary functions as $n \to \infty$ and $\beta \to 0$. Pre-functions also possess some of the properties possessed by the elementary functions but not properties like periodicity. Pre-functions were found to be very simple and useful in the study of differential equations. For the methods and solutions of various differential equations one can refer [3-10].

2. The pre-exponential function of a complex variable

Exponential functions play a significant role in almost every branch of Mathematics. In this section, we have determined a set of functions called pre-exponential functions, owning a sequence that generalizes the exponential function $exp(z)$.

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For any complex number z, the series form of pre-exponential function is given by,

$$\text{pexp}(z,\beta) = 1 + \frac{z^{1+\beta}}{\Gamma(2+\beta)} + \frac{z^{2+\beta}}{\Gamma(3+\beta)} + \frac{z^{3+\beta}}{\Gamma(4+\beta)} + ...$$
$$= 1 + \sum_{n=1}^{\infty} \frac{z^{n+\beta}}{\Gamma(n+1+\beta)}, \quad \beta \geq 0, \qquad (1)$$

β being the parameter.

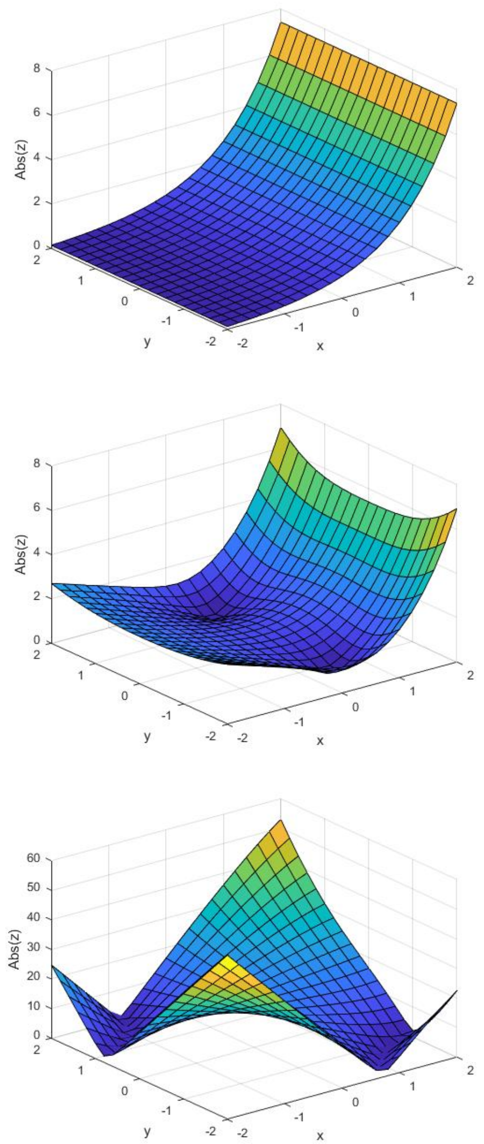

Figure 1: Graphs of $\text{pexp}(z, 0), \text{pexp}(z, 1)$ and $\text{pexp}(z, 2)$

Figure 1 represents the behaviour of the pre-exponential functions $\text{pexp}(z, 0), \text{pexp}(z, 1)$ and $\text{pexp}(z, 2)$ in order. Note that when $\beta = 0$, $\text{pexp}(z, 0) = \exp(z)$. In general,

$$\text{pexp}(z,n) = * \, \text{pexp}(z,n-1) - \frac{z^n}{n!}$$
$$= \text{pexp}(z,n-2) - \frac{z^{n-1}}{(n-1)!} ..., \quad n = 1,2,3,..$$

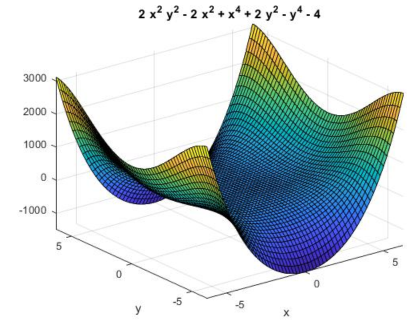

Figure 2: Z-Plane

Specifically,

$$\text{pexp}(z,1) = \exp(z) - z$$
$$\text{pexp}(z,2) = \exp(z) - z - \frac{z^2}{2!}$$
$$\text{pexp}(z,3) = \exp(z) - z - \frac{z^2}{2!} - \frac{z^3}{3!} = \exp(z) - S_3$$

where S_3 is the partial sum of $\text{pexp}(z, 3)$. For $n \in N$,

$$\text{pexp}(z,n) = \exp(z) - S_n \qquad (2)$$

where $S_n = \sum_{r=1}^{n} \frac{z^r}{r!}$. Also, note that $\text{pexp}(z, n) = 1, \forall\, z \in C$ as $n \to \infty$. Replacing z by $-z$ in (1), we have,

$$\text{pexp}(-z,\beta) = 1 - (-1)^{\beta}\left\{\frac{z^{1+\beta}}{\Gamma(2+\beta)} - \frac{z^{2+\beta}}{\Gamma(3+\beta)} + \frac{z^{3+\beta}}{\Gamma(4+\beta)} - ...\right\}$$
$$= 1 + (-1)^{\beta} \sum_{n=1}^{\infty} (-1)^n \frac{z^{n+\beta}}{\Gamma(n+1+\beta)}. \qquad (3)$$

In (3), replacing β by 0

$$\text{pexp}(-z,0) = 1 - \frac{z}{1!} + \frac{z^2}{2!} - \frac{z^3}{3!} + ... = \exp(-z) \qquad (4)$$

In short,

$$\text{pexp}(-z,n) = \exp(-z) - S_n$$

where $S_n = \sum_{r=1}^{n} (-1)^r \frac{z^r}{r!}$.

3. Pre-trigonometric Functions of a Complex Variable

The pre-trigonometric functions of a complex variable are defined by

$$Y_1(z,\beta) = \text{pcos}(z,\beta) = 1 - \frac{z^{2+\beta}}{\Gamma(3+\beta)} + \frac{z^{4+\beta}}{\Gamma(5+\beta)} - \frac{z^{6+\beta}}{\Gamma(7+\beta)} + ...$$
$$= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n+\beta}}{\Gamma(2n+1+\beta)}, \quad z \in C, \quad \beta \geq 0 \qquad (5)$$

and

$$Y_2(z,\beta) = \text{psin}(z,\beta) = \frac{z^{1+\beta}}{\Gamma(2+\beta)} - \frac{z^{3+\beta}}{\Gamma(4+\beta)} + \frac{z^{5+\beta}}{\Gamma(6+\beta)} - ...$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1+\beta}}{\Gamma(2n+2+\beta)}, \quad z \in C, \quad \beta \geq 0. \qquad (6)$$

Table 1 gives the expressions for the pre-trigonometric functions when β takes the values 0,1,2 and 3.

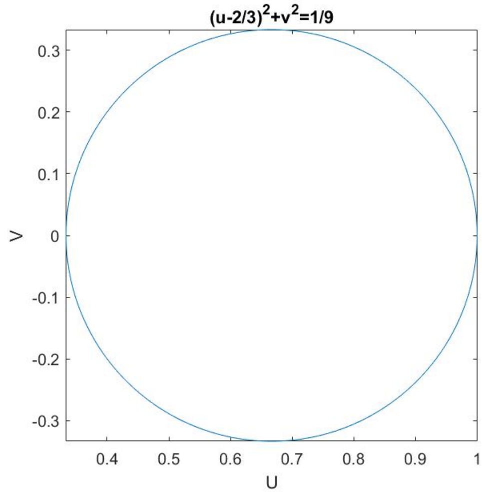

Figure 3: W-Plane

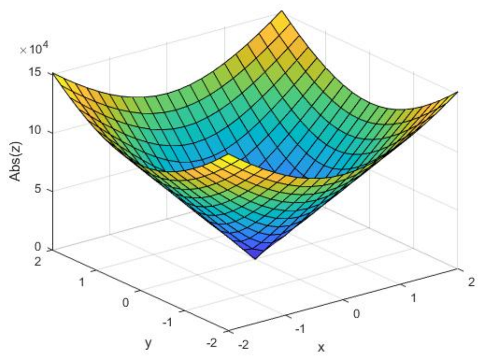

Figure 4: $\text{pcos}(z,1) = 1 - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!}$

Table 1: Expressions for the pre-trigonometric functions

S.No	(z,β)	$\text{pcos}(z,\beta)$	$\text{psin}(z,\beta)$
1	$(z,0)$	$\cos(z)$	$\sin(z)$
2	$(z,1)$	$\sin z - z + 1$	$1 - \cos z$
3	$(z,2)$	$-cosz - \frac{z^2}{2} + 2$	$z - \sin z$
4	$(z,3)$	$-\sin z + z - \frac{z^3}{6} + 1$	$\cos z + \frac{z^2}{2} - 1$

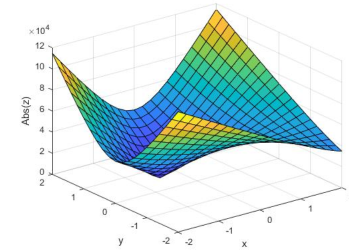

Figure 5: $\text{pcos}(z,0.6) = 1 - \frac{z^{2.6}}{\Gamma(3.6)} + \frac{z^{4.6}}{\Gamma(5.6)} - \frac{z^{6.6}}{\Gamma(7.6)}$

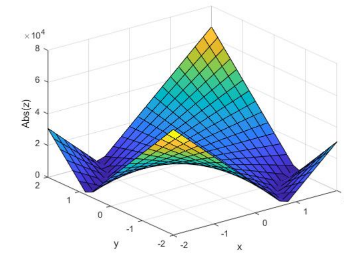

Figure 6: $\text{psin}(z,1) = \frac{z^2}{2!} - \frac{z^4}{4!} - \frac{z^6}{6!}$

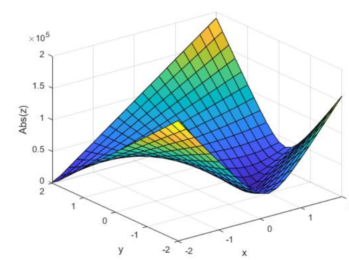

Figure 7: $\text{psin}(z,0.6) = \frac{z^{1.6}}{\Gamma(2.6)} - \frac{z^{3.6}}{\Gamma(4.6)} - \frac{z^{5.6}}{\Gamma(6.6)}$

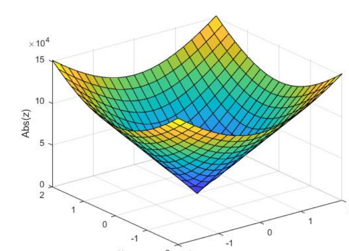

Figure 8: $\text{pcosh}(z,1) = 1 + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!}$

4. Euler's Formula

For a complex number z,

$$\exp(iz) = \cos z + i \sin z. \tag{7}$$

Using (1), the above expression can be rewritten as

$$\begin{aligned} \text{pexp}(iz,\beta) &= 1 - (i)^{\beta}\left\{\frac{z^{2+\beta}}{\Gamma(3+\beta)} - \frac{z^{4+\beta}}{\Gamma(5+\beta)} + \frac{z^{6+\beta}}{\Gamma(7+\beta)} + \ldots\right\} \\ &+ i(i)^{\beta}\left\{\frac{z^{1+\beta}}{\Gamma(2+\beta)} - \frac{z^{3+\beta}}{\Gamma(4+\beta)} + \frac{z^{5+\beta}}{\Gamma(6+\beta)} - \ldots\right\} \\ &= 1 - (i)^{\beta}\{1 - \text{pcos}(z,\beta) - i\,\text{psin}(z,\beta)\}, \quad \beta \geq 0 \end{aligned} \tag{8}$$

which is the general form of Euler's Formula for pre-exponential function. Clearly

$$e^{iz} = \text{pexp}(iz,0) = \text{pcos}(z,0) + i\,\text{psin}(z,0) = \cos z + i \sin z.$$

Using $-iz$ in place of iz, we have

$$\text{pexp}(-iz,\beta) = 1 - (i)^{\beta}\{1 - \text{pcos}(z,\beta) + i\,\text{psin}(z,\beta)\}, \tag{9}$$

which results in $e^{-iz} = \cos z - i \sin z$, when $\beta = 0$.

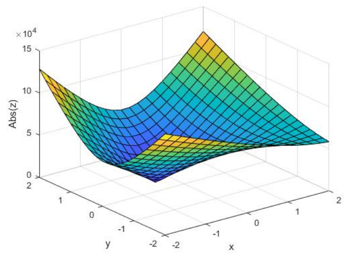

Figure 9: $\mathrm{pcosh}(z, 0.7) = 1 + \frac{z^{2.7}}{\Gamma(3.7)} + \frac{z^{4.7}}{\Gamma(5.7)} + \frac{z^{6.7}}{\Gamma(7.7)}$

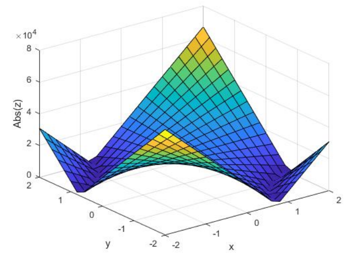

Figure 10: $\mathrm{psinh}(z, 1) = \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!}$

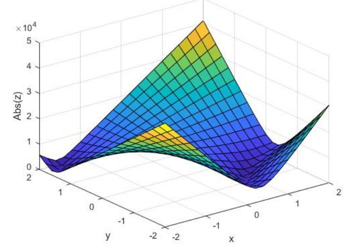

Figure 11: $\mathrm{psinh}(z, 0.7) = \frac{z^{1.6}}{\Gamma(2.6)} - \frac{z^{3.6}}{\Gamma(4.6)} - \frac{z^{5.6}}{\Gamma(6.6)}$

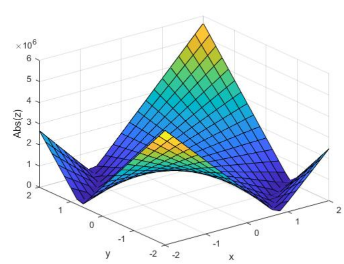

Figure 12: $M_{3,0}(z, 1) = 1 - \frac{z^4}{4!} + \frac{z^7}{7!} - \frac{z^10}{10!}$

5. Relation between pre-circular and pre- exponential functions

Using Euler's formula, the relation between circular and exponential functions, we arrive at the following:

$$\mathrm{pcos}(z,\beta) = \frac{(-i)^\beta \,\mathrm{pexp}(iz,\beta) + (i)^\beta \,\mathrm{pexp}(-iz,\beta)}{2} - \frac{(i)^\beta + (-i)^\beta}{2} + 1$$

$$\mathrm{psin}(z,\beta) = \frac{(-i)^\beta \,\mathrm{pexp}(iz,\beta) - (i)^\beta \,\mathrm{pexp}(-iz,\beta)}{2i} - \frac{(-i)^\beta - (i)^\beta}{2i}.$$

$$\mathrm{ptan}(z,\beta) = i\,\frac{(-i)^\beta - (i)^\beta - (-i)^\beta \,\mathrm{pexp}(iz,\beta) + (i)^\beta \,\mathrm{pexp}(-iz,\beta)}{(-i)^\beta \,\mathrm{pexp}(iz,\beta) + (i)^\beta \,\mathrm{pexp}(-iz,\beta) - (i)^\beta - (-i)^\beta + 2}$$

$$\mathrm{psec}(z,\beta) = \frac{2}{(-i)^\beta \,\mathrm{pexp}(iz,\beta) + (i)^\beta \,\mathrm{pexp}(-iz,\beta) - (i)^\beta - (-i)^\beta + 2}$$

Table 2: Expressions for the pre-hyperbolic functions

S.No	(z,β)	$\mathrm{pcosh}(z,\beta)$	$\mathrm{psinh}(z,\beta)$
1	$(z,0)$	$\cosh(z)$	$\sinh(z)$
2	(z,1)	$\sinh z - z + 1$	$\cosh z - 1$
3	(z,2)	$\cosh z - \frac{z^2}{2}$	$\sinh z - z$
4	$(z, 2n)$	$\cosh z - \sum_{r=1}^{n} \frac{z^{2r}}{(2r)!}$	$\sinh z - \sum_{r=1}^{n} \frac{z^{2r-1}}{(2r-1)!}$

$$\mathrm{pcosec}(z,\beta) = \frac{2i}{(-i)^\beta \,\mathrm{pexp}(iz,\beta) - (i)^\beta \,\mathrm{pexp}(-iz,\beta) + (i)^\beta - (-i)^\beta}$$

$$\mathrm{pcot}(z,\beta) = -i\,\frac{(-i)^\beta \,\mathrm{pexp}(iz,\beta) + (i)^\beta \,\mathrm{pexp}(-iz,\beta) - (i)^\beta - (-i)^\beta + 2}{(-i)^\beta - (i)^\beta - (-i)^\beta \,\mathrm{pexp}(iz,\beta) + (i)^\beta \,\mathrm{pexp}(-iz,\beta)} \quad (10)$$

whenever they exist. For $\beta = 0$, these results give the relation between circular functions and exponential functions.

6. Pre-hyperbolic Functions of a Complex Variable

The pre-hyperbolic sine and cosine functions are defined by

$$\begin{aligned} H_1(z,\beta) = \mathrm{pcosh}(z,\beta) &= 1 + \frac{z^{2+\beta}}{\Gamma(3+\beta)} \\ &+ \frac{z^{4+\beta}}{\Gamma(5+\beta)} + \frac{z^{6+\beta}}{\Gamma(7+\beta)} + \ldots \\ &= 1 + \sum_{n=1}^{\infty} \frac{z^{2n+\beta}}{\Gamma(2n+1+\beta)}, \quad z \in C, \end{aligned} \quad (11)$$

and

$$\begin{aligned} H_2(z,\beta) = \mathrm{psinh}(z,\beta) &= \frac{z^{1+\beta}}{\Gamma(2+\beta)} + \frac{z^{3+\beta}}{\Gamma(4+\beta)} + \frac{z^{5+\beta}}{\Gamma(6+\beta)} + \ldots \\ &= \sum_{n=0}^{\infty} \frac{z^{2n+1+\beta}}{\Gamma(2n+2+\beta)}, \quad z \in C. \end{aligned} \quad (12)$$

Table 2 gives the expressions for the pre-hyperbolic functions when β takes the values 0, 1, 2 and $2n$.

7. Relation between pre-hyperbolic and pre- exponential functions

$$\mathrm{pcosh}(z,\beta) = \frac{(-1)^\beta \,\mathrm{pexp}(z,\beta) + \mathrm{pexp}(-z,\beta)}{2} - \frac{1-(-1)^\beta}{2},$$

$$\mathrm{psinh}(z,\beta) = \frac{(-1)^\beta \,\mathrm{pexp}(z,\beta) - \mathrm{pexp}(-z,\beta)}{2} - \frac{(-1)^\beta - 1}{2},$$

$$\mathrm{ptanh}(z,\beta) = \frac{(-1)^\beta \,\mathrm{pexp}(z,\beta) - \mathrm{pexp}(-z,\beta) - (-1)^\beta + 1}{(-1)^\beta \,\mathrm{pexp}(z,\beta) + \mathrm{pexp}(-z,\beta) - 1 + (-1)^\beta},$$

$$\mathrm{psech}(z,\beta) = \frac{2}{(-1)^\beta \,\mathrm{pexp}(z,\beta) + \mathrm{pexp}(-z,\beta) - 1 + (-1)^\beta}$$

$$\mathrm{pcosech}(z,\beta) = \frac{2}{(-1)^\beta \,\mathrm{pexp}(z,\beta) - \mathrm{pexp}(-z,\beta) - (-1)^\beta + 1},$$

$$\mathrm{pcoth}(z,\beta) = \frac{(-1)^\beta \,\mathrm{pexp}(z,\beta) + \mathrm{pexp}(-z,\beta) - 1 + (-1)^\beta}{(-1)^\beta \,\mathrm{pexp}(z,\beta) - \mathrm{pexp}(-z,\beta) - (-1)^\beta + 1},$$

if exists. Assigning β the value 0, we find these relations reducing to the relations between Exponential and Hyperbolic functions.

8. Extended pre-functions of a Complex Variable

From generalized pre-trigonometric and pre-hyperbolic functions of a complex variable we have,

$$\begin{aligned}\text{pexp}(-z,\beta) &= 1-(-1)^{\beta}\left\{\frac{z^{1+\beta}}{\Gamma(2+\beta)}-\frac{z^{2+\beta}}{\Gamma(3+\beta)}+\frac{z^{3+\beta}}{\Gamma(4+\beta)}-....\right\}\\ &= 1+(-1)^{\beta}\sum_{n=1}^{\infty}(-1)^{n}\frac{z^{n+\beta}}{\Gamma(n+1+\beta)}.\end{aligned}$$

Trisection of the above series leads us to three infinite absolutely convergent series for $z \in C$ and $\beta \geq 0$. They are

$$\begin{aligned}M_{3,0}(z,\beta) &= 1+\sum_{n=1}^{\infty}(-1)^{n}\frac{z^{3n+\beta}}{\Gamma(3n+1+\beta)}\\ M_{3,1}(z,\beta) &= \sum_{n=0}^{\infty}(-1)^{n}\frac{z^{3n+1+\beta}}{\Gamma(3n+2+\beta)}\\ M_{3,2}(z,\beta) &= \sum_{n=0}^{\infty}(-1)^{n}\frac{z^{3n+2+\beta}}{\Gamma(3n+3+\beta)},\end{aligned} \tag{13}$$

with the initial conditions $M_{3,0}(0,\beta) = 1, M_{3,1}(0,\beta) = 0$, $M_{3,2}(0,\beta) = 0$. One can easily verify

$$\begin{aligned}M'_{3,0}(z,\beta) &= -M_{3,2}(z,\beta)\\ M'_{3,1}(z,\beta) &= (-1)^{\beta}\frac{z^{\beta}}{\Gamma(1+\beta)}+M_{3,0}(z,\beta)-1\\ M'_{3,2}(z,\beta) &= M_{3,1}(z,\beta).\end{aligned}$$

Rewriting the above system in matrix form, we have

$$\begin{bmatrix}M_{3,0}(z,\beta)\\M_{3,1}(z,\beta)\\M_{3,2}(z,\beta)\end{bmatrix}' = \begin{bmatrix}0&0&-1\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}M_{3,0}(z,\beta)\\M_{3,1}(z,\beta)\\M_{3,2}(z,\beta)\end{bmatrix}+\begin{bmatrix}0\\(-1)^{\beta}\frac{z^{\beta}}{\Gamma(z+\beta)}-1\\0\end{bmatrix}$$

$$\begin{bmatrix}M_{3,0}(z,\beta)\\M_{3,1}(z,\beta)\\M_{3,2}(z,\beta)\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} \tag{14}$$

We find the infinite series represented by $M_{3,0}(0,\beta) = 1, M_{3,1}(0,\beta) = 0, M_{3,2}(0,\beta) = 0$ to be the solutions of the system of non-homogeneous equations given by (12). The expressions in (11) define the extended pre-trigonometric functions for $n = 3$. Specifically when $\beta = 1$, we have

$$\begin{aligned}M_{3,0}(z,1) &= 1+\sum_{n=1}^{\infty}(-1)^{n}\frac{z^{3n+1}}{\Gamma(3n+2)} = M_{3,1}(z,0)-z+1\\ M_{3,1}(z,1) &= \sum_{n=0}^{\infty}(-1)^{n}\frac{z^{3n+2}}{\Gamma(3n+3)} = M_{3,2}(z,0)\\ M_{3,2}(z,1) &= \sum_{n=0}^{\infty}(-1)^{n}\frac{z^{3n+3}}{\Gamma(3n+4)} = -M_{3,0}(z,0)+1\end{aligned} \tag{15}$$

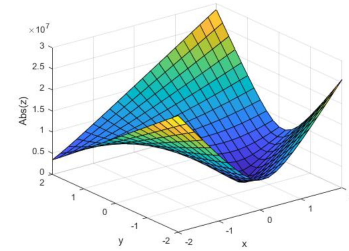

Figure 13: $M_{3,0}(z,0.5) = 1-\frac{z^{3.5}}{\Gamma(4.5)}+\frac{z^{6.5}}{\Gamma(7.5)}-\frac{z^{10.5}}{\Gamma(11.5)}$

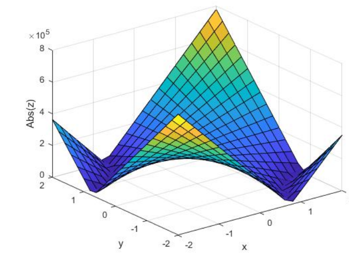

Figure 14: $M_{3,1}(z,1) = \frac{z^{2}}{2!}-\frac{z^{5}}{5!}+\frac{z^{8}}{8!}$

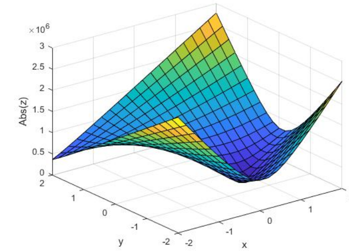

Figure 15: $M_{3,1}(z,0.5) = \frac{z^{1.5}}{\Gamma(2.5)}-\frac{z^{4.5}}{\Gamma(5.5)}-\frac{z^{7.5}}{\Gamma(8.5)}$

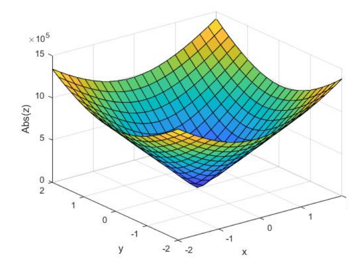

Figure 16: $M_{3,2}(z,1) = \frac{z^{3}}{3!}-\frac{z^{6}}{6!}+\frac{z^{9}}{9!}$

9. Absolute Convergence, analyticity and univalence of pre-functions

We know that every absolute convergent series is convergent. But the converse is not true. In this section we have discussed about the absolute convergence of pre-exponential function using Ratio test.

$$\text{pexp}(z,\beta) = 1+\sum_{n=1}^{\infty}\frac{z^{n+\beta}}{\Gamma(n+1+\beta)}$$

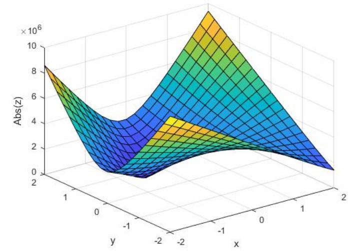

Figure 17: $M_{3,2}(z, 0.5) = \frac{z^{2.5}}{\Gamma(3.5)} - \frac{z^{5.5}}{\Gamma(6.5)} - \frac{z^{8.5}}{\Gamma(9.5)}$

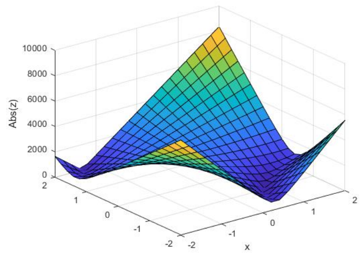

Figure 18: $\text{pexp}(z, 0.8) = 1 + \frac{z^{1.8}}{\Gamma(2.8)} + \frac{z^{2.8}}{\Gamma(3.8)} + \frac{z^{3.8}}{\Gamma(4.8)}$

Consider

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{\Gamma(n+1+\beta)}{\Gamma(n+2+\beta)}\right| \\
&= \left|\frac{(n+\beta)!}{(n+1+\beta)!}\right| \\
&= \left|\frac{(n+\beta)!}{(n+\beta+1)(n+\beta)!}\right| \\
&= \left|\frac{1}{n+\beta+1}\right| \\
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= 0
\end{aligned}$$

In similar lines the absolute convergence of the other pre-functions can also be proved. As the pre-exponential, pre-trigonometric, pre-hyperbolic and extended pre-functions are all polynomials with infinite number of terms, they are analytic throughout the complex plane, (i.e.) they are entire functions. Also they are univalent. As any function that is both analytic and univalent is conformal, so are pre-functions.

10. The transformation $w = \frac{1}{\text{pexp}(z,\beta)}$

In this section, we have obtained the image of $|\text{pexp}(z,\beta) - 2| = 1$ under the transformation $w = \frac{1}{\text{pexp}(z,\beta)}$.

$$w = \frac{1}{\text{pexp}(z,\beta)} \Rightarrow \text{pexp}(z,\beta) = \frac{1}{w}$$

and $\text{pexp}(z,\beta) = 1 + \sum_{n=1}^{\infty} \frac{z^{n+\beta}}{(n+\beta)!}$.

$$\begin{aligned}
|\text{pexp}(z,\beta) - 2| = 1 &\Rightarrow \left|1 + \sum_{n=1}^{\infty} \frac{z^{n+\beta}}{(n+\beta)!} - 2\right| = \left|\frac{1}{w} - 2\right| \\
&\Rightarrow \left|\sum_{n=1}^{\infty} \frac{z^{n+\beta}}{(n+\beta)!} - 1\right| = \left|\frac{1-2w}{w}\right|
\end{aligned}$$

Approximating the series to only one term we have

$$\left|\frac{z^2}{2} - 1\right| = \left|\frac{1-2w}{w}\right| \quad [n = 1, \beta = 1]$$

Now

$$\begin{aligned}
\left|\frac{1-2w}{w}\right| &= 1 \\
\left|1-2w\right| &= |w| \\
|1 - 2(u+iv)| &= |u+iv| \\
(u - \frac{2}{3})^2 + v^2 - \frac{1}{9} &= 0
\end{aligned}$$

which is a circle in the w-plane. Figures 2 and 3 are visualization of the given transformation.

11. Visualization of Certain pre-functions

The extended trigonometric functions $M_{3,0}(z), M_{3,1}(z)$ and $M_{3,2}(z)$ are found to be the linear independent solutions of the differential equation $z''' + z = 0$. The properties possessed by these functions are similar to that of the classical trigonometric functions but for periodicity. Due to lack of periodicity we see the graph to be oscillating with interlacing zeros. The parametric equations

$$y_1 = M_{3,0}(z), \quad y_2 = M_{3,1}(z), \quad y_3 = M_{3,2}(z)$$

will generate a surface $y_1^3 - y_2^3 + y_3^3 + 3y_1y_2y_3 = r^3$. For $\beta = 1$, the graphs of pre-trigonometric and extended pre-trigonometric functions are found to be oscillating and at the same time loosing periodicity. The graphs in Figures 4-18 show how specific pre-functions behave for some fixed values of β.

11.1. Special Cases

Following are some of the special cases obtained as a result of our study about the pre-functions of a complex variable.

1. From the first identity of (13), we obtain

   $$M_{3,0}(z_1 + z_2, 1) = M_{3,1}(z_1 + z_2, 0) - (z_1 + z_2) + 1 \quad (16)$$

   by replacing z by $z_1 + z_2$ in it.

2. Trisecting the series (1) for $\text{pexp}(z,\beta)$, three infinite absolutely convergent series namely $N_{3,0}(z,\beta), N_{3,1}(z,\beta), N_{3,2}(z,\beta)$ for $z \in C$, $\beta \geq 0$ have been obtained and these series define extended hyperbolic functions for $n = 3$. Proceeding in similar lines as it has been done for $n = 2$ and $n = 3$, n-section of the infinite series $\text{pexp}(-z,\beta)$ and $\text{pexp}(z,\beta)$ give rise to generalized extended trigonometric and hyperbolic functions.

3. Generating function for pre-Laguerre polynomial can be obtained from pexp(z,β), by replacing z using $\frac{-zy}{z-1}$.

$$\frac{1}{(1-z)}\,\text{pexp}\left(\frac{-zy}{1-z},\beta\right)$$
$$=\frac{1}{(1-z)}\left\{1+(-1)^{\beta}\sum_{r=0}^{\infty}(-1)^{r}\frac{z^{r+\beta}y^{r+\beta}}{(1-z)^{r+\beta}\Gamma(r+1+\beta)}\right\}$$
$$=\frac{1}{(1-z)}+\sum_{r=0}^{\infty}\frac{(-1)^{r+\beta}}{\Gamma(r+1+\beta)}\frac{z^{r+\beta}y^{r+\beta}}{(1-z)^{r+\beta+1}}$$
$$=\frac{1}{(1-z)}+\sum_{r=0}^{\infty}\frac{(-1)^{r+\beta}}{\Gamma(r+1+\beta)}z^{r+\beta}y^{r+\beta}(1-z)^{-(r+\beta+1)}$$
$$=\frac{1}{(1-z)}+\sum_{r=0}^{\infty}\frac{(-1)^{r+\beta}}{\Gamma(r+1+\beta)}z^{r+\beta}y^{r+\beta}\sum_{t=0}^{\infty}\frac{(r+t+\beta)!}{(r+\beta)!t!}z^{t}$$
$$=\frac{1}{(1-z)}+\sum_{r,t=0}^{\infty}(-1)^{r+\beta}\frac{(r+t+\beta)!}{\Gamma(r+\beta+1)(r+\beta)!t!}y^{r+\beta}\sum_{t=0}^{\infty}z^{r+t+\beta}$$

For a fixed value of r and taking $r+t=n$, the coefficient of z^n is

$$(-1)^{r+\beta}\frac{(n+\beta)!}{\Gamma(r+\beta+1)(r+\beta)!(n-r)!}y^{r+\beta}.$$

Taking all possible values of r into account, the total coefficient of z^n is obtained to be

$$\sum_{r=0}^{n}(-1)^{r+\beta}\frac{(n+\beta)!}{\Gamma(r+\beta+1)(r+\beta)!(n-r)!}y^{r+\beta}=L_n(y,\beta),$$

$s=n-r\geq 0$ or $r\leq n$. Here $L_n(y,\beta)$ represents the Laguerre Polynomial when $\beta=0$.

$$L_n(y,0)=\sum_{r=0}^{n}(-1)^{r}\frac{n!}{(r)!(n-r)!}y^{r}$$
$$=L_n(y)\frac{1}{(1-z)}\,\text{pexp}\left\{\left(\frac{-zy}{1-z},\beta\right)-1\right\}=\sum_{n=0}^{\infty}z^{n+\beta}L_n(y,\beta)$$

4. We can also obtain the generating function for pre-Bessel polynomial using pre-hyperbolic *sine* function. In psinh(z,β), replacing z by $\frac{zx}{2}$, we have

$$\frac{(3n+1+\beta)!}{-(n+1-\beta)!}\,\text{psinh}\left(\frac{zx}{2},\beta\right)$$
$$=\frac{(3n+1+\beta)!}{-(n+1-\beta)!}\left\{\sum_{n=0}^{\infty}\frac{z^{2n+1+\beta}x^{2n+1+\beta}}{2^{2n+1+\beta}\Gamma(2n+2+\beta)}\right\}$$
$$=\sum_{n=0}^{\infty}\frac{(3n+1+\beta)!z^{2n+1+\beta}x^{2n+1+\beta}}{-(n+1-\beta)!2^{2n+1+\beta}\Gamma(2n+1+\beta)}$$
$$=\sum_{n=0}^{\infty}\frac{(3n+1+\beta)!z^{2n+1+\beta}x^{2n+1+\beta}}{-(n+1-\beta)!2^{2n+1+\beta}(2n+1+\beta)!}$$

For a fixed n and setting $k=2n+1$, the coefficient of z^n is

$$\sum_{k=0}^{n}\frac{(k+n+\beta)!}{(n-k+\beta)!(k+\beta)!}\frac{x^{k+\beta}}{2^{k+\beta}}=Y_n(x,\beta),$$

$k\leq n$. Here $Y_n(x,\beta)$ represents the Bessel Polynomial when $\beta=0$.

$$Y_n(x,0)=\sum_{k=0}^{n}\frac{(k+n)!}{(n-k)!(k)!}(\frac{x}{2})^{k}=Y_n(x)$$
$$\frac{(3n+1+\beta)!}{-(n+1-\beta)!}\,\text{psinh}\left(\frac{zx}{2},\beta\right)=\sum_{n=0}^{\infty}z^{n+\beta}Y_n(x,\beta)$$

5. Replacement of z using $\frac{zx}{2}$ yields the generating function for pre-Legendre polynomial

$$\begin{aligned}&\frac{(3n+\alpha+1)!}{(3m+n+1+\alpha)!(2n+m+1+\alpha)!}\,\text{psin}\left(\frac{zx}{2},\alpha\right)\\&=\frac{(3n+\alpha+1)!}{(3m+n+1+\alpha)!(2n+m+1+\alpha)!}\\&*\left\{\sum_{n=0}^{\infty}(-1)^{n}\frac{z^{2n+1+\alpha}x^{2n+1+\alpha}}{2^{2n+1+\alpha}\Gamma(2n+2+\alpha)}\right\}\\&=\sum_{n=0}^{\infty}\frac{(-1)^{n}(3n+\alpha+1)!z^{2n+1+\alpha}x^{2n+1+\alpha}}{(3m+n+1+\alpha)!(2n+m+1+\alpha)!2^{2n+1+\alpha}\Gamma(2n+1+\alpha)}\\&=\sum_{n=0}^{\infty}\frac{(-1)^{n}(3n+\alpha+1)!z^{2n+1+\alpha}x^{2n+1+\alpha}}{(3m+n+1+\alpha)!(2n+m+1+\alpha)!2^{2n+1+\alpha}(2n+1+\alpha)!}\end{aligned}\tag{17}$$

Fixing n and taking $\alpha=-(n+2m+1-\beta)$, the coefficient of z^n is

$$\begin{aligned}&\sum_{m=0}^{M}(-1)^{m+\beta}\frac{(2n-2m+\beta)!x^{n-2m+\beta}}{(m+\beta)!(n-m+\beta)!(2^{n-2m+\beta})(n-2m+\beta)!}\\&=P_n(x,\beta),\end{aligned}\tag{18}$$

$m\leq n$. Here $P_n(x,\beta)$ represents the Legendre polynomial when $\beta=0$.

$$\begin{aligned}&P_n(x,0)=\sum_{m=0}^{M}(-1)^{m}\frac{(2n-2m)!x^{n-2m}}{m!(n-m)!(2^{n-2m})(n-2m)!}=P_n(x)\\&\Longrightarrow\frac{(3n+\alpha+1)!}{(3m+n+1+\alpha)!(2n+m+1+\alpha)!}\,\text{psin}\left(\frac{zx}{2},\alpha\right)\\&=\sum_{n=0}^{\infty}z^{n-2m+\beta}P_n(x,\beta)\end{aligned}\tag{19}$$

12. Conclusion

In this paper, we have introduced and investigated the properties of pre-functions and extended pre-functions for a complex variable. By fixing the value of β, we were able to graph these pre-functions and extended pre-functions. For suitable choices of z, we observe that these functions reducing to Leguerre, Bessel, and Legendre polynomials, among other special functions. Some more special functions can be derived by assuming simple functions for the variable z.

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