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Application of the Stampacchia lemma to anisotropic degenerate elliptic equations

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Abstract. In this paper, we prove the existence and regularity of solutions for a class of nonlinear anisotropic degenerate elliptic equations with the data f belonging to certain Marcinkiewicz spaces $\mathcal{M}^m(\Omega)$ with $m > 1$. We use a generalized Stampacchia Lemma version to establish the main results.

Keywords: Anisotropic problem, Degenerate elliptic, Generalized Stampacchia Lemma, Marcinkiewicz space.
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1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2)$. We consider the following problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \left[a_i(x, u, \nabla u) \right] = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where f belongs to some Marcinkiewicz space $\mathcal{M}^m(\Omega)$ with $m > 1$. We assume that $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, for all $i = 1, ..., N$, are Carathéodory functions satisfying the following conditions for almost every $x \in \Omega$, all $s \in \mathbb{R}$, and all $\xi, \eta \in \mathbb{R}^N$:

$$|a_i(x, s, \xi)| \leq \beta |\xi_i|^{p_i - 1}, \tag{1.2}$$

$$[a_i(x, s, \xi) - a_i(x, s, \eta)].(\xi_i - \eta_i) > 0, \quad \xi_i \neq \eta_i, \tag{1.3}$$

$$a_i(x, s, \xi) \cdot \xi_i \geq b(s) |\xi_i|^{p_i}, \tag{1.4}$$

where β is a positive constant, $b : \mathbb{R} \to \mathbb{R}$ is a continuous function, such that

$$\frac{\alpha}{(1 + |s|)^\theta} \leq b(s) \leq \gamma, \quad \forall\, 0 \leq \theta < 1, \tag{1.5}$$

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where $0 \le \theta < 1$ and α, γ are two positive constants.

Our inspiration for this paper is derived from [8], where the author addressed elliptic problems described by the following model:

$$\begin{cases} -\mathrm{div}(a(x,u)\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where

$$\frac{\alpha}{(1+|s|)^{\theta}} \le a(x,s) \le \beta,$$

with $0 < \alpha \le \beta < \infty$ and $0 \le \theta < 1$. The authors in [8] mainly consider the regularity of u to vary with m: Let $u \in W_0^{1,2}(\Omega)$ be a weak solution to (1.6) and $f \in \mathcal{M}^m(\Omega)$. Then

(R1) If $m > \frac{N}{2}$, then there exists $L > 0$, such that $|u| \le 2L$ a.e. in Ω;

(R2) If $m = \frac{N}{2}$, then there exists $\lambda > 0$, such that $e^{\lambda |u|^{1-\theta}} \in L^1(\Omega)$;

(R3) If $(2^*)' < m < \frac{N}{2}$, then $u \in \mathcal{M}^{m^{**}(1-\theta)}(\Omega)$; with $m^{**} = \frac{Nm}{N-2m}$.
Let u be an entropy solution of (1.6) and $f \in \mathcal{M}^m(\Omega)$. Then

(R4) If $1 < m \le (2^*)'$, then $u \in \mathcal{M}^{m^{**}(1-\theta)}(\Omega)$.

In [3] under the hypotheses $\theta = 0$ and $a_i(x,s,\xi) = |\xi_i|^{p_i-2}\xi_i$, the author proved that

(R1) If $m > \frac{N}{\overline{p}}$, then $u \in L^{\infty}(\Omega)$;

(R2) If $m = \frac{N}{\overline{p}}$, then there exists $\lambda > 0$, such that $e^{\lambda |u|} \in L^1(\Omega)$;

(R3) If $(\overline{p}^*)' < m < \frac{N}{\overline{p}}$, then $u \in L^{\frac{mN(\overline{p}-1)}{N-m\overline{p}}}(\Omega)$;

(R4) If $1 < m \le (\overline{p}^*)'$, then $u \in W_0^{1,p_i\frac{mN(\overline{p}-1)}{\overline{p}(N-m)}}(\Omega)$.

Existence and regularity results for the problem (1.1) have been obtained in [1] with $f \in L^m(\Omega)$, $m \ge 1$, $a_i(x,s,\xi) = \frac{a_i(x,\xi)}{(1+|s|)^{\theta(p_i-1)}}$, where $\theta \ge 0$ and $p_i \in (1,+\infty)$ for all $i = 1,...,N$.

Let Ω be a bounded open set in $\mathbb{R}^N$, where $N \ge 2$ and $1 < p_1 \le p_2 \le \ldots \le p_N$. The natural functional framework of the problem (1.1) is anisotropic Sobolev spaces $W^{1,(p_i)}(\Omega)$ and $W_0^{1,(p_i)}(\Omega)$, which are defined by

$$W^{1,(p_i)}(\Omega) = \left\{v \in W^{1,1}(\Omega) : \partial_i v \in L^{p_i}(\Omega),\ i = 1,...,N\right\},$$
$$W_0^{1,(p_i)}(\Omega) = W^{1,(p_i)}(\Omega) \cap W_0^{1,1}(\Omega).$$

The space $W_0^{1,(p_i)}(\Omega)$ can also be defined as the closure of $C_0^{\infty}(\Omega)$ in $W^{1,(p_i)}(\Omega)$ with respect to the norm

$$\|v\|_{W_0^{1,(p_i)}(\Omega)} = \sum_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)}.$$

Now we will recall some lemmas that are known and needed for the subsequent analysis.

Lemma 1.1. *[10] There exists a positive constant C, depending only on Ω, such that for $v \in W_0^{1,(p_i)}(\Omega)$, $\overline{p} < N$ we have*

$$\|v\|_{L^r(\Omega)} \leq C \prod_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)}^{\frac{1}{N}}, \quad \forall r \in [1, \overline{p}^*], \tag{1.7}$$

where $\overline{p}^* = \frac{N\overline{p}}{N-\overline{p}}$, $\frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{p_i}$.

Definition 1.2. [2] Let Ω be a bounded open subset of $\mathbb{R}^N$. Let $p \geq 0$. The Marcinkiewicz space $\mathcal{M}^p(\Omega)$ is the space of all measurable functions $f : \Omega \to \mathbb{R}$ with the following property: there exists a constant $C > 0$ such that

$$\text{meas}(\{|f| > \lambda\}) \leq \frac{C}{\lambda^p}, \quad \forall \lambda > 0, \tag{1.8}$$

where meas(E) is the Lebesgue measure of the set E in $\mathbb{R}^N$. The norm of $f \in \mathcal{M}^p(\Omega)$ is defined by

$$\|f\|^p_{\mathcal{M}^p(\Omega)} = \inf\{C > 0 : \ (1.8) \text{ holds}\}.$$

It is immediate that the following inclusions hold, $1 \leq q < p < \infty$,

$$L^p(\Omega) \subset \mathcal{M}^p(\Omega) \subset L^q(\Omega).$$

A Hölder inequality holds true for $f \in \mathcal{M}^m(\Omega)$, $m > 1$: there exists $B = B(\|f\|_{\mathcal{M}^m(\Omega)}, m) > 0$ such that for every measurable subset $E \subset \Omega$,

$$\int_E |f| dx \leq B|E|^{1-\frac{1}{m}}. \tag{1.9}$$

We now present a generalization of Lemma 4.1 from [9] (see [5]), which can be applied in the analysis of degenerate anisotropic elliptic equations of divergence type.

Lemma 1.3. *[8] Let c, τ_1, τ_2, k_0 be positive constants and $0 \leq \theta < 1$. Let $\Phi : [k_0, +\infty) \to [0, +\infty)$ be nonincreasing and such that*

$$\Phi(h) \leq \frac{c h^{\theta\tau_1}}{(h-k)^{\tau_1}} [\Phi(k)]^{\tau_2}, \tag{1.10}$$

for every h, k with $h > k \geq k_0 > 0$. It results that:

(i) *if $\tau_2 > 1$, then*

$$\Phi(2L) = 0,$$

where

$$L = \max\left\{2k_0, c^{\frac{1}{(1-\theta)\tau_1}} [\Phi(k_0)]^{\frac{\tau_2-1}{(1-\theta)\tau_1}} 2^{\frac{1}{(1-\theta)\tau_2}\left(\tau_2+\theta+\frac{1}{\tau_2-1}\right)}\right\}, \tag{1.11}$$

(ii) *if $\tau_2 = 1$, then for any $k \geq k_0$,*

$$\Phi(k) \leq \Phi(k_0) e^{1-\left(\frac{k-k_0}{\tau}\right)^{1-\theta}},$$

where

$$\tau = \max\left\{k_0, \left(c 2^{\frac{(2-\theta)\theta\tau_1}{1-\theta}} (1-\theta)^{\tau_1}\right)^{\frac{1}{(1-\theta)\tau_1}}\right\},$$

(iii) *if* $0 < \tau_2 < 1$, *then for any* $k \geq k_0$,

$$\Phi(k) \leq 2^{\frac{(1-\theta)\tau_1}{(1-\tau_2)^2}} \left\{ \left(c_1 2^{\theta\tau_1}\right)^{\frac{1}{1-\tau_2}} + (2c_2k_0)^{\frac{(1-\theta)\tau_1}{1-\tau_2}} \Phi(k_0) \right\} \left(\frac{1}{k}\right)^{\frac{\tau_1(1-\theta)}{1-\tau_2}}, \tag{1.12}$$

where

$$c_1 = \max\left\{4^{(1-\theta)\tau_1} c 2^{\theta\tau_1}, c_2^{1-\tau_2}\right\}, \quad c_2 = 2^{\frac{(1-\theta)\tau_1}{(1-\tau_2)^2}} \left[(c2^{\theta\tau_1})^{\frac{1}{1-\tau_2}} + (2k_0)^{\frac{(1-\theta)\tau_1}{1-\tau_2}} \Phi(k_0)\right].$$

Let $k > 0$, we will use the truncation T_k defined as

$$T_k(s) = \begin{cases} -k, & \text{if } s \leq -k, \\ s, & \text{if } -k \leq s \leq k, \\ k, & \text{if } s \geq k, \end{cases} \quad \text{and} \quad G_k(s) = s - T_k(s). \tag{1.13}$$

2 The main results and their proof

We define the notion of a weak solution to the problem (1.1) as follows:

Definition 2.1. Let $f \in L^m(\Omega)$ with $m > (\overline{p}^*)'$. We define a weak solution of (1.1) as a function u in $W_0^{1,(p_i)}(\Omega)$ satisfying the following identity for all $\varphi \in W_0^{1,(p_i)}(\Omega)$:

$$\sum_{i=1}^{N} \int_{\Omega} a_i(x, u, \nabla u)\partial_i\varphi dx = \int_{\Omega} f\varphi dx. \tag{2.1}$$

Theorem 2.2. *Under the hypotheses* (1.2)-(1.5), *if* $f \in \mathcal{M}^m(\Omega)$, *with* $m > (\overline{p}^*)'$ *and* $u \in W_0^{1,(p_i)}(\Omega)$ *be a weak solution to* (1.1) *in the sense of* (2.1). *Then*

(i) *If* $m > \frac{N}{\overline{p}}$, *then there exists a constant* L *that can depend on the data, such that* $|u| \leq 2L$ *a.e.* $x \in \Omega$.

(ii) *If* $m = \frac{N}{\overline{p}}$, *then there exists a constant* $\lambda > 0$ *that can depend on the data, such that*

$$e^{\lambda|u|^{1-\theta}} \in L^1(\Omega).$$

(iii) *If* $(\overline{p}^*)' < m < \frac{N}{\overline{p}}$, *then* $u \in \mathcal{M}^{\frac{Nm(\overline{p}-1)(1-\theta)}{N-\overline{p}m}}(\Omega)$.

Proof of Theorem 2.2. Let $h > k > 0$. We use $\varphi = T_{h-k}(G_k(u))$ as a test function in (2.1), we obtain

$$\sum_{i=1}^{N} \int_{\Omega} a_i(x, u, \nabla u)\partial_i T_{h-k}(G_k(u))dx = \int_{\Omega} f T_{h-k}(G_k(u))dx. \tag{2.2}$$

Note that $\varphi = 0$ for $x \in \{|u_n| \leq k\}$, $|\varphi| \leq h - k$ and

$$\nabla\varphi = \begin{cases} 0, & \text{if } |u_n| \leq k, \\ \nabla u_n, & \text{if } k < |u_n| \leq h, \\ 0, & \text{if } |u_n| > h, \end{cases}$$

then (1.4),(1.5) and (2.2) yield

$$\alpha\int_{B_{k,h}}\frac{|\partial_i u|^{p_i}}{(1+|u|)^{\theta}}dx\le(h-k)\int_{A_k}|f|dx\quad\forall i=1,...,N, \tag{2.3}$$

where

$$A_k=\{x\in\Omega:\ |u(x)|>k\},\quad \text{and}\quad B_{k,h}=\{x\in\Omega:\ k<|u(x)|\le h\}.$$

Using Hölder's inequality with exponent m in the right-hand side and the fact that $\frac{1}{(1+|u|)^{\theta}}\ge\frac{1}{(1+h)^{\theta}}$ if $x\in B_{k,h}$ on the left-hand side of (2.3), we have

$$\begin{aligned}\frac{\alpha}{(1+h)^{\theta}}\int_{\Omega}|\partial_i T_{h-k}(G_k(u))|^{p_i}dx&=\frac{\alpha}{(1+h)^{\theta}}\int_{B_{k,h}}|\partial_i u|^{p_i}dx\\&\le\alpha\int_{B_{k,h}}\frac{|\partial_i u|^{p_i}}{(1+|u|)^{\theta}}dx\\&\le(h-k)\|f\|_{\mathcal{M}^m(\Omega)}|A_k|^{\frac{1}{m'}}\\&\le C_1(h-k)|A_k|^{\frac{1}{m'}}\quad\forall i=1,...,N,\end{aligned}$$

the above estimate implies

$$\prod_{i=1}^{N}\frac{1}{(1+h)^{\frac{\theta}{Np_i}}}\left(\int_{\Omega}|\partial_i T_{h-k}(G_k(u))|^{p_i}dx\right)^{\frac{1}{Np_i}}\le C_2\prod_{i=1}^{N}(h-k)^{\frac{1}{Np_i}}|A_k|^{\frac{1}{Np_i m'}},$$

hence,

$$\frac{1}{(1+h)^{\frac{\theta}{\overline{p}}}}\prod_{i=1}^{N}\left(\int_{\Omega}|\partial_i T_{h-k}(G_k(u))|^{p_i}dx\right)^{\frac{1}{Np_i}}\le C_2(h-k)^{\frac{1}{\overline{p}}}|A_k|^{\frac{1}{\overline{p}m'}}. \tag{2.4}$$

Applying 1.1 with $v=T_{h-k}(G_k(u))$, $r=\overline{p}^*$, and by (2.4), we find

$$\begin{aligned}\frac{1}{(1+h)^{\frac{\theta\overline{p}^*}{\overline{p}}}}(h-k)^{\overline{p}^*}|A_h|&=\frac{1}{(1+h)^{\frac{\theta\overline{p}^*}{\overline{p}}}}\int_{\Omega}|T_{h-k}(G_k(u))|^{\overline{p}^*}dx\\&\le C_2(h-k)^{\frac{\overline{p}^*}{\overline{p}}}|A_k|^{\frac{\overline{p}^*}{\overline{p}m'}}.\end{aligned} \tag{2.5}$$

Thus, from (2.5), it follows that for all $h>k\ge1$

$$\begin{aligned}\Phi(h)&\le C_3\frac{(1+h)^{\frac{\theta\overline{p}^*}{\overline{p}}}}{(h-k)^{\left(1-\frac{1}{\overline{p}}\right)\overline{p}^*}}\Phi(k)^{\frac{\overline{p}^*}{\overline{p}m'}}\\&\le C_3\frac{h^{\theta\frac{\overline{p}^*}{\overline{p}}}}{(h-k)^{\overline{p}^*\left(1-\frac{1}{\overline{p}}\right)}}\Phi(k)^{\frac{\overline{p}^*}{\overline{p}m'}},\end{aligned}$$

where $\Phi(k)=|A_k|$. The assumption (1.10) of Lemma 1.3 holds with

$$c=C_3,\quad \tau_1=\overline{p}^*\left(1-\frac{1}{\overline{p}}\right),\quad \tau_2=\frac{\overline{p}^*}{\overline{p}m'}\quad\text{and}\quad k_0=1.$$

We use Lemma 1.3, and we have:

(i) If $m > \frac{N}{\overline{p}}$, then $\tau_2 > 1$. We use Lemma 1.3 ($\mathbf{i}$), and we get $\Phi(2L) = 0$ for some constant L is defined as in (1.11), from which we derive $|u| \leq 2L$ a.e. $x \in \Omega$.

(ii) If $m = \frac{N}{\overline{p}}$, then

$$\tau_2 = \frac{\overline{p}^*}{\overline{p}m'} = \frac{N(m-1)}{(N-\overline{p})m} = 1.$$

By Lemma 1.3 ($\mathbf{ii}$), we obtain

$$\Phi(k) \leq \Phi(1)e^{1-\left(\frac{k-1}{\tau}\right)^{1-\theta}} \leq |\Omega|e^{1-\left(\frac{k-1}{\tau}\right)^{1-\theta}} \quad \forall k \geq 1,$$

Hence, if $k \geq 2$ (i.e. $k - 1 \geq \frac{k}{2}$), we have

$$\Phi(k) \leq |\Omega|e^{1-\left(\frac{k}{2\tau}\right)^{1-\theta}} \leq C_4 e^{-(2\tau)^{\theta-1}k^{1-\theta}}. \tag{2.6}$$

We let $\tau^{\theta-1} = 2^{2-\theta}\lambda$, by (2.6), we get

$$\text{meas}\{e^{\lambda|u|^{1-\theta}} > e^{\lambda k^{1-\theta}}\} = \Phi(k) \leq C_4 e^{-2\lambda k^{1-\theta}}, \tag{2.7}$$

choosing $\tilde{k} = e^{\lambda k^{1-\theta}}$ in (2.7), we obtain

$$\text{meas}\{e^{\lambda|u|^{1-\theta}} > \tilde{k}\} \leq \frac{C_4}{\tilde{k}^2}, \quad \forall \tilde{k} \geq e^{\lambda 2^{1-\theta}} = k_1. \tag{2.8}$$

Let us now use Lemma 3.11 from [2], which says that a sufficient and necessary condition for $g \in L^1(\Omega)$ is

$$\sum_{k=0}^{\infty} \text{meas}\{|h| > k\} < +\infty.$$

Finally we choose $g = e^{\lambda|u|^{1-\theta}}$, by (2.8), we deduce that

$$\begin{aligned}
\sum_{\tilde{k}=0}^{\infty} \text{meas}\{e^{\lambda|u|^{1-\theta}} > \tilde{k}\} &= \sum_{\tilde{k}=0}^{k_1} \text{meas}\{e^{\lambda|u|^{1-\theta}} > \tilde{k}\} + \sum_{\tilde{k}=k_1+1}^{\infty} \text{meas}\{e^{\lambda|u|^{1-\theta}} > \tilde{k}\} \\
&\leq (1+k_1)|\Omega| + C_4 \sum_{\tilde{k}=k_1+1}^{\infty} \frac{1}{\tilde{k}^2} \\
&\leq C_5 < \infty,
\end{aligned}$$

then $e^{\lambda|u|^{1-\theta}} \in L^1(\Omega)$.

(iii) If $(\overline{p}^*)' < m < \frac{N}{\overline{p}}$, then $\tau_2 < 1$. We use Lemma 1.3 ($\mathbf{iii}$), and we have for all $k \geq 1$

$$\begin{aligned}
\Phi(k) &\leq C_6 \left(\frac{1}{k}\right)^{\frac{\tau_1(1-\theta)}{1-\tau_2}} \\
&\leq C_6 \left(\frac{1}{k}\right)^{\frac{Nm(\overline{p}-1)(1-\theta)}{N-\overline{p}m}},
\end{aligned}$$

that is $u \in \mathcal{M}^{\frac{Nm(\overline{p}-1)(1-\theta)}{N-\overline{p}m}}(\Omega)$ as desired.

□

If $f \in \mathcal{M}^m(\Omega)$, with $1 < m \le (\overline{p}^*)'$, then it is possible to give a meaning to the solution for problem (1.1), using the concept of entropy solutions, which has been introduced in [1].

Definition 2.3. A measurable function u is an entropy solution to the problem (1.1) if $a_i(x, u, \nabla u) \in L^1(\Omega)$, $T_l(u)$ belongs to $W_0^{1,(p_i)}(\Omega)$ for every $l > 0$ and the inequality

$$\sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u) \partial_i T_l(u - \varphi) dx \le \int_\Omega f T_l(u - \varphi) dx, \tag{2.9}$$

holds for every $l > 0$ and every $\varphi \in W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega)$.

Theorem 2.4. *Let $f \in \mathcal{M}^m(\Omega)$ with $1 < m \le (\overline{p}^*)'$. Then the problem (1.1) admits at least one entropy solution $u \in \mathcal{M}^{\frac{Nm(\overline{p}-1)(1-\theta)}{N-\overline{p}m}}(\Omega)$ in the sense of (2.9).*

Proof of Theorem 2.4. The proof is similar to that one of Theorem 1.1 in [6]. Let $h > k > 0$. We use $\varphi = T_k(u) \in W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega)$, $l = h - k$, as a test function in (2.9). By (1.4) and (1.5), we obtain (2.3). The result follows from the proof of Theorem 2.2 (**iii**). □

Conflict of interest

The author has no conflicts of interest to declare.

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