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Capturing and communicating advanced mathematical activity

Peter Samuels
Birmingham City University, UK

Abstract

Unlike most other subjects, mathematical activity resides almost entirely within the cognitive processes of a mathematics practitioner and is therefore difficult to characterise. Despite recent interest, the nature of advanced mathematical activity remains something of a black box to educational researchers. In addition, the production of advanced mathematical texts, such as theses or journal articles, is often divorced from mathematicians' experiences of creating mathematics which can lead them to a sense of personal alienation from their work. This article proposes four practical techniques for capturing advanced mathematical activity. The timing of the use of these techniques is compared against a model of mathematical creativity and the writing process. The establishment of a new genre for communicating advanced mathematics is proposed which combines the product of the activity with the process of creating it.

Keywords: mathematical creativity; advanced mathematical activity; mathematical writing; data capturing techniques; mathematical activity corpus.

Introduction

Creating mathematics can be an alienating experience for mathematicians. This is because the product of mathematical activity can be far removed from the process by which it is created. Marx (1974, p. 64) described **alienation from product** as:

> The worker is related to the product of his labour as to an alien object. For on this premise it is clear that the more the worker spends himself, the more powerful the alien objective world becomes which he creates over against himself – his inner world – becomes, the less belongs to himself as his own.

Rowan (1981) applied Marx's theory of alienation to the traditional paradigm of research in the human sciences. I claim that it is even more applicable to mathematics research.

The **four stage model** of the creative process is widely attributed to Poincaré (1908) who described a four stage process for his own mathematical creativity: **preparation**, **incubation**, **illumination**, and **verification**. According to Lubart (2001), by 1950 there was considerable agreement that this model describes the creative process in general (Guilford, 1950). Crowley (1977) suggested the application of this model to describing the writing process with **writing** replacing illumination and **revision** replacing verification. Other writing process researchers have reclassified preparation and incubation as **pre-writing**, writing as **drafting** and have added **editing** after revision. Whilst the four stage model has been criticised for lacking psychological depth by Guilford (1950) in its application to general creativity, and by Flower and Hayes (1981) in its application to the writing process, it remains an important initial model for understanding the sequence of psychological processes in these activities.

Providing accurate data on the process of creating advanced mathematics has proved notoriously difficult. Mathematicians are generally private individuals with a large cultural gap between their logical research paradigm and the paradigms of educational research (Nardi and Iannone, 2004). Apart from introspective reports like Poincaré's (1908) perhaps the most noteworthy example of an account of actual advanced mathematical activity is that of Tall (1980) who was in the almost unique position of being a mathematician as well as a mathematics educator. Tall's approach could be described as **analytical autoethnography** in that he was 'a full member in the research group or setting, visible as such a member in the researcher's published texts, and committed to an analytic research agenda focused on improving theoretical understandings of broader social phenomena' (Anderson, 2006, p.375). The lack of any similar papers to Tall's since its publication, apart from Chick's (1998) deconstruction of her own thought processes in advanced mathematics, indicates that such an approach is difficult for mathematicians to emulate as they generally lack the ability to qualitatively reflect on and interpret the significance of their actions.

Rather than expecting mathematicians to qualitatively reflect on their own experiences or mathematics educators to capture advanced mathematical activity as it occurs, I propose that a more practical solution is to develop techniques by which mathematicians can capture their process of creating mathematics without distracting their concentration but not necessarily expect them to analyse it themselves. In this paper I present four techniques which I have used in the process of creating and writing up advanced mathematics. These techniques can be mapped onto different stages in the mathematics and writing creativity process, which I view as generally sequential, as shown in Table 1. Finally, I argue that the way mathematics research is communicated in journals should provide a means for mathematicians to present their process data as well as the product of their research which could then be analysed by mathematics educators or cognitive psychologists.

Table 1. Data capturing techniques for advanced mathematics and their relationship with the different stages of the mathematical creativity and writing processes.

Process	Stage	Data capture technique	Reference
Mathematical creativity	Preparation	Proof plan	(Wolska et al., 2004)
	Incubation	Concept map	(Bolte, 1999; Lavigne et al., 2008)
	Illumination	Activity transcript	(Tall, 1980)
	Verification		
Writing	Preparation		
	Incubation	Concept map	(Margerum-Leys, 1999)
	Writing		
	Revision	Annotated draft and transcript	(Eliot, 1971)

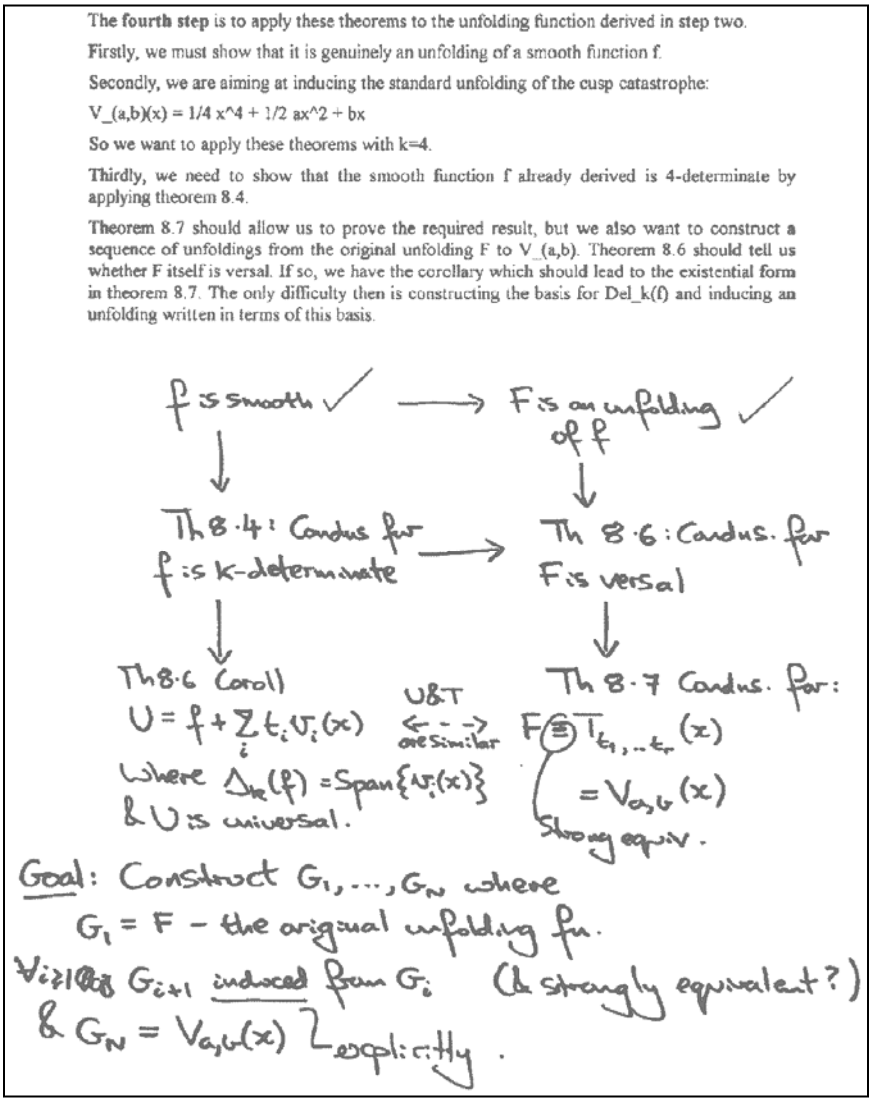
The data capturing techniques

The four data capturing techniques are described in the order that they might be used in the production of a mathematical research article. Each technique has been illustrated by an example from my own PhD research work. Apart from illustrating these techniques, the second and fourth examples were chosen because their content appeared to be interesting from more than a mathematical perspective: the former contains an account of making a mistake, getting stuck and then overcoming it; the latter indicates my emotional state when reviewing my research writing and my struggles to remember something I had done in the past.

Proof planning

This first data capture technique proposed is proof **planning**. This term was probably originally used by Wallen (1983) in computational mathematics to describe automated theorem proving techniques and is now commonly used in this field. However, little research has been carried out to compare computational proof planning with the actual proving methods of mathematicians (although, see Wolska et al., 2004). It is used here to describe the process of creating an informal plan for a proof. An example of its use in advanced mathematics is provided in Figure 1. I created this plan in preparation for a meeting with my PhD supervisor. It is likely that this technique will be motivated by some kind of social interaction for the plan to be written down. In this case, the structure of the actual final proof was remarkably similar to the plan.

Figure 1. Part of plan for a proof that a one dimensional unsteady wave equation is an example of a cusp catastrophe.



The fourth step is to apply these theorems to the unfolding function derived in step two.

Firstly, we must show that it is genuinely an unfolding of a smooth function f.

Secondly, we are aiming at inducing the standard unfolding of the cusp catastrophe:

V_(a,b)(x) = 1/4 x^4 + 1/2 ax^2 + bx

So we want to apply these theorems with k=4.

Thirdly, we need to show that the smooth function f already derived is 4-determinate by applying theorem 8.4.

Theorem 8.7 should allow us to prove the required result, but we also want to construct a sequence of unfoldings from the original unfolding F to V_(a,b). Theorem 8.6 should tell us whether F itself is versal. If so, we have the corollary which should lead to the existential form in theorem 8.7. The only difficulty then is constructing the basis for Del_k(f) and inducing an unfolding written in terms of this basis.

Activity transcripts

An activity transcript is an account of an episode of mathematical activity. It combines notes from the mathematical activity with a journal style account. Journal writing has been suggested for undergraduate mathematics classes (Rosenthal, 1995) but is unusual in more advanced mathematics research. An example of an activity transcript for advanced mathematics is given in Table 2. It is divided into four parts: a background to the mathematical activity (written eight days after it took place), notes from the activity itself, a written-up version of these notes (also written after eight days) and reflections on the activity (written after three weeks and written up after two and a half months). The time delay in writing up a mathematical activity appears to be important but an appropriate length of time may depend upon the writer's context and personal preferences. This approach is similar to Tall's (1980) article but it records a single mathematical activity in more detail and does not include qualitative analysis of the significance of the experience. It is therefore more practical for a mathematician to use if they wish to write up a single episode of mathematical activity which may include mistakes, such as the one shown in Table 2.

Concept maps

Concept maps are:

> Graphical tools for organizing and representing knowledge. They include concepts, usually enclosed in circles or boxes of some type, and relationships between concepts indicated by a connecting line linking two concepts. Words on the line, referred to as linking words or linking phrases, specify the relationship between the two concepts. (Novak and Cañas, 2008, p.1)

Bolte (1999) has proposed the use of concept maps for assessment in undergraduate mathematics courses. They have also been used as a research tool by Lavigne and others (2008) for exploring students' understanding of undergraduate statistics. Margerum-Leys (1999) promotes their use as an aid to understanding in the pre-writing stage of the writing process. Figure 2 provides a concept map for a chapter of one of my mathematics research reports (Samuels, 1989) on the method of characteristics applied to unsteady wave equations.

Table 2. Four extracts from an example activity.

<table>
<tr>
<td>My general aim had been to explicitly characterise regions to the solution of the one-dimensional unsteady wave equation by the number of solutions the wave equation has for each point in terms of derivatives of the initial wave speed. The standard solution technique is to plot lines on which the solution is constant whose slope is related to the wave speed, known as characteristic curves. The generally accepted result is that when the initial wave speed has an inflection point, is decreasing with respect to the base line and has positive third derivative then a breaking point will occur at which the solution surface initially starts to overturn. After this point, it is possible to locally obtain two curves called caustics which mark the boundary to the region in which the solution is triple-valued. Outside this region the solution locally remains single-valued.</td>
<td>

ξ becomes infinite when:

$$C_0(\xi_B)\tilde{t} - \left[\frac{1 - C_0^{(1)}(\xi_B)\tilde{t}}{C_0^{(1)}(\xi_B)}\right]\int_0^{\tilde{\xi}} (\tilde{\xi}-\eta)C_0^{(3)}(\xi_B+\eta)d\eta = 0 \quad (B)$$

Let $\int_0^{\tilde{\xi}} (\tilde{\xi}-\eta)C_0^{(3)}(\xi_B+\eta)d\eta = \tilde{\xi}^2 I(\tilde{\xi})$

So $\tilde{t}\left[C_0(\xi_B) + \tilde{\xi}^2 I(\tilde{\xi})\right] = \frac{1}{C_0^{(1)}(\xi_B)}\tilde{\xi}^2 I(\tilde{\xi})$

$$\tilde{t} = \frac{\tilde{\xi}^2 I(\tilde{\xi})}{C_0^{(1)}(\xi_B)\left[C_0(\xi_B) + \tilde{\xi}^2 I(\tilde{\xi})\right]}$$

</td>
</tr>
<tr>
<td>

Combining (8) with (10) allowed me to state that the integral $\frac{\partial\xi}{\partial\tilde{z}}$ becomes infinite when:

$$c_0(\xi_B)\tilde{t} - \left[\frac{1 - c_0^{(1)}(\xi_B)\tilde{t}}{c_0^{(1)}(\xi_B)}\right]\int_0^{\tilde{\xi}} (\tilde{\xi}-\eta)c_0^{(3)}(\xi_B+\eta)d\eta = 0 \qquad (11)$$

I wanted to rearrange this equation to make $\tilde{t}$ the subject. In order to simplify the working I decided to introduce an intermediate variable by defining:

$$\int_0^{\tilde{\xi}} (\tilde{\xi}-\eta)c_0^{(3)}(\xi_B+\eta)d\eta = \tilde{\xi}^2 I(\tilde{\xi}) \qquad (12)$$

(The introduction of the $\tilde{\xi}^2$ term was to ensure that the integral $I(\tilde{\xi})$ was of the right order.)

Using this definition, I inferred that:

$$\tilde{t}\left[c_0(\xi_B) + \tilde{\xi}^2 I(\tilde{\xi})\right] = \frac{1}{c_0^{(1)}(\xi_B)}\tilde{\xi}^2 I(\tilde{\xi}) \qquad (13)$$

</td>
<td>

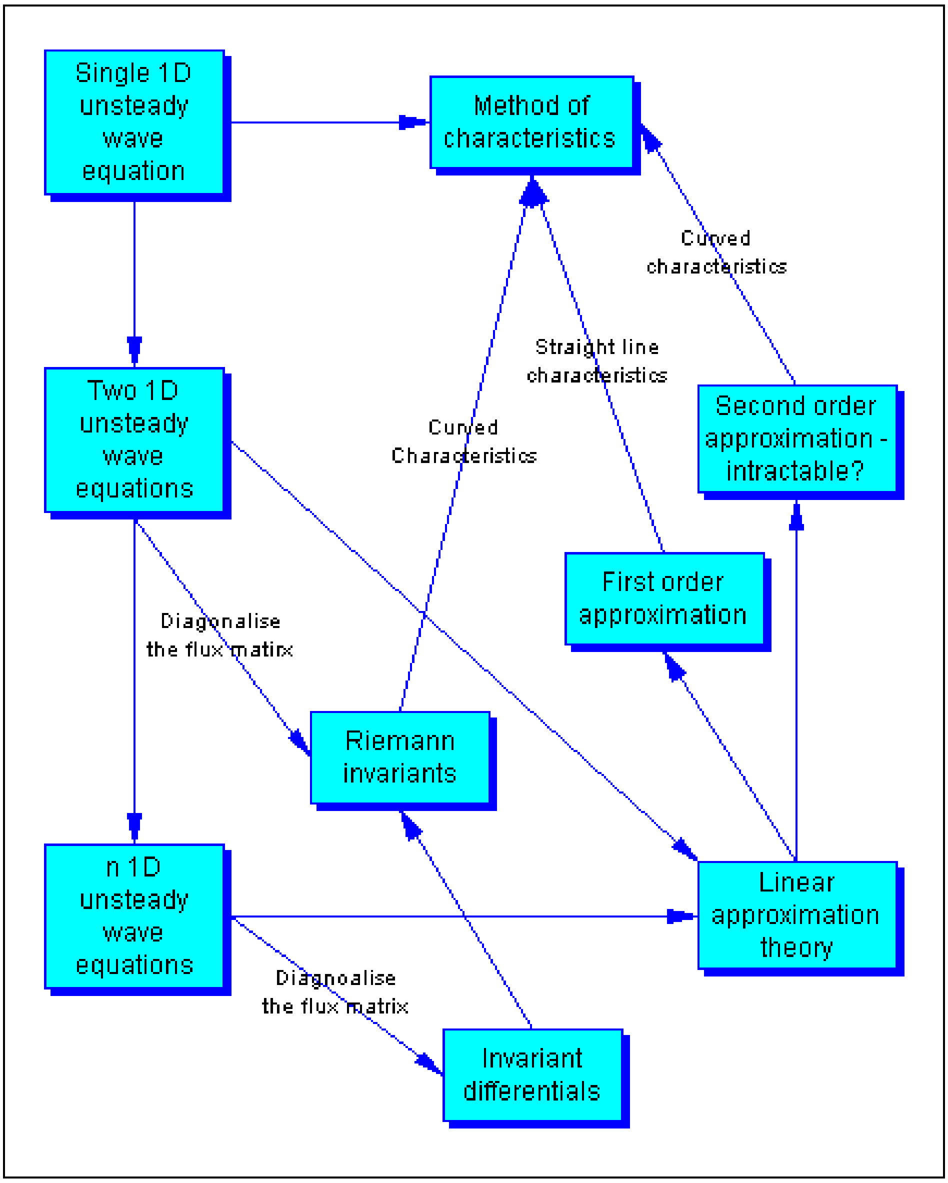
In trying to find a more explicit relationship between $\tilde{t}$ and $\tilde{\xi}$, I found I had made a mistake in equation (13): the term $c_0(\xi_B)$ in the first bracket should have been $c_0^{(1)}(\xi_B)$. My mistake became evident when I tried to calculate the sign of $\tilde{t}$. Although I could have confirmed this with a dimensional analysis, I decided to make completely sure by going back to the parametric definition of the caustic curves. I applied the partial derivative method to the original characteristic equation as in the above analysis. This was an improvement over the previous method I had used which had involved calculating the equations of the caustics using neighbouring characteristics.

This gave me the symmetrical relationship:

$$\tilde{t} = \frac{c_0^{(1)}(\xi_B+\tilde{\xi}) - c_0^{(1)}(\xi_B)}{c_0^{(1)}(\xi_B)c_0^{(1)}(\xi_B+\tilde{\xi})} \qquad (17)$$

</td>
</tr>
</table>

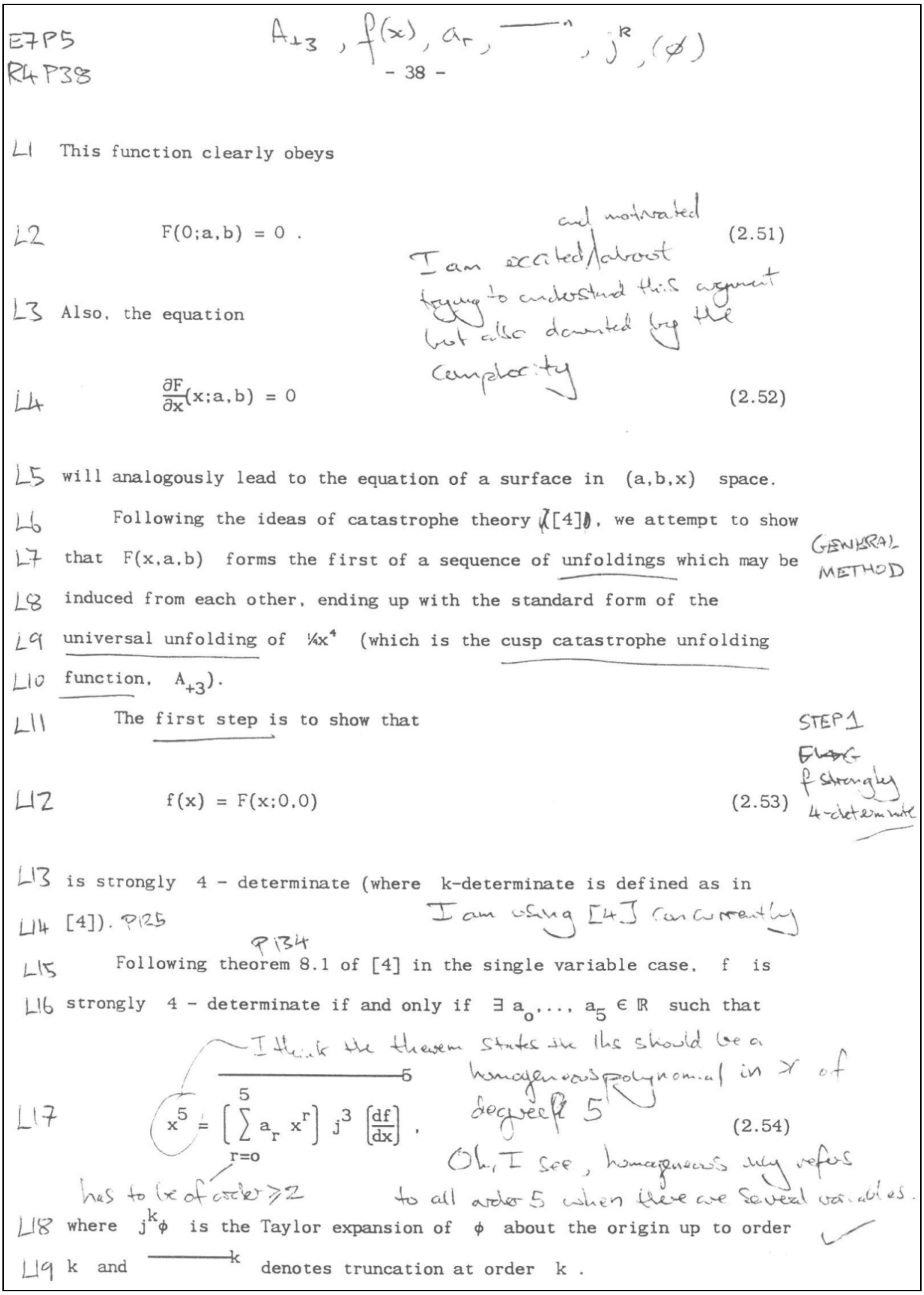
Figure 2. Concept map of the method of characteristics applied to unsteady wave equations (produced using Inspiration® – see http://www.inspiration.com/).



Annotated draft and transcript

The final data capturing technique proposed here is an **annotated draft and transcript**. It is similar to the approach used by Valerie Eliot to represent the annotated draft of her late husband's poem, The Waste Land (Eliot, 1971). I developed this technique as a means to capture my 'thinking aloud' as I re-read my internally published research reports. An example of an annotated draft of an extract from Samuels (1989) with its transcription is given in Figures 3 and 4.

Figure 3. Facsimile of a research report with 'thinking aloud' comments.


E7P5
R4P38

A_{+3} , $f(x)$, a_r, $\overline{}^{n}$, j^k, (ϕ)

– 38 –

L1 This function clearly obeys

L2 $$F(0;a,b) = 0 \;. \qquad (2.51)$$

I am excited/about trying to understand this argument and motivated but also daunted by the complexity

L3 Also, the equation

L4 $$\frac{\partial F}{\partial x}(x;a,b) = 0 \qquad (2.52)$$

L5 will analogously lead to the equation of a surface in (a,b,x) space.

L6 Following the ideas of catastrophe theory ([4]), we attempt to show

L7 that F(x,a,b) forms the first of a sequence of unfoldings which may be GENERAL METHOD

L8 induced from each other, ending up with the standard form of the

L9 universal unfolding of $\frac{1}{4}x^4$ (which is the cusp catastrophe unfolding

L10 function, A_{+3}).

L11 The first step is to show that STEP 1

L12 $$f(x) = F(x;0,0) \qquad (2.53)$$

f strongly 4-determinate

L13 is strongly 4 - determinate (where k-determinate is defined as in

L14 [4]). P125 I am using [4] concurrently

L15 Following theorem 8.1 of [4] in the single variable case, f is P134

L16 strongly 4 - determinate if and only if $\exists\, a_0, \ldots, a_5 \in \mathbb{R}$ such that

I think the theorem states the this should be a homogeneous polynomial in x of degree 5

L17 $$\overline{x^5 = \left[\sum_{r=0}^{5} a_r\, x^r\right] j^3 \left[\frac{df}{dx}\right]}^{\,5} \;, \qquad (2.54)$$

has to be of order ≥ 2

Oh, I see, homogeneous only refers to all order 5 when there are several variables.

L18 where $j^k\phi$ is the Taylor expansion of ϕ about the origin up to order

L19 k and $\overline{}^{\,k}$ denotes truncation at order k .

Figure 4. Transcript of the ‘thinking aloud’ comments on the extract in Figure 3.

Note:

1. ‘Top’ refers to the top of the page – this is a list of variables defined on this page which indicates the short term memory load for the reader.
2. ‘L*n*’ refers to line number *n*.
3. Text to the left of the colon in bold is a positional descriptor for an action or an annotation.
4. Words in italics describe an action.
5. Words in normal font are an annotation.

Exract 7 page 5

Report 4 page 38

Top: $A_{+3}, f(x), a_r, \overline{\quad}^{n}, j^k, (\phi)$

L1–L4: I am excited *(‘and motivated’ inserted)* about trying to understand this argument but also daunted by the complexity

L6, ([4]): *brackets removed*

L7, unfoldings: *underlined*

L7: GENERAL METHOD

L9, universal unfolding: *underlined*

L9–L10, cusp catastrophe unfolding function: *underlined*

L11, first step: *underlined*

L11: STEP 1
(F → G) crossed out
f strongly 4-determinate

L14: P125

L13–L14: I am using [4] concurrently

L15, theorem 8.1: P134

L17, x^5: I think the theorem states that lhs should be a homogeneous polynomialin x of degree *(‘f’ crossed out)* 5
Oh, I see, homogeneous only refers to all order 5 when there are several variables.

L17, $\sum_{r=0}^{5} a_r x^r$: has to be of order ≥ 2

L18: *ticked*

Discussion

In this discussion paper I have presented four data capturing techniques for advanced mathematical activities with examples of each from my own research. I have not attempted to analyse the data provided through these techniques as I believe this would be done more appropriately and objectively by another educational or cognitive psychologist researcher. However, as explained above, there was an additional non-mathematical motivation behind including the second and fourth examples. Rather, they exemplify how a mathematician might present **content laden data**, with each technique potentially providing insight into their thought processes when carrying out advanced mathematics research as well as conveying their mathematical meaning. These techniques are easy to learn and are not time consuming to use. Their use may actually enhance mathematical activity and the process of writing it up rather than detract from it, overcoming mathematicians' sense of alienation from the production of mathematics research.

I propose the establishment of a corpus of advanced mathematical process data, similar to the Digital Variants corpus (Björk, 1998, see http://www.digitalvariants.org/) where mathematicians are able to supply their content laden data of different stages of the mathematics creativity process and the process of writing it up, and other researchers are able to analyse their thought processes. Alternatively, this could partially be achieved by using the **additional online resources** facility available with some journals.

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Author details

Dr Peter Samuels is an Academic Skills Tutor at Birmingham City University, UK, and a Visiting Research Fellow at the Serious Games Institute at Coventry University, UK. His research interests include university-level mathematics education (especially the use of new technologies and serious games) and learning development.