Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. 5 (2023) 1148

Journal of the Nigerian Society of Physical Sciences

An Alleviation of Cloud Congestion Analysis of Fluid Retrial User on Matrix Analytic Method in IoT-based Application

K. Nandhini, V. Vidhya*

Division of Mathematics, Vellore Institute of Technology, Chennai, Tamilnadu, India

Abstract

Cloud Computing (CC) and Internet of Things (IoT) are upgrowing human intervention to enhance the daily lifestyle. Currently, the heavy loaded traffic congestion is a very big challenge over IoT-based applications. For that purpose, the researchers approached various ways to overcome the congestion mechanism in recent years. Even though, they have futile to acheive the best resource storage accessing capacity expectation other than, Cloud Computing. Data sharing is a key impediment of Cloud Computing as well as Internet of Things. These are the constituent that give rise to the combination of the IoT and cloud computing paradigm as IoT Cloud. Though, preserving the missed data during the execution time is a key factor to indulge the Retrial Queueing Theory (RQT), who is facing issue upon accessing Cloud Service Provider (CSP) enter into virtual pool to preserve the data for reuse. The paper imposes Markov Fluid analysis with Matrix Analytic Method (MAM) allows the data as continuous length of data rather than individual data to avoid the congestion. The virtual orbit queue follow constant retrial rate discipline, that is, head of the orbital users makes attempt to occupy the server are assumed to be independent and identically distributed (i.i.d). Steady-state expression presented to study the behaviour of congestion. An illustrative analysis is produced to gain deep perception into the system model.

DOI:10.46481/jnsps.2023.1148

Keywords: Retrial queue, Stationary distribution, Congestion, Markov Fluid queue, Matrix analytic Method, IoT cloud computing

Article History :
Received: 26 October 2022
Received in revised form: 13 January 2023
Accepted for publication: 18 January 2023
Published: 04 April 2023

Communicated by: S. Fadugba

1. Introduction

The rising prevalence of cloud users and their anticipations has resulted in an incredible growth in the state-of-the-art of cloud computing approaches in recent years. A key primary characteristics of cloud computing and the IoT environment is better communication. The cloud is a stand-alone platform that enables users to access cloud data and resources via the Internet from any location or device. Similarly, IoT devices may access data and resources from any location. The necessary components of IoT technology include cloud properties like resource pooling, flexibility, and ubiquitous network connectivity. The on-demand and elastic nature of the exertion service enhance service reliability, which is further enhanced by massive resource pooled storage. All of these strong arguments support the necessity to combine IoT with Cloud Computing standards. Researchers have worked on numerous projects in this field. The potential and stigma associated with the bifurcation of the IoT and cloud were extensively investigated by [15] and [16]. In order to govern and monitor the complex cloud infrastructure, [25] conducted a survey on the cloud integrated IoT

*Corresponding author tel. no: +91 9791020444
Email address: vidhya.v@vit.ac.in (V. Vidhya)

dependent supervisory control and data collection security systems. The premium approach was created by [26] and is based on the amount of time the user data should wait whilst being uploaded to the cloud.

Cloud storage is a critical aspect of cloud-based technologies. As both the user's data volume and the network bandwidth are increasing at an exponential rate periodically, the device's capacity cannot keep up with the user's requirements. People were seeking a brand-new alternative to save their data, and they uncovered cloud storage, which has more powerful floor space. The practice of using cloud storage becomes increasingly sophisticated, and in the foreseeable years ahead, users' information will be held there instead. Cloud storage merely refers to a computing environment where processing and information information is made available. The cloud storage consists of a cluster of distributed file systems, applications, and cooperating network technology. The firms that provide cloud storage services include iCloud, Dropbox, Baidu Cloud, Amazon Web Services S3, Azure Cloud, Google Drive, and so forth. All such enterprises are successful in persuading an immense number of users by offering sustainable throughputs and many administrations pertaining to well-known applications. The underlying storage in this instance ensures resource pooling for different kinds of data sources. Contributions to [17],[19] as well as several sources of literature where numerous sorts of cloud storage and forthcoming concerns have been emphasised and reviewed.

Traffic congestion is the main concern among these problems. When significant amounts of heterogeneous data are transferred to sensor nodes concurrently in a cloud environment, network bandwidth congestion ensues, which in turn affects cloud application service or a contravene of the service level agreement (SLA). Few background studies on network congestion were previously known; see[20]. As in meantime, there is indeed a considerable amount of traffic whenever the utilization of cloud data has increased substantially. Now, the usage of data of cloud uses have been raised significantly leads to heavy traffic. The user has access to store data directly in the cloud, and the cloud service provider (CSP) administers the data there. The partitioning, positioning, and management of information do occurs without the user presiding over the actual storage of their data. Due to the heterogeneous dataset with diverse interface and various geographical paths to store the data, the CSP independently accesses the data in the cloud, and there is a significant risk that the user will receive inaccurate transmission. As previously stated, data losses will result from the instances taken. Information is repeated as a consequence of it though.

Accessing data from the virtual pool and waiting for CSP for web-based services are crucial components of cloud data repetition. Due to massive infrastructure and increasing number of cloud primary user and repeated user, collision among the user have been increased. New collision prevention techniques are developing in response to environmental changes. Fluid Cloud model is the ongoing model used to freely opt the CSP providing accessible data into clusters omitting the individual performance and reduce the blocking of Network bandwidth congestion, see [21]. Therefore, we proposed the model of Markov fluid queue (MFQ) approach to simplify the Matrix Analytic Method based on Retrial queueing theory to validate the result.

There is some hope for effective background traffic production in mathematical representations of traffic congestion. The idea behind abstraction is that it aggregates activity requiring less computational work. [22] established the practise of fluid queue analysis in Markovian environments. This research focuses on the distinctive elements of network behaviour, like the buffer. Use of important sampling is an additional intriguing method for quick stochastic modelling and simulation. e.g. [23]; nevertheless, given that it focuses on its quick estimation of statistics like Probability of packet loss, this is unlikely to suit our goals of effectively creating realistic background traffic. The phenomenon of fluid queues has garnered a lot of attention over the last two decades. Whilst also modelling "packet trains" as instead of individual packets, networks can be simulated at a lower computing cost. Such fluid queues have been widely regarded as valuable mathematical tools for modelling, for instance, packet speech, video systems with or without background data, computer networks, including call admission control, traffic shaping and modelling of Traffic Control Protocol (TCP), and production and inventory control. Refer to [7],[8] . The research presented in this article effectively employs the traffic model used in the previous two articles, where a traffic flow is defined by a piecewise-constant rate function.

According to [24], an effort with a specific intent analogous to our own has been made. In order to demonstrate how to make packets and the fluid model work together, a simulation model that is packet-oriented and is based on SDE was incorporated. As a packet travels through a region characterised by the fluid approach, the fluid model was utilised to estimate the overall queueing delay. Although packet flow data was used to feed the fluid model's input for the time step, neither the packets themselves nor the fluid in the fluid network directly interacted with the packet streams that passed the fluid networks at the same time.

As indicated earlier, fluid models enabled by traditional queuing systems make some progress. There is, however, no article on a fluid queue system that is driven over retrial queue background. The analytical theory of fluid models can be provided by studies on the fluid model over various queueing models. We can offer the fluid model more diversity and flexibility in the design and control of input and output rate by putting forth several retry strategies. For instance, user's requests are organized into packets and relayed via a network of routers. Requests are blocked unless the router is accessible to send packets to the server for decoding the data. In order to minimize the packet latency, the user request flow must therefore be continuous to neglect the individual effect on system performance over the Retrial Queueing paradigm. The network bandwidth congestion describes with repeated transmission using MAM to obtain system stability.

The work [1], [2] the author instigated the homogeneity of the fluid queue by explicitly defined the closed form solution. [3], have studied the buffer content distribution by expressing closed form solution of fluid over two distict queues. The study

[4] approaches a catastrophic queueing model in random environment over fluid queue investigates the stability of the system. Mao et al. [5] and [6] have studied the fluid vacation model expressing Laplacian of the stationary distribution. Kapoor, S., and Dharmaraja, S. [9] have obtained the transient analysis of fluid queue by exploiting the explicit solution of eigenvalue decomposition and subsequently computing the eigenvector interms of bisection algorithm. N.J Starreveld, R. Bekker and M. Mandjes [10] approaches an fluid queue with finite buffer along with workload process by martingale method. In [11] encouraged Markov Modulated arrival process by obtaining steady state probabiliy distribution with phase type service and impatient customers. Reader can refer from [12],[13],[14] for further reference of fluid queues

This paper makes two significant contributions. First, (i) To present a Matrix analytic method (MAM) based on data repetition phenomena to acquire the data from cloud data warehouse. (ii) to introduce a Markov fluid queue (MFQ) approach-based Retrial queueing mechanism which is used to access the heterogeneous data from various interfaces. The proposed model (iii) developed modified Bessel function of first kind which is used to show linearly independent solution leads to gradual stability and (iv) Using the fluid queue technique, we show a methodological benefit that might be used to intuitively get the stationary probability of the system states using the matrix-analytic method. Finally, by conducting extensive numerical trials, we significantly broaden the stationary performance analyses' applicability.

Following of how the remaining of this work is organized: The more pertinent and related research in the field of fluid flow approach and cloud storage has been discussed. System design is discussed with real-world instance in Section 1.1 The conceptual mathematical framework is thoroughly explained in Section 2 and Section 3. Section 4 presents the experimental setup and provides justifications with practical augmentation. The conclusion of the suggested work is provided in Section 5 with potential improvement.

1.1. System Architecture

A real-world instance of an online education cloud system is provided for illustration in Fig 1. The major components of the system including the user interface module (UI), a cloud database contains Virtual Machine Manager (VMM), a cloud broker, a Cloud Information Service (CIS), a cloud server. Here cloud user send and receive the data via User interface module. The data storage and retrieval process of the data in the cloud server are the primary responsibility of the user interface module. The cloud database contains the volume of information of cloud users. VMM is responsible for processing the assigned task, and a cloud broker partition the task into cloudlets but in our case, it is evenly spaced packet trunks where each trunk represents multiple packets, that is, "fluid packet trunks", a cloud information service (CIS) corresponds to obtain the list of resources for the assigned task to provide service and virtual server to fulfill the request of cloud users in connection for online-learning. Each Virtual Machine receives one of the 16 CPU cores that are available. A 1 TB Hadoop distributed file system storage will be deployed in the interim to increase the I/O effectiveness of the online learning system. Each VM has a preinstalled Linux operating system and "Moodle" online learning platform with a root file system size of 50GB. Each VM has a 300 GB additional root filesystem mounted in order to retain any lost data and provide an orbital storage pool in case of transmission errors. Thus, a maximum of 8 missing information in VMs can be transmitted to the server to wait for service. To guarantee the quality of service, unsuccessful task are transmitted to a VM server, which can inspect and make the corrupt root filesystem accessible again. If the server is accessible, a failing virtual machine will proceed right away to access the service in accordance with its system log file. In the event that this does not occur, the failing VM is sent to the server's storage pool to wait for a filesystem check and reconfiguration.

When a failed VM does not access the server, the cloudstack has a 50% (treated as q_1) chance to inhibit any failed information due to non-availability of server. The cloudstack may failed to prevent a impatient re-accessed information with probability 50% (treated as $\bar{p}_1$) and cause an error due to excessive congestion of cloud user.

In the beginning, all the VM are bootstrapped to speed up the process. Let the arrival rate of the failed VM be stepwise function with parameter $\lambda = 1.2/hr$. The failed VM can be re-accessed at an instant if the server is accessible. The service time is exponentially distributed with service rate $\mu = 2.0/hr$ if the server is in busy period. On the contrary, the failed information in VM is sent to storage pool if the server is busy and repeat its service with exponential amount of retrial time with rate $\theta = 0.9/hr$.

2. A framework of Mathematical structure

As a way to examine the behaviour of congestion, we take into account a single standby server Markovian queueing mechanism with retrial users. Figure 2 represents the state transition diagram.

The system consist of single server Markov Fluid queue model with constant retrial rate policy. The customer arrive according to the piecewise-constant process and the inter-arrival time distribution follows an i.i.d exponential distribution with parameter λ. The service time of a user are i.i.d follows an exponential distribution with parameter μ. If a user encounters a free server upon arrival, this user receives service immediately. If a server are busy during an arbitary user arrival epoch, the user join in virtual buffer and wait until a server becomes available with probability q_1 and if the server is not accessible, they again enter the so-called orbit after an arbitary amount of time. A user who arrive to the system from orbit is called a retrial user or exit the sytem with complementary probability $\bar{p}_1$, $0 \leq \bar{p}_1 \leq 1$, $\bar{p}_1 = 1 - p_1$.

2.1. The process of background queueing model

The behaviour of the process under examination can be derived as the conventional irreducible continuous-time Markov Chain (CTMC) $\zeta_t = \{(k_t, i_t)\}, t \geq 0$, where $k_t \in K_t$ signify the architecture.png

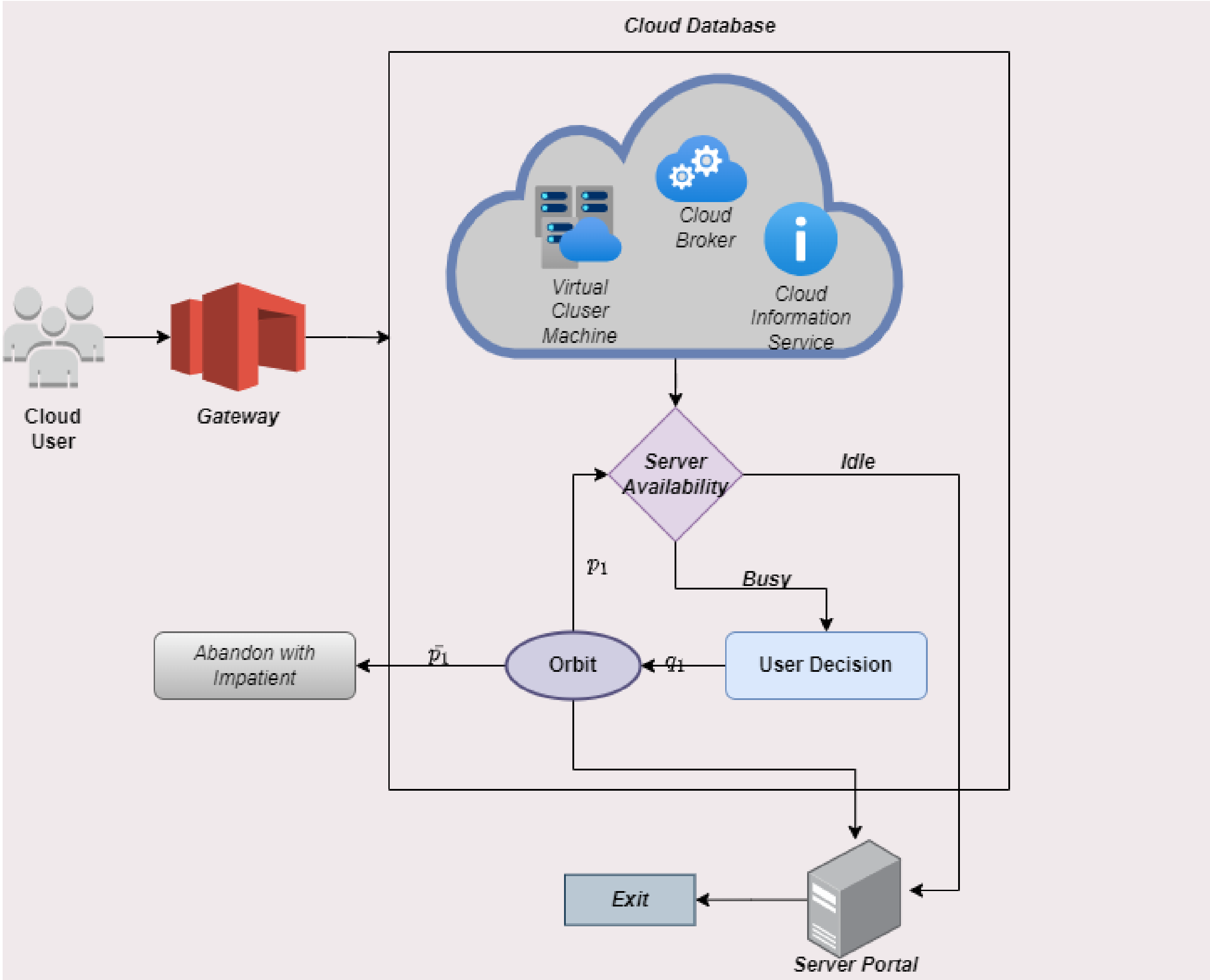

Figure 1. Overview of system design of cloud data storage

transition diagram 2.png

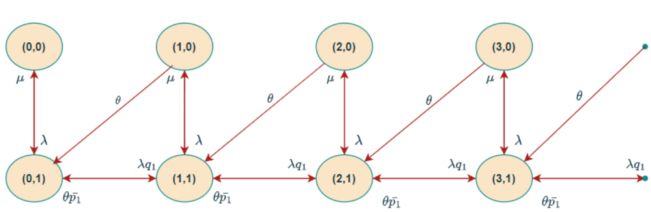

Figure 2. Transition state diagram $\zeta_t = \{(k_t, i_t)\}, t \geq 0$

$$Q = \begin{pmatrix} A_0 & C & . & & & \\ A & B & C & . & & \\ & A & B & C & . & \\ & & A & B & C & . \\ & & . & . & . & \\ & & & . & . & . \end{pmatrix}$$

amount of data packets accumulated in the orbit, $K_t \geq 0$; $i_t \in I_t$ designate the state of the underlying process of QBD process, $i_t \in \{0, 1\}$, where, $i_t = 0$, the state is in operational mode; $i_t = 1$, the state is in non-operational mode.

Theorem 1. *The infinitesimal generator Q structured as block-tridiagonal matrices of the Markov chain* $\zeta_t, t \geq 0$

where

$$A_0 = \begin{pmatrix} -\lambda & \lambda \\ \mu & -(\lambda q_1 + \mu) \end{pmatrix}$$
$$C = \begin{pmatrix} 0 & 0 \\ 0 & \lambda q_1 \end{pmatrix}$$
$$A = \begin{pmatrix} 0 & \theta \\ 0 & \theta \bar{p}_1 \end{pmatrix}$$
$$B = \begin{pmatrix} -(\lambda + \theta) & \lambda \\ \mu & -(\lambda q_1 + \mu + \theta \bar{p}_1) \end{pmatrix}$$

Proof. Theorem 2.1 is demonstrated by analysing each transition in the Markov chain $\zeta_t, t \geq 0$ during the interval of infintesimal length, and by reconfiguring the intensities into block matrices.
Here the intensities of the transition assumed to be continuous entity, termed as fluid. Individual users impact will be negligible since the flow will be continuous in the system during an interval of an infintesimal length, the non-diagonal blocks are zero matrix, thus the generator Q has the block-tridiagonal structure.
The diagonal entries of the 2×2 square block matrix A_0, define the intensities of the transition of the chain $\zeta_t, t \geq 0$ that does not affect the number of users in the orbit. The intensities of an event are given by $-(\lambda q_1 + \mu)$ defines the user enter orbit with probability q_1 and ends with customers service. The intensities of the former event are defined by the entries of the event that insist arrival of users in the orbit while the latter event defines the service of the user μ.
The super-diagonal 2×2 matrix C, defines the intensities of the event are given by λq_1, that is, the probability of joining the so-called orbit who finds server busy. The sub-diagonal 2×2 matrix A, defines the intensities of the event are given by θ, enter into a retrial orbit with the transition probability $(1, 0) \rightarrow (0, 1)$ and leave with impatience with $\theta \bar{p}_1$
The diagonal entries of the matrix B, defines the intensities of the event are given by $-(\lambda + \theta)$ that the arrival of user encounters an busy server, enter into a retrial orbit with probability θ and probability of joining the so-called orbit who finds server busy λq_1. The intensity of the former event is arrival of users in the orbit while the latter event describes zero
Hence the proof □

3. Markov Fluid Analysis

[1] This section includes the set of governing equation driven by QBD process over fluid retrial queue with non-persistent users. Let x be the buffer occupancy at time $'t'$. Let $B(t)$ denotes the amount of fluid data packets in the buffer at time $'t'$.

[1]The efficacious action of the buffer is defined as

$$\eta(B(t), K(t), I(t)) = \frac{d(B(t))}{dt} = \begin{cases} r_0, (K(t), I(t)) = (k, 0), k \geq 0 \\ r, (K(t), I(t)) = (k, 1), k \geq 0 \\ 0, B(t) = 0 \end{cases}$$

We know that, $r_0 \geq r \geq 0$. The occupancy of the buffer when the server is in non-operational (idle) period is much greater than the buffer occupancy in the operational (busy) period. Clearly, buffer occupancy is linear increasing at the rate of r_0, as soon as users begin the driving process and the process is still in progress; buffer occupancy is linear increasing at the rate of r according to the driving process stays in operational period. Also, the buffer occupancy level cannot depleted until the buffer is empty. As soon as the buffer occupancy level varies, the effective request-service rate of fluid remains unstable. In order to ensure the stability condition, define the average drift condition should satisfy $d < 0$

$$d = r_0 \sum_{k=1}^{\infty} \mathcal{F}_{k0} + r \sum_{k=1}^{\infty} \mathcal{F}_{k1} < 0$$

The following set of differential equations that satisfy a quasi-birth and death process for a stationary joint distribution $\mathcal{F}_{ki}(t)$ can easily be solved using the conventional approach. Define $\mathcal{F}_{ki}(t)$ is the stationary joint distribution when $k_t \in K_t$ users in the retrial orbit at time $'t'$ and $i_t \in I_t$ represents the affiction of the server

$$r_0 \frac{d\mathcal{F}_{00}(t)}{dt} = -\lambda \mathcal{F}_{00}(t) + \mu \mathcal{F}_{01}(t) \tag{1}$$
$$r_0 \frac{d\mathcal{F}_{k0}(t)}{dt} = -\lambda \mathcal{F}_{k0}(t) + \mu \mathcal{F}_{k1}(t) - \theta \mathcal{F}_{k+1,0}(t) \tag{2}$$
$$r \frac{d\mathcal{F}_{01}(t)}{dt} = \lambda \mathcal{F}_{00}(t) - (\lambda q_1 + \mu)\mathcal{F}_{01}(t) - \theta \mathcal{F}_{1,0}(t) + \theta \bar{p}_1 F_{11}(t) \tag{3}$$
$$r \frac{d\mathcal{F}_{11}(t)}{dt} = \lambda q_1 \mathcal{F}_{01}(t) + \lambda \mathcal{F}_{10}(t) + \theta \mathcal{F}_{2,0}(t) - (\lambda p_1 + \mu + \theta \bar{p}_1)\mathcal{F}_{11}(t) + \theta \bar{p}_1 \mathcal{F}_{21}(t) \tag{4}$$

with boundary conditions

$$\mathcal{F}_{00}(0) = a_1$$
$$\mathcal{F}_{01}(0) = a_2$$
$$\mathcal{F}_{k_t,i_t}(0) = (k_t, i_t), \quad (k_t, i_t) \in \Omega\{(k_t, 0) \cup (k_t, 1)\}$$

where $0 < a_1, a_2 < 1$

The boundary conditions are feasible to the solution since there is linear growth of customers in the background driving process and it has some non-negative probability distributed around the boundary region. Also the buffer gets empty when the buffer occupancy level decreases at the rate of r_0. Thus the boundary conditions are satisfied.

3.1. Markov Decision Process

In this theoretical evaluation, the steady state analysis of the fluid retrial queue is presented with the detailed study of average buffer occupancy expressed in an closed form solution of stationary distribution and buffer occupancy is determined in the form of modified Bessel function of first kind. The set of differential equations are governed by Quasi Birth-Death process evaluated interms of infintesimal generator matrix. The set of matrix quadratic equations defined to provide an explicit solution for joint steady state analytic distribution. Now define

$$\mathcal{F}_0(t) = \{\mathcal{F}_{00}(t), \mathcal{F}_{01}(t)\}$$
$$\mathcal{F}_k(t) = \{\mathcal{F}_{k,0}(t), \mathcal{F}_{k,1}(t)\}, \quad j = 1, 2, ...$$

Therefore,

$$\mathcal{F}(t) = (\mathcal{F}_0(t), \mathcal{F}_1(t), \mathcal{F}_2(t), ...)$$

The set of differential equation can be rewritten with the joint probability density sequence in the matrix form as

$$\frac{d\mathcal{F}(t)}{dt}\Lambda = \mathcal{F}(t)Q \quad (5)$$

where

$$\Lambda = \begin{pmatrix} \Lambda_0 & . & . & & & \\ & \Lambda_1 & & . & & \\ & & & \Lambda_1 & . & \\ & & & & \Lambda_1 & . \\ & & . & . & . & \end{pmatrix}$$

Here

$$\Lambda_0 = \begin{pmatrix} r_0 & 0 \\ 0 & r_0 \end{pmatrix}, \quad \Lambda_1 = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$$

To depict the Laplace transform of the system of matrix solutions to provide an analytical interpretation to the stationary analysis of the buffer capacity distribution.

$$\hat{\mathcal{F}}(s) = (\hat{\mathcal{F}}_0(s), \hat{\mathcal{F}}_1(s), \hat{\mathcal{F}}_2(s), \dots)$$

then the above matrix form is given as

$$\hat{\mathcal{F}}(s)(Q - s\Lambda) = (-\bar{a}\Lambda, 0, 0...), \quad \bar{a} = (r_0 a_1, r_0 a_2)$$

The set of differential equations is given by

$$\hat{\mathcal{F}}_0(s)(A - s\Lambda_0) - \hat{\mathcal{F}}_1(s)A = -\bar{a}\Lambda \quad (6)$$
$$\hat{\mathcal{F}}_0(s)C + \hat{\mathcal{F}}_1(s)(B - s\Lambda_1) + \hat{\mathcal{F}}_2(s)A = 0 \quad (7)$$
$$\hat{\mathcal{F}}_{k-1}(s)C + \hat{\mathcal{F}}_k(s)(B - s\Lambda_1) + \hat{\mathcal{F}}_{k+1}(s)A = 0 \quad (8)$$

Theorem 2. *A necessary sufficient condition for the driving queueing process* $\{K_t = k_t, I_t = i_t; t \geq 0\}$ *is given by* $\rho = \frac{\lambda q_1(\lambda+\theta)}{\theta\mu_p}$ *where* $\mu_p = \mu(\lambda + \theta)\bar{p}_1$

Proof. From the framework of the generator matrix, we know that $\{K_t = k_t, I_t = i_t; t \geq 0\}$ is a Quasi Birth-Death process. Let us define $L = A + B + C$.

$$L = \begin{pmatrix} 0 & 0 \\ 0 & \lambda q_1 \end{pmatrix} + \begin{pmatrix} 0 & \theta \\ 0 & \theta\bar{p}_1 \end{pmatrix} + \begin{pmatrix} -(\lambda + \theta) & \lambda \\ 0 & \lambda q_1 \end{pmatrix}$$
$$= \begin{pmatrix} -(\lambda + \theta) & (\lambda + \theta) \\ \mu & -\mu \end{pmatrix}$$

Clearly, L is a finite integrable matrix. Its steady-state probability vector defined as $\pi = (\pi_0, \pi_1)$ satisfies $(\pi_0, \pi_1)L = 0$ where $\pi_0 + \pi_1 = 1$ The obtained solution is $\pi_0 = \frac{\mu}{\lambda+\theta}$ and $\pi_1 = \frac{\lambda+\theta}{\mu}$ A necessary sufficient condition for the driving queueing process $\{K_t = k_t, I_t = i_t; t \geq 0\}$ is $\pi Ce < \pi Ae$, where e is a two dimensional column vector whose elements are all equal to one. Then using simple expression, we obtained the result $\rho = \frac{\lambda q_1(\lambda+\theta)}{\theta\mu_p}$ where $\mu_p = \mu(\lambda + \theta)\bar{p}_1$

□

Theorem 3. *The evaluation of rate matrix is determined in the form of matrix quadratic equation is given by* $R^2(s)A + R(s)(B - s\Lambda_1) + C = 0$ *yields minimal non-negative solution such as*

$$R(s) = \begin{pmatrix} \chi(s) & \beta(s) \\ 0 & r(s) \end{pmatrix}$$

Proof. We make an assumption for the rate matrix $R(s)$ has the structure is of the 2×2 matrix form such that

$$R = \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix}$$

From the equation above, we obtain,

$$\begin{pmatrix} r_{11}^2 & r_{12}(r_{11} + r_{22}) \\ 0 & r_{22}^2 \end{pmatrix}\begin{pmatrix} 0 & \theta \\ 0 & \theta\bar{p}_1 \end{pmatrix} + \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix}\begin{pmatrix} -(\lambda + \theta + sr) & \lambda \\ \mu & -(\lambda q_1 + \mu + \theta\bar{p}_1) + sr \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \lambda q_1 \end{pmatrix} = 0$$

The set of differential equations are determined as follows:

$$r_{11}(\lambda + \theta + sr) + r_{12}\mu = 0 \quad (9)$$
$$r_{11}^2\theta + r_{12}(r_{11} + r_{22})\theta\bar{p}_1 + r_{11}(\lambda q_1 + \mu + \theta\bar{p}_1 + sr) = 0 \quad (10)$$
$$r_{22}\mu = 0 \quad (11)$$
$$r_{22}^2\theta\bar{p}_1 - r_{22}(\lambda q_1 + \mu + \theta\bar{p}_1 + sr) + \lambda q_1 = 0 \quad (12)$$

The solution from Equation 7 expressed as

$$r_{22} = r(s)$$
$$= (\lambda q_1 + \mu + \theta\bar{p}_1 + sr) \pm \frac{\sqrt{(\lambda q_1 + \mu + \theta\bar{p}_1 + sr)^2 - 4\theta\bar{p}_1\lambda q_1}}{2\theta\bar{p}_1} \quad (13)$$

Let the above equation has positive and negative eigenvalues namely,$r(s)$ and $r_1(s)$ .Since the root that lies inside the unit disc, we consider $r(s)$ for further reference. Therefore it satisfies the following recurrsion relation as follows:

$$(\lambda q_1 + \mu + sr + \theta\bar{p}_1(1 - r)) = \frac{\mu}{1 - r} + \theta\bar{p}_1 + \frac{sr}{1 - r} = \frac{\lambda q_1}{r} \quad (14)$$

From the equation above, we get,

$$\chi(s) = \frac{\rho + sr}{r(s)} - \frac{\lambda\mu}{\theta\mu_p} \quad (15)$$

where $\rho = \frac{\lambda q_1(\lambda+\theta+sr)}{\theta\mu_p}$ and $\mu_p = \mu + (\lambda + \theta + sr)\bar{p}_1$.
Using the expression given in Equation 8, the solution yields,

$$\beta(s) = \frac{(\lambda + \theta + sr)\rho}{\mu r(s)} - \lambda\mu\frac{\lambda + \theta + sr}{\mu\theta\mu_p} \quad (16)$$

Hence, the rate matrix is given by

$$R(s) = \begin{pmatrix} \frac{\rho}{r(s)} - \frac{\lambda\mu}{\theta\mu_p} & \frac{(\lambda+\theta+sr)\rho}{\mu r(s)} - \lambda\mu\frac{\lambda+\theta+sr}{\mu\theta\mu_p} \\ 0 & (\lambda q_1 + \mu + \theta\bar{p}_1 + sr) \pm \frac{\sqrt{(\lambda q_1+\mu+\theta\bar{p}_1+sr)^2 - 4\theta\bar{p}_1\lambda q_1}}{2\theta\bar{p}_1} \end{pmatrix}$$

$$= \begin{pmatrix} \chi(s) & \beta(s) \\ 0 & r(s) \end{pmatrix}$$

$$\vdots$$

$$R^{n-1} = \begin{pmatrix} \chi(s)^{n-1} & \beta(s)(\chi(s)^{n-2} + r(s)^{n-2} + \sum_{i=1}^{n-3}\chi(s)^i r(s)^{n-2-i}) \\ 0 & r(s)^{n-1} \end{pmatrix}$$

In general,

$$R^n = \begin{pmatrix} \chi(s)^n & \beta(s)(\chi(s)^{n-1} + r(s)^{n-1} + \sum_{i=1}^{n-2}\chi(s)^i r(s)^{n-1-i}) \\ 0 & r(s)^n \end{pmatrix}$$

Hence the result is proved □

Theorem 4. *The steady state probability distribution of the buffer occupancy in the driving queueing model gives an analytical solution in an Laplace domain determined as*

$$\hat{\mathcal{F}}_{k0}(s) = \hat{\mathcal{F}}_{00}(s)\chi(s)^n$$

$$\hat{\mathcal{F}}_{k1}(s) = \hat{\mathcal{F}}_{00}(s)\beta(s)(\chi(s)^{n-1} + r(s)^{n-1}\sum_{i=0}^{n-2}\chi(s)^i r(s)^{n-1-i}) + \hat{\mathcal{F}}_{01}(s)r(s)^n$$

Proof. Without loss of generality, let us assume $\mathcal{F}_k(s) = \mathcal{F}_{k-1}(s)R(s)$ can be rewritten as $\mathcal{F}_k(s) = \mathcal{F}_0(s)R^k(s)$
From Equation 6,7, we have

$$\begin{aligned} &= \hat{\mathcal{F}}_{k-1}(s)C + \hat{\mathcal{F}}_k(s)(B - sR) + \hat{\mathcal{F}}_{k+1}(s)A \\ &= \hat{\mathcal{F}}_{k-1}(s)C + \hat{\mathcal{F}}_{k-1}(s)R(s)(B - sR) + \hat{\mathcal{F}}_{k-1}R^2(s)A \\ &= \hat{\mathcal{F}}_{k-1}(s)\{C + R(s)(B - sR) + R^2(s)A\} = 0 \end{aligned}$$

$$\begin{aligned} &= \hat{\mathcal{F}}_0(s)C + \hat{\mathcal{F}}_1(s)(B - sR) + \hat{\mathcal{F}}_2(s)A \\ &= \hat{\mathcal{F}}_0(s)C + \hat{\mathcal{F}}_0(s)R(s)(B - sR) + \hat{\mathcal{F}}_0(s)R^2(s)A \\ &= \hat{\mathcal{F}}_0(s)\{C + R(s)(B - sR) + R^2(s)A\} = 0 \end{aligned}$$

From Equation 6, we get

$$\hat{\mathcal{F}}_0(s)(A - sR_1) - \hat{\mathcal{F}}_1(s)A = -\bar{a}$$

$$\hat{\mathcal{F}}_0(s) = \frac{\bar{a}}{sR_1 - A_0 - R(s)A} \tag{17}$$

The proof is provided in the Appendix with all sufficient conditions. □

3.2. Stationary Analysis of Buffer Content Distribution

Theorem 5. *The stationary distribution of the buffer occupancy level can be expressed analytically to provide explicit solution to the joint steady probability vector[1]*

$$\hat{\mathcal{F}}(s) = P(X \le x) = \sum_{k=0}^{\infty}\{\hat{\mathcal{F}}_{k0}(s) + \hat{\mathcal{F}}_{k1}(s)\} \tag{18}$$

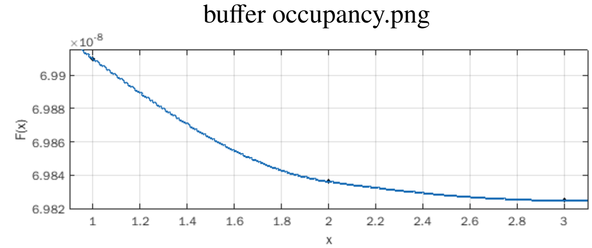

Figure 3. A Stationary buffer occupancy distribution $F(x)$ against x

Proof. Taking Laplace transform on the above equation, we get

$$\begin{aligned} \hat{\mathcal{F}}(s) &= \hat{\mathcal{F}}_0(s)e_1 + \hat{\mathcal{F}}_0(s)\sum_{k=1}^{\infty}R^n(s)e_k \\ &= \hat{\mathcal{F}}_0(s)e_1 + \hat{\mathcal{F}}_0(s)e(I - R(s))^{-1}e_2 \end{aligned} \tag{19}$$

where

$$\begin{aligned} (I - R(s))^{-1} &= \frac{1}{(I - \chi(s))(I - r(s))}\begin{pmatrix} I_r(s) & \beta(s) \\ 0 & I - \chi(s) \end{pmatrix} \\ &= \begin{pmatrix} (I_\chi(s))^{-1} & \beta(s)(I - \chi(s))(I - r(s))^{-1} \\ 0 & (I - r(s))^{-1} \end{pmatrix} \end{aligned}$$

$$\hat{\mathcal{F}}(s) = \hat{\mathcal{F}}_{00}(s)\sum_{j=0}^{n-1}\chi(s)^j + \hat{\mathcal{F}}_{00}(s)\sum_{j=0}^{n-1}\chi(s)^j\beta(s)\sum_{k=0}^{\infty}r(s)^n + \hat{\mathcal{F}}_{01}(s)r(s)^n$$

$$\begin{aligned} &= \sum_{k=0}^{\infty}\left(\rho e^{-\frac{(\lambda q_1+\mu+\theta\bar{p}_1)}{r}x}\frac{jI_j(\beta x)\beta^j}{x}\left(\frac{r}{2\theta\bar{p}_1}\right)^j - \frac{\lambda\mu}{\theta\mu_p}\right)^* kF_{00}(s) \\ &+ \beta(x) * \sum_{k=0}^{\infty}\left(\frac{r}{2\theta\bar{p}_1}\right)^k\frac{kI_k(\beta x)\beta^j}{x}e^{-\frac{(\lambda q+\mu+\theta\bar{p}_1)}{r}x} \\ &* \sum_{k=0}^{\infty}\left(\rho e^{-\frac{(\lambda q_1+\mu+\theta\bar{p}_1)}{r}x}\frac{jI_j(\beta x)\beta^j}{x}\left(\frac{r}{2\theta\bar{p}_1}\right)^j - \frac{\lambda\mu}{\theta\mu_p}\right)^* kF_{00}(s) \\ &+ \hat{\mathcal{F}}_{01}(s) * \sum_{k=0}^{\infty}\left(\frac{r}{2\theta\bar{p}_1}\right)^k\frac{kI_k(\beta x)\beta^j}{x}e^{-\frac{(\lambda q_1+\mu+\theta\bar{p}_1)}{r}x} \end{aligned} \tag{20}$$

Hence, evaluation of buffer content distribution are expressed in terms of $\hat{\mathcal{F}}_{00}(s)$ and $\hat{\mathcal{F}}_{01}(s)$. Therefore the measure of performance can be given using this explicit analytical solution given in the form of modified Bessel function of first kind. □

4. Analysis of Statistical illustration

To acquire a holistic view of the model stability and performance, this section describe the findings of certain numerical experiments in the aspect of system stability with intrinsic of Markov fluid analysis. We examine the steady-state convergence of several measures to their exact theoretical values of buffer occupancy distribution.png

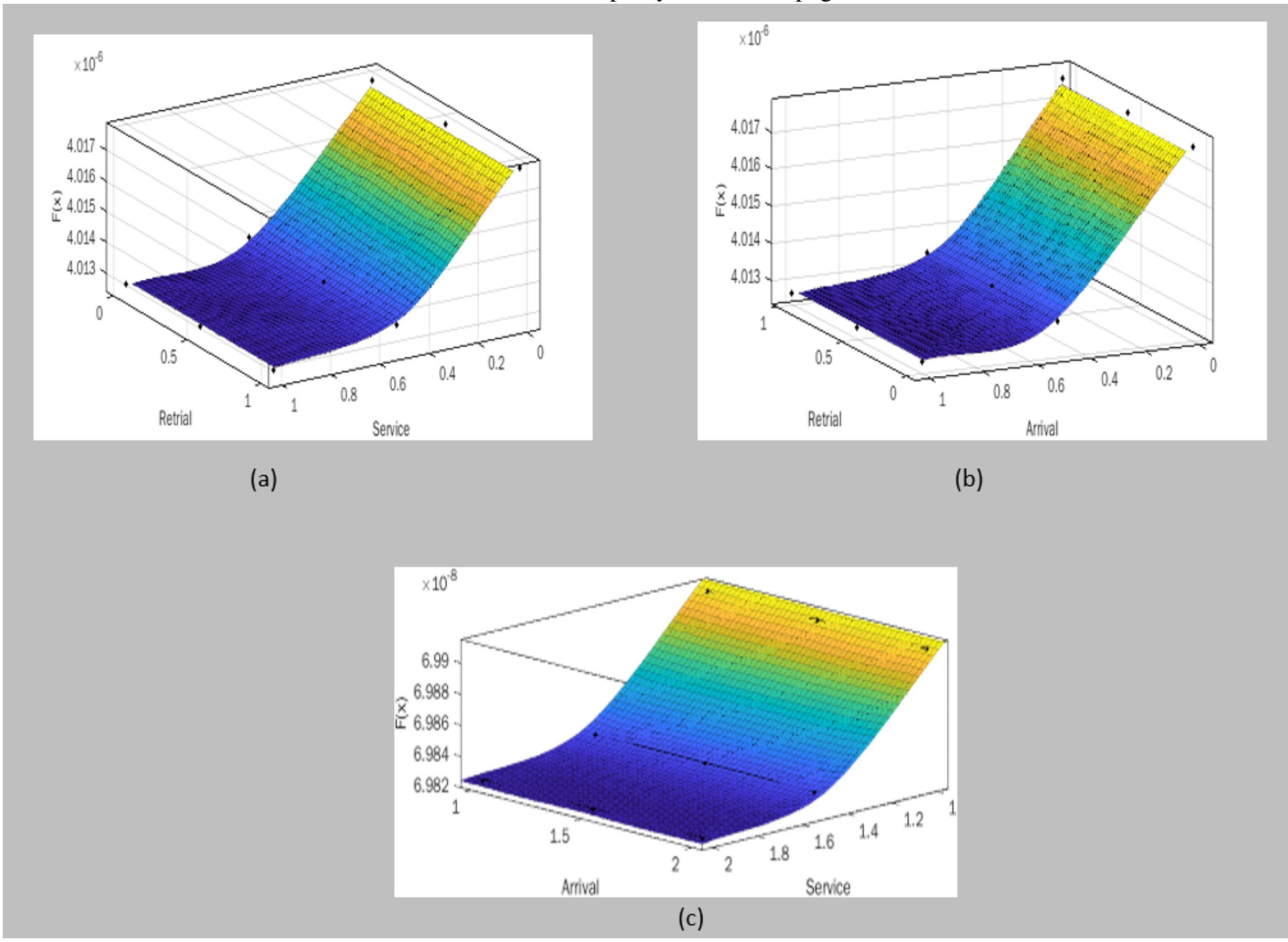

Figure 4. Dependency of buffer occupancy distribution against the variation of parameters

where analytical expressions are given in the depiction for the purpose of model validation. Following the model's validation, we use empirical correlation to conduct a numerical investigation of a few auxiliary measures for the dependence of orbits, arrivals, and service patterns. The simulation is carried out using MATLAB and is based on the continuous event procedure.

As a desired set approach, the following preceding parameters are chosen for the illustrated example of an online education cloud system shown in Section 1.1: $\lambda = 1.2/hr$, $\mu = 2.0/hr$, $\theta = 0.9/hr$, $q_1 = 0.5$, $\bar{p}_1 = 0.5$. The values are selected in such a way to guarantee sufficient stability condition Equation 20 holds true.

The graph displayed in Figure 3 displays the difference of buffer capacity distribution against time t with retrial rate θ using the appropriate value of θ considered for the purpose of comparison. It is to be noted that the retrial rate θ value increases with increases of buffer capacity distribution, particularly at the values of buffer capacity t more than 0.4. It is strictly convex montonically increasing function proves stability in relation with the dependent variable of λ, μ and θ.

In Figure 4 depicts the behaviour of buffer content distribution in three dimensions with variation of the system estimation parameters dependence of λ, μ and θ. In particular, Figure 4(a) explains the variation of buffer content distribution along with service rate μ and retrial rate θ has validate the result. Figure 4(b) depicts the evolution of stationary distribution in three dimension with variation of the arrival λ and retrial rate θ. This figure explains the variation of buffer content distribution increase along with increment of the arrival rate λ and retrial rate θ. It is concluded that the numerical illustration provided has validate the result. Figure 4(c) depicts the behaviour of buffer content distribution in three dimensions with variation of the arrival λ and service μ. This figure explains the evolution of buffer content distribution is strictly increasing along with increment of the arrival rate λ and service rate μ. It is to be concluded that the numerical illustration provided has validate the result.

To illustrate our result, we simulate M/M/1 type queueing system. The main motivation of our simulation study is to learn the behaviour of the performance measure which are associated with the buffer content distribution against the variation of some system parameters.

5. Conclusion

In this research, steady-state evaluation of the Markovian queueing model with constant retrial rate with non-persistent users is demonstrated. It is displayed that the service and inter-retrial times are linearly independent of one another. A Markov fluid queue model serves as a foundation for analytic viewpoint. Besides, we demonstrate how an interaction of retrial correlation with matrix analytic method allows to optimize the performance efficiency by reducing the packet service delay and congestion avoidance. In future, the extended result will contemplate to control the congestion using time-inhomogeneous in a finite source retrial queue.

Acknowledgments

The authors also acknowledge the Vellore Institute of Technology (VIT) at Chennai for providing resources that contributed to the research results reported within this paper.

References

[1] K. Vijayashree & A. Anjuka, "Fluid Queue Driven by an M/M/1 Queue Subject to Bernoulli-Schedule-Controlled Vacation and Vacation Interruption", Advances in Operations Research (2016) 1.
[2] B. Mao, F. Wang & N. Tian, "Fluid model driven by an M/M/1 queue with multiple vacations and N-policy", Journal of Applied Mathematics and Computing **38** (2010) 119.
[3] A. H. El-Baz, A. M. Tarabia & A. M. Darwiesh, "Cloud Storage Facility as a Fluid queue controlled by Markovian Queue", Probability in the Engineering and Informational Sciences (2020) 1.
[4] S. Ammar, "Fluid M/M/1 catastrophic queue in a random environment", RAIRO - Operations Research **55** (2021) S2677.
[5] S. Kapoor & S. Dharmaraja, "Steady state analysis of fluid queues driven by birth death processes with rational rates", International Journal of Operational Research 37 (2020) 562.
[6] B. Mao, F. Wang & N. Tian, "Fluid model driven by an M/M/1 queue with multiple exponential vacations and N-policy", Journal of Applied Mathematics and Computing **38** (2012) 119.
[7] Q.M. He, *Matrix-analytic methods in stochastic models*, Springer (2008).
[8] R. Malhotra, M. Mandjes, W. Scheinhardt & W, J. Van den Berg, "A feedback fluid queue with two congestion control thresholds", Mathematical Methods of Operations Research **70** (2008) 149.
[9] V. Arunachalam, S. Kapoor & S. Dharmaraja, "Transient solution of fluid queue modulated by two independent birth-death processes", International Journal of Operational Research **36** (2019) 1.
[10] N. Starreveld, R. Bekker & M. Mandjes, "Occupation times for the finite buffer fluid queue with phase-type ON-times", Operations Research Letters **46** (2018) 27.
[11] G. Horv'th & B. Van Houdt, "A Multi-layer Fluid Queue with Boundary Phase Transitions and Its Application to the Analysis of Multi-type Queues with General Customer Impatience", in 2012 Ninth International Conference on Quantitative Evaluation of Systems (2012) 23.
[12] S. Kapoor & D. Selvamuthu, "On the exact transient solution of fluid queue driven by a birth death process with specific rational rates and absorption", Operation Research **52** (2015) 746.
[13] J. Bosman & R. Núñez-Queija, "A spectral theory approach for extreme value analysis in a tandem of fluid queues", Queueing Systems **78** (2014) 121.
[14] V. Arunachalam, V. Gupta & S. Dharmaraja, "A fluid queue modulated by two independent birth–death processes", Computers and Mathematics with Applications **60** (2010) 2433.
[15] A. R. Biswas, R. Giaffreda, "Iot and cloud convergence: Opportunities and challenges", in 2014 IEEE World Forum on Internet of Things (WF-IoT) (2014) 375.
[16] P. P. Ray, "A survey of iot cloud platforms", Future Computing and Informatics Journal **1** (2016) 35.
[17] A. Sajid, H. Abbas & K. Saleem, "Cloud-assisted iot-based scada systems security: A review of the state of the art and future challenges", IEEE Access **4** (2016) 1375.
[18] W. Wang, P. Xu & L. T. Yang, "Secure data collection, storage and access in cloud-assisted iot", IEEE Cloud Computing, 5 (2018) 77.
[19] J. S. Fu, Y. Liu, H. C. Chao, B. K. Bhargava & Z. J. Zhang, "Secure data storage and searching for industrial iot by integrating fog computing and cloud computing", IEEE Transactions on Industrial Informatics **14** (2018) 4519.
[20] K. Liu & L. J. Dong, "Research on cloud data storage technology and its architecture implementation", Procedia Engineering **29** (2012) 133.
[21] I. Odun-Ayo, O. Ajayi, B. Akanle, R. Ahuja, "An overview of data storage in cloud computing", in 2017 International Conference on Next Generation Computing and Information Systems (ICNGCIS) (2017) 29. https://doi.org/10.1109/ICNGCIS.2017.9.
[22] S. Kuppusamy, V. Kaniappan, D. Thirupathi & T. Ramasubramanian, "Switch bandwidth congestion prediction in cloud environment", Big Data, Cloud and Computing Challenges, Procedia Computer Science **50** (2015) 235.
[23] R. Gu, K. Zhang, Z. Xu, Y. Che, B. Fan, H. Hou, H. Dai, L. Yi, Y. Ding, G. Chen & Y. Huang, "Fluid: Dataset abstraction and elastic acceleration for cloud-native deep learning training jobs", in 2022 IEEE 38th International Conference on Data Engineering (ICDE) (2022) 2182. https://doi.org/10.1109/ICDE53745.2022.00209.
[24] T. Yung, J. Martin, M. Takai & R. Bagrodia, "Integration of fluid-based analytical model with packet-level simulation for analysis of computer networks", in Proceedings of SPIE - The International Society for Optical Engineering (2001).
[25] M. Mandjes, D. Mitra & W. R. Scheinhardt, "Models of network access using feedback fluid queues", Queueing Systems **44** (2003) 365.
[26] O. E. Ojo, M. K. Kareem, O. Samuel & C. O. Ugwunna, "An internet-of-things based real-time monitoring system for smart classroom", Journal of the Nigerian Society of Physical Sciences **4** (2022) 573.
[27] E. E. Akpanibah & U. O. Ini, "An Investor's Investment Plan with Stochastic Interest Rate under the CEV Model and the Ornstein-Uhlenbeck Process", Journal of the Nigerian Society of Physical Sciences **3** (2021) 172.
[28] A. Vinodhini & V. Varadharajan, "Computational analysis of queues with catastrophes in a multiphase random environment", Mathematical Problems in Engineering (2016) 1.
[29] R. Sakthi, V. Varadharajan & K. Mahaboob, "Performance measures of state dependent mmpp/m/1 queue", International Journal of Engineering and Technology **4** (2018) 942.

Appendix

Proof of Theorem 4. The proof is built on the analysis of buffer occupancy distribution given by

$$\hat{\mathcal{F}}_0(s) = \frac{a_1 r_0, a_2 r_0}{s\begin{pmatrix} r_0 & 0 \\ 0 & r_0 \end{pmatrix} - \begin{pmatrix} -\lambda & \lambda \\ \mu & -(\lambda q + \mu) \end{pmatrix} - \begin{pmatrix} \chi(s) & \beta(s) \\ 0 & r(s) \end{pmatrix}\begin{pmatrix} 0 & \theta \\ 0 & \theta\bar{p} \end{pmatrix}}$$

$$= (a_1 r_0, a_2 r_0)\begin{pmatrix} s r_0 + \lambda & -(\lambda\theta\chi(s) + \theta\bar{p}\beta(s)) \\ -\mu & s r_0 + (\lambda q + \mu)\theta\bar{p} r(s) \end{pmatrix}^{-1}$$

$$\hat{\mathcal{F}}_{00}(s) = \frac{a_1 r_0(s r_0 + \lambda q_1 + \mu - \theta\bar{p}_1 r(s)) + a_2 r_0 \mu}{(s r_0 + \lambda)(s r_0 + \lambda q_1 + \mu - \theta\bar{p}_1 r(s)) - \mu(\lambda + \theta\chi(s) + \theta\bar{p}_1\beta(s))}$$

$$\hat{\mathcal{F}}_{01}(s) = \frac{a_1 r_0(\lambda + \theta\chi(s) + \theta\bar{p}_1\beta(s)) + a_2 r_0(s r_0 + \lambda)}{(s r_0 + \lambda)(s r_0 + \lambda q + \mu - \theta\bar{p}_1 r(s)) - \mu(\lambda + \theta\chi(s) + \theta\bar{p}_1\beta(s))}$$

Upon factorization which leads to

$$\hat{\mathcal{F}}_{00}(s) = \frac{a_1 r_0(sr_0+\lambda q_1+\mu-\theta\bar{p}_1 r(s)) + a_2 r_0\mu}{(sr_0+\lambda)(sr_0+\lambda q_1+\mu-\theta\bar{p}_1 r(s)) - \mu(\lambda+\theta\chi(s)+\theta\bar{p}_1\beta(s))}$$

$$= \sum_{k=0}^{\infty}\omega(s)^k\{a_2\sigma_0\mu D(s) - \frac{a_1\sigma_0}{s\sigma_0+\lambda}\} \quad (21)$$

Similarly for $\hat{\mathcal{F}}_{01}(s)$

$$\hat{\mathcal{F}}_{01}(s) = \frac{a_1 r_0(\lambda+\theta\chi(s)+\theta\bar{p}\beta(s)) + a_2 r_0(sr_0+\lambda)}{(sr_0+\lambda)(sr_0+\lambda q+\mu-\theta\bar{p}_1 r(s)) - \mu(\lambda+\theta\chi(s)+\theta\bar{p}_1\beta(s))}$$

$$= (a_1 r_0(\lambda+\theta\chi(s)+\theta\bar{p}_1\beta(s))D(s)) + \frac{a_2 r_0}{sr_0+\lambda q_1+\mu-\theta\bar{p}_1 r(s)}\sum_{k=0}^{\infty}\omega(s)^k \quad (22)$$

where

$$\omega(s) = \frac{\mu(\lambda+\theta\chi(s)+\theta\bar{p}_1\beta(s))}{(sr_0+\lambda)(sr_0+\lambda q_1+\mu-\theta\bar{p}_1 r(s))}$$

$$D(s) = \frac{1}{(sr_0+\lambda)(sr_0+\lambda q_1+\mu-\theta\bar{p}_1 r(s))}$$

From inversion of (12),(14),(15) with $\beta = 2\frac{\sqrt{\lambda\theta\bar{p}q}}{\sigma}$, we have,

$$\beta(s) = \frac{(\lambda+\theta+sr)\rho}{\mu r(s)} - \lambda\mu\frac{\lambda+\theta+sr}{\mu\theta\mu_p} = e^{-\frac{(\lambda+\theta)}{r}x}\left(\frac{r}{\mu}\right)^k\frac{x^k}{k!} * \chi(x)^* i$$

$$\omega(s) = \frac{\mu(\lambda+\theta\chi(s)+\theta\bar{p}_1\beta(s))}{(sr_0+\lambda)(sr_0+\lambda q_1+\mu-\theta\bar{p}_1 r(s))}$$

$$= \mu(\lambda+\theta\chi(x)+\theta\bar{p}_1\beta(x)) * \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\theta\bar{p}_1^i}{r^{i+2}}e^{-\frac{\lambda}{r_0}x}e^{-\frac{(\lambda q_1+\mu)}{r_0}x}\frac{x^i}{i!} \frac{iI_i(\beta x)\beta^i}{x}(\frac{r}{2\theta\bar{p}_1})^i e^{-(\frac{\lambda q_1+\mu+\theta\bar{p}_1}{r})x}$$

$$\chi(s) = \frac{\rho+sr}{r(s)} - \frac{\lambda\mu}{\theta\mu_p}$$

$$= \frac{1}{\theta p_1}\sum_{k=0}^{\infty}\left(\frac{\mu}{\bar{p_1}r}\right)^k\frac{x^{k-1}}{(k-1)!}e^{-\frac{(\lambda+\theta)}{r}x}\left(\frac{\lambda\mu}{r}\right)e^{-\frac{(\lambda+\theta)}{r}x} - \frac{1}{\theta p_1} * \frac{kI_k(\beta x)\beta_k}{x}e^{-\frac{(\lambda q_1+\mu+\theta p_1)}{r}x}\left(\frac{r}{2\theta\bar{p}_1}\right)^k$$

$$D(s) = \frac{1}{(sr_0+\lambda)(sr_0+\lambda q_1+\mu-\theta\bar{p}_1 r(s))}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\frac{\theta\bar{p}_1^i}{r^{i+1}}e^{-\frac{(\lambda q_1+\mu)}{r_0}x}\frac{x_{i+j}}{(i+j)!}\right] * \frac{iI_j(\beta x)\beta^i}{x}(\frac{r}{2\theta\bar{p}_1})^i e^{-\frac{(\lambda q_1+\mu+\theta\bar{p}_1)}{r}x}$$

From Equation 17, we arrive at the solution in terms of the modified Bessel function of the first kind, which signifies explicitly the exponentially growing function achieved the characteristic of the proposed work by significantly increasing the function of the number of packets/signals in the virtual buffer (Retrial)

$$\hat{\mathcal{F}}_{00}(s) = \left(\frac{-a_1 r_0}{(sr_0+\lambda)} + a_2 r_0\mu D(s)\right)\sum_{k=0}^{\infty}\omega(s)^k$$

$$= \sum_{k=0}^{\infty}(\omega(s))^{*k} * \left(a_2\sum_{v=0}^{\infty}(\frac{\theta\bar{p}_1}{r_0})^v e^{-\frac{(\lambda q_1+\mu)}{r_0}x}\frac{x^v}{v!} * \frac{vI_v(\beta x)\beta^v}{x}(\frac{r}{2\theta\bar{p}_1})^v e^{-\frac{(\lambda q_1+\mu+\theta\bar{p}_1)}{r}x}\right) + a_1 r_0(\lambda+\theta\chi(x)+\theta\bar{p}_1\beta(x)) * D(x)$$

$$\hat{\mathcal{F}}_{01}(s) = (a_1 r_0(\lambda+\theta\chi(s)+\theta\bar{p}_1\beta(s))D(s)) + \frac{a_2 r_0}{sr_0+\lambda q_1+\mu-\theta\bar{p}_1 r(s)}\sum_{k=0}^{\infty}\omega(s)^k$$

$$= \sum_{k=0}^{\infty}(\omega(s))^{*k} * \left(a_2 r_0\mu * D(x) - a_1 e^{-\frac{\lambda}{r_0}x}\right)$$

$$(\hat{\mathcal{F}}_{k0}(s), \hat{\mathcal{F}}_{k1}(s)) = (\hat{\mathcal{F}}_{00}(s), \hat{\mathcal{F}}_{01}(s))R^n(s)$$

$$= (\hat{\mathcal{F}}_{00}(s), \hat{\mathcal{F}}_{01}(s))\begin{pmatrix}\chi(s)^n & \sum_{i=0}^{n-1}\chi(s)^i\beta(s)r(s)^{n-1-i} \\ 0 & r(s)^n\end{pmatrix}$$

$$= \begin{pmatrix}\hat{\mathcal{F}}_{00}(s)\chi(s)^n & \hat{\mathcal{F}}_{00}(s)\sum_{i=0}^{n-1}\chi(s)^i\beta(s)r(s)^{n-1-i} + \hat{\mathcal{F}}_{01}(s)r(s)^n\end{pmatrix}$$

which is simplified in the form of

$$\hat{\mathcal{F}}_{k0}(s) = \hat{\mathcal{F}}_{00}(s)\chi(s)^n \quad (23)$$

$$\hat{\mathcal{F}}_{k1}(s) = \hat{\mathcal{F}}_{00}(s)\sum_{i=0}^{n-1}\chi(s)^i\beta(s)r(s)^{n-1-i} + \hat{\mathcal{F}}_{01}(s)r(s)^n \quad (24)$$

Thus all steady state probabilities are obtained interms of modified Bessel function of first kind to determine the flow of fluid in retrial queue along with impatient customers are observed.

□