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Non polynomial fractional spline method for solving Fredholm integral equations

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Abstract. A new type of non-polynomial fractional spline function for approximating solutions of Fredholm-integral equations has been presented. For this purpose, we used a new idea of fractional continuity conditions by using the Caputo fractional derivative and the Riemann Liouville fractional integration to generate fractional spline derivatives. Moreover, the convergence analysis is studied with proven theorems. The approach is also well-explained and supported by four computational numerical findings, which show that it is both accurate and simple to apply.

Keywords: Non-polynomial fractional spline method, Fredholm integral equations, Fractional derivative.

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1 Introduction

Consider the second kind of linear integral equation [5-8].

$$y(t) = f(t) + \int_a^b k(t,x)y(x)dx, \tag{1.1}$$

The kernel function of two variables t and x is $k(t,x)$, a and b are constants, $y(t)$ is the unknown function, and $f(t)$ is given. Integral equations can be used to describe some difficulties as well Bellour, A. [5] solving Fredholm integral equations by using two cubic spline methods, in [10] D. Hammad, a new general form of Ten non-polynomial cubic splines for some classes of Fredholm integral equations are presented, Maleknejad, Khosrow, Jalil Rashidinia, and Hamed Jalilian in [22], solved Fredholm integral equation via Quintic Spline functions and in [27] S. Saha Ray, and P. K. Sahu. proposed Numerical methods for solving Fredholm integral equations of the second kind. And for other works see [4] and [25] Non-polynomial spline functions are used to find approximate solutions to a variety of problems, including integral equations [10], [26], [23], and [13], and differential equations [3], [16], [30], [11] and [12],

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wave equations [8], Burgers equation [1], etc.
We employ a similar technique outlined in [10], [22], and [5], but fractional derivative and fractional models are not used there, it is clear that fractional calculus is one of the most reliable processes for managing complex systems and there are still many models to be suggested, analyzed and used in real-world applications in many fields of science and engineering where locality plays a significant role. Several fractional derivatives and integral definitions have recently been offered. [28]- [20], [7], and [29]. It also contributes significantly to the progress of other fields of science, such as engineering [31], chemistry [14], physics [6], and biology [15]. We hope that much better work on this technique will be done in the future and that this will be the beginning of doing better work.
The following sections of this paper are organized in the given sequence: In Section 2, we give some basic definitions, derivations, and formulations of the non-polynomial fractional spline function for solving second-order integral equations. In section 3, we present the methodology of our technique for Fredholm-integral equations (FIE). In sections 4 and 5, the method's convergence is discussed, and some numerical results are shown for the accurate and simple techniques, respectively. Finally, section 6 consists of the conclusion.

2 Mathematical preliminaries and Non-polynomial fractional spline construction

Here are some key fractional definitions before we get into the details of our approach. Different definitions are available for fractional derivatives. In this paper, both the Riemann-Liouville fractional derivative and the Caputo fractional derivative will be used.

Definition 2.1. [24] The RLD (Riemann- Liouville fractional derivative) of order β can be defined as:

$$ {}_aD_t^{\beta}f(t) = \frac{1}{\Gamma(n-\beta)}\left(\frac{d}{dt}\right)^n \int_a^t (t-u)^{n-\beta-1}f(u)du. $$

for every β, and $n = \lceil \beta \rceil$

Definition 2.2. [24] The CD (Caputo fractional derivative) of order β is defined as:

$$ {}_a^CD_t^{\beta}f(t) = \frac{1}{\Gamma(n-\beta)}\int_a^t (t-u)^{n-\beta-1}\left(\frac{d}{du}\right)^n f(u)du \ \ , n = \lceil \beta \rceil \text{ and } \beta > 0. $$

For $\beta = 0$, we introduce the notation:

$$ {}^CD_t^{\beta}f(t) = D^{\beta}f(t). $$

We used the non-polynomial fractional spline function to approximate a solution to the integral equation. For this reason, we consider a finite set of points $\Theta = [a,b]$ with $\Delta : a = t_0 < \cdots < t_m = b$, where $t_i = a + ih$. Let $S_i(t)$ be the interpolating non-polynomial FS (fractional Spline) function with interpolates y at t_i with new fractional continuity conditions, defined on $[t_i, t_{i+1}], i = 0, \ldots, m-1$, as:

$$ S_i(t) = a_i sin(\tau(t-t_i)) + b_i cos(\tau(t-t_i)) + c_i(t-t_i) + d_i(t-t_i)^{\frac{1}{2}} + e_i. \tag{2.1} $$

Where a_i, b_i, c_i, d_i, and e_i are real numbers and τ is the frequency of trigonometric functions. To derive the coefficients a_i, b_i, c_i, d_i and e_i we define boundary conditions:

$$ S_i(t_i) = y_i,\ S_i(t_{i+1}) = y_{i+1},\ S_i'(t_i) = M_i,\ S_i'(t_{i+1}) = M_{i+1}, \text{ and } S_i''(t_i) = y_i''(t_i). \tag{2.2} $$

Then, using algebraic manipulation and a Python program, we get the following expression:

$$a_i = \frac{(\tau M_{i+1} - \tau M_i - \alpha_1 y_i'')}{\tau^2(\alpha_0 - 1)},\ b_i = \frac{-y_i''}{\tau^2},\ c_i = \frac{\tau\alpha_0 M_i - \tau M_{i+1} + \alpha_1 y_i''}{\tau(\alpha_0 - 1)},\ e_i = \frac{(\tau^2 y_i + y_i'')}{\tau^2}, \text{and}$$
$$d_i = \frac{\tau^2(\alpha_0 - 1)y_{i+1} - \tau^2(\alpha_0 - 1)y_i - (\tau^2 h\alpha_0 - \tau\alpha_1)M_i - (\tau\alpha_1 - \tau^2 h)M_{i+1} - (2\alpha_0 + \tau h\alpha_1 - 2)y_i''}{\tau^2\sqrt{h}(\alpha_0 - 1)}. \tag{2.3}$$

Where $\alpha_0 = cos(\tau h)$, and $\alpha_1 = sin(\tau h)$.
We obtained the continuity conditions using fractional derivative from the Caputo fractional derivative:
$D^{(1/2)}S_i(t_i) = D^{(1/2)}S_{i-1}(t_i)$, then we get the following relations:

$$\mu_1 M_{i-1} - \mu_2 M_i + \mu_3 M_{i+1} = \mu_4(y_{i-1} - 2y_i + y_{i+1}) - \mu_5 y_{i-1}'' + \mu_6 y_i''. \tag{2.4}$$

Where
$\mu_1 = \frac{2\sqrt{\pi\tau h}\alpha_2 + (\pi-4)\theta\alpha_0 - \pi\alpha_1}{\sqrt{\tau\pi}}$, $\mu_2 = \frac{2\sqrt{h\pi\theta}(\sqrt{2}\alpha_2 - 1) - \sqrt{h}(2\pi\alpha_1 - \alpha\theta\alpha_0 + (4-\pi)\theta)}{\sqrt{\pi\theta}}$, $\mu_3 = \frac{(\sqrt{\pi\tau}(\theta - \alpha_1) + \sqrt{2h\tau})}{\tau}$,
$\mu_4 = -\alpha_4\sqrt{\theta}(\alpha_0 - 1)$, $\mu_5 = \frac{2\sqrt{\pi\theta}(\alpha_1\alpha_2 + \alpha_3(\alpha_0 - 1)) + 2\pi(\alpha_0 - 1) + (\pi - 4)\theta\alpha_1}{\sqrt{\pi\tau^3}}$, $\mu_6 = \frac{(\sqrt{\pi}(\alpha_1\theta + 2\alpha_0 - 2) + \sqrt{2\theta}(\alpha_1 + \alpha_0 - 1))}{\sqrt{\tau^3}}$
,
$\theta = h\tau$, $\alpha_0 = cos(h\tau)$, $\alpha_1 = sin(h\tau)$, $\alpha_2 = sin((4h\tau + \pi)/4)$, $\alpha_3 = cos((4h\tau h + \pi)/4)$
and $\alpha_4 = \sqrt{(\pi/h)}$.
The following local truncation error was observed by expanding Eq. (2.4) with Taylor series about t_i:
$T_i = \beta_1 y_i' + \beta_2 y_i'' + \beta_3 y_i''' + \beta_4 y_i^{(4)} + \beta_5 y_i^{(5)} + \beta_6 y_i^{(6)} + O(h^6)$.
Where
$\beta_1 = (-\mu_1 - \mu_2 - \mu_3)$, $\beta_2 = (\mu_1 h - \mu_3 h - \mu_4 h^2 + \mu_5 + \mu_6)$, $\beta_3 = (-\mu_1\frac{h^2}{2!} - \mu_3\frac{h^2}{2!} - \mu_5 h)$,
$\beta_4 = (\mu_1\frac{h^3}{3!} - \mu_3\frac{h^3}{3!} - \mu_4\frac{h^4}{12} + \mu_5\frac{h^2}{2!})$, $\beta_5 = (\mu_1\frac{h^4}{4!} - \mu_3\frac{h^4}{4!} - \mu_4\frac{2h^5}{5!} - \mu_5\frac{h^3}{3!})$,
and $\beta_6 = \mu_5\frac{h^4}{4!}$. Two more equations are required to get the unique solution of the linear system (2.4). Using the Taylor series and the undetermined coefficients technique, which is shown below.

$$\begin{aligned} \sum_{k=1}^{2} \gamma_k y_k' &= \frac{1}{6h}\sum_{k=0}^{4} \eta_k y_k + O(h^5), \\ \sum_{k=2}^{3} \gamma_{k-1} y_k' &= \frac{1}{12h}\sum_{k=0}^{5} \sigma_k y_k + O(h^5). \end{aligned} \tag{2.5}$$

The unknown coefficients in Eq. (2.5) are obtained as follows by using Taylor's expansion:
$(\gamma_1, \gamma_2) = (-\mu_2, \mu_3)$
$(\eta_0, \eta_1, \eta_2, \eta_3, \eta_4) = (-2\mu_2, -3\mu_2 - 2\mu_3, 6\mu_2 - 3\mu_3, 6\mu_3 - \mu_2, -\mu_3)$
$(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) = (\mu_2, \mu_3 - 8\mu_2, -8\mu_3, 8\mu_2, 8\mu_3 - \mu_2, -\mu_3)$
Rewriting equation (2.4) we get the following in matrix form: $LM = L_1 y + L_2\bar{y}$
And hence

$$M = L^{-1}L_1 y + L^{-1}L_2 L_3 y, \tag{2.6}$$

Where $M = (y_0', y_1', \ldots, y_n')^T, y = (y_0, y_1, \ldots, y_n)^T, \bar{y} = (y_0'', y_1'', \ldots, y_n'')^T$ and $L_3 y = \bar{y}$.
Also L_1 is three-diagonal matrix, L_2 is two-diagonal matrix, L_3 is an integration diagonal

matrix, and

$$L=\begin{bmatrix} \mu_1 & -\mu_2 & \mu_3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_1 & -\mu_2 & \mu_3 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & -\mu_2 & \mu_3 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_1 & -\mu_2 & \mu_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_1 & -\mu_2 & \mu_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_1 & -\mu_2 & \mu_3 \end{bmatrix}, \tag{2.7}$$

3 Method of Non polynomial Analysis

Using the spline polynomial technique, we investigate the second type of integral equation. For Eq. (1.1), a problem has been derived, which discusses the existence and uniqueness of the solution.
From Eq. (1.1) and Eqs. (2.1)-(2.3) we have:

$$\begin{aligned} y(t_i) &\approx f(t_i)+\sum_{j=0}^{m-1}\int_{x_j}^{x_{j+1}} k\left(t_i,x\right)S_j(x)dx, \\ &= f(t_i)+\sum_{j=0}^{m-1}\int_{x_j}^{x_{j+1}} k\left(t_i,x\right)\left[a_j sin(\tau(x-x_j))+b_j cos(\tau(x-x_j))+c_j(x-x_j)+d_j(x-x_j)^{1/2}+e_j\right]dx, \\ &= f(t_i)+\sum_{j=0}^{m-1}\int_{x_j}^{x_{j+1}} k\left(t_i,x\right)\Big[\frac{\sqrt{(x-x_j)}}{h}y_{j+1}+(1-\frac{\sqrt{(x-x_j)}}{h})y_j+(\frac{\alpha_0}{(\alpha_0-1)}(x-x_j)- \\ &\quad \frac{sin(\tau(x-x_j))}{\tau(\alpha_0-1)}-\frac{\theta\alpha_0-\alpha_1}{\sqrt{\theta\tau}(\alpha_0-1)})M_j+(\frac{sin(\tau(x-x_j)}{\tau(\alpha_0-1)}-\frac{(x-x_j)}{(\alpha_0-1)}-\frac{(\alpha_1-\theta)}{\sqrt{\tau\theta}(\alpha_0-1)}\sqrt{x-x_j})M_{j+1} \\ &\quad +(\frac{\alpha}{\theta(\alpha_0-1)}(x-x_j)-\frac{\alpha_1 sin(\tau(x-x_j))}{\tau^2(\alpha_0-1)}-\frac{cos(\tau(x-x_j))}{\tau^2}-\frac{2\alpha_0+\theta\alpha_1-2}{\tau^2\sqrt{h}(\alpha_0-1)}\sqrt{x-x_j}+\frac{1}{\tau^2})y_j''\Big]dx \end{aligned}$$

$$= f(t_i) + \sum_{j=0}^{m-1} \left(\frac{y_{j+1}}{\sqrt{h}} - \frac{\alpha_1 - \theta}{\sqrt{\tau\theta}(\alpha_0 - 1)} M_{j+1} \right) \int_{x_j}^{x_{j+1}} k(t_i, x)\ \sqrt{x - x_j} dx + \sum_{j=0}^{m-1} \Big(-y_j -$$

$$\frac{\theta\alpha_0 - \alpha_1}{\sqrt{\tau\theta}(\alpha_0 - 1)} M_j - \frac{(2\alpha_0 + \theta\alpha_1 - 2)}{\tau^2 \sqrt{h}(\alpha_0 - 1)} y_j'' \Big) \int_{x_j}^{x_{j+1}} k(t_i, x)\ \sqrt{x - x_j} dx + \sum_{j=0}^{m-1} \Big(\frac{\alpha_0}{\alpha_0 - 1} M_j +$$

$$\frac{\alpha_1}{\tau(\alpha_0 - 1)} y_j'' \Big) \int_{x_j}^{x_{j+1}} k(t_i, x)\left(x - x_j\right) dx - \sum_{j=0}^{m-1} \frac{M_{j+1}}{(\alpha_0 - 1)} \int_{x_j}^{x_{j+1}} k(t_i, x)\left(x - x_j\right) dx + \sum_{j=0}^{m-1} \frac{M_{j+1}}{\tau(\alpha_0 - 1)}$$

$$\int_{x_j}^{x_{j+1}} k(t_i, x) \sin\left(\tau\left(x - x_j\right)\right) dx + \sum_{j=0}^{m-1} \Big(\frac{-M_j}{\tau(\alpha_0 - 1)} - \frac{\alpha_1}{\tau^2(\alpha_0 - 1)} y_j'' \Big) \int_{x_j}^{x_{j+1}} k(t_i, x)$$

$$\sin\left(\tau\left(x - x_j\right)\right) dx + \sum_{j=0}^{m-1} \left(y_j + \frac{y_j''}{\tau^2} \right) \int_{x_j}^{x_{j+1}} k(t_i, x) dx - \sum_{j=0}^{m-1} \frac{y_j''}{\tau^2} \int_{x_j}^{x_{j+1}} k(t_i, x) \cos\left(\tau\left(x - x_j\right)\right) dx.$$

Let
$a(i,j) = \int_{x_j}^{x_{j+1}} k(t_i, x)\ \sqrt{x - x_j} dx = b(i, j+1)$, $c(i,j) = \int_{x_j}^{x_{j+1}} k(t_i, x)\left(x - x_j\right) dx = d(i, j+1)$,
$q(i,j) = \int_{x_j}^{x_{j+1}} k(t_i, x)\, Sin\left(\tau\left(x - x_j\right)\right) dx = r(i, j+1)$, $g(i,j) = \int_{x_j}^{x_{j+1}} k(t_i, x) dx$
and $p(i,j) = \int_{x_j}^{x_{j+1}} k(t_i, x) \cos\left(\tau\left(x - x_j\right)\right) dx$
Suppose that $A = a(i,j), B = b(i,j), C = c(i,j), D = d(i,j), Q = q(i,j), R = r(i,j), G = g(i,j)$ and $P = p(i,j)$.
Also $\hat{y}_j, \hat{M}_j, \hat{F}_l$ and $\hat{\ddot{y}}_j$ are approximations for $\mathrm{y_j}, \mathrm{M_j}, \mathrm{f_i}$ and $\hat{y}_i$ respectively such satisfy in Eq. (2.4) for i=0, 1, . . . ,m then we get:

$$\hat{y}_j = \hat{F}_i + \frac{B}{\sqrt{h}} \hat{y}_j - \frac{\alpha_1 - \theta}{\sqrt{\tau\theta}(\alpha_0 - 1)} B\hat{M}_j - A\hat{y}_j - \frac{\theta\alpha_0 - \alpha_1}{\sqrt{\tau\theta}(\alpha_0 - 1)} A\hat{M}_j - \frac{(2\alpha_0 + \theta\alpha_1 - 2)}{\tau^2 \sqrt{h}(\alpha_0 - 1)} A\hat{\ddot{y}}_j +$$
$$\frac{\alpha_0}{\alpha_0 - 1} C\hat{M}_j + \frac{\alpha_1}{\tau(\alpha_0 - 1)} C\hat{\ddot{y}}_j - \frac{1}{\alpha_0 - 1} D\hat{M}_j + \frac{1}{\tau(\alpha_0 - 1)} R\hat{M}_j - \frac{1}{\tau(\alpha_0 - 1)} Q\hat{M}_j - \frac{\alpha_1}{\tau^2(\alpha_0 - 1)} Q$$
$$\hat{\ddot{y}}_j + G\hat{y}_j + \frac{G}{\tau^2} \hat{\ddot{y}}_j - \frac{P}{\tau^2} \hat{\ddot{y}}_j. \tag{3.1}$$

Then we get:

$$\hat{y}_j = \hat{F}_i + \frac{1}{\sqrt{h}} (B - \sqrt{h} A + G) \hat{y}_j + \frac{1}{(\alpha_0 - 1)} \left(\frac{\alpha_1 - \theta}{\sqrt{\tau\theta}} B - \frac{\theta\alpha_0 - \alpha_1}{\sqrt{\tau\theta}} A + \alpha_0 C - D + \frac{R}{\tau} - \frac{Q}{\tau} \right) \hat{M}_j$$
$$+ \frac{1}{\tau} \left(\frac{2\alpha_0 + \theta\alpha_1 - 2}{\tau\sqrt{h}(\alpha_0 - 1)} A + \frac{\alpha_1}{(\alpha_0 - 1)} C - \frac{\alpha_1}{\tau(\alpha_0 - 1)} Q + \frac{G}{\tau} - \frac{P}{\tau} \right) \hat{\ddot{y}}_j.$$

Let

$$A_1 = \frac{1}{\sqrt{h}} (B - \sqrt{h} A + G),\ A_2 = \frac{1}{(\alpha_0 - 1)} \left(\frac{\alpha_1 - \theta}{\sqrt{\tau\theta}} B - \frac{\theta\alpha_0 - \alpha_1}{\sqrt{\tau\theta}} A + \alpha_0 C - D + \frac{R}{\tau} - \frac{Q}{\tau} \right),$$
$$A_3 = \frac{1}{\tau} \left(\frac{2\alpha_0 + \theta\alpha_1 - 2}{\tau\sqrt{h}(\alpha_0 - 1)} A + \frac{\alpha_1}{(\alpha_0 - 1)} C - \frac{\alpha_1}{\tau(\alpha_0 - 1)} Q + \frac{G}{\tau} - \frac{P}{\tau} \right).$$

Then

$$\hat{y}_j = \hat{F}_i + A_1 \hat{y}_j + A_2 \widehat{M}_j + A_3 \hat{\ddot{y}}_j \tag{3.2}$$

Substituting Eq. (2.6) in Eq. (3.2) we get:

$$[I - A_1 - L^{-1}L_1A_2 + L^{-1}L_2L_3A_2 + A_3L_3]\hat{y} = \hat{F} + T. \tag{3.3}$$

The vector of local truncation error is $T = [t_0, t_1, \dots, t_m]$, displayed as the (m+1) dimensional column vector of the exact solution. $\hat{y} = [y_0, y_1, \cdots, y_m]^T$.
According to Eqs. (3.2) and (3.3) we get:

$$[I - A_1 - L^{-1}L_1A_2 + L^{-1}L_2L_3A_2 + A_3L_3]E = T. \tag{3.4}$$

By solving Eq. (3.1), an approximation of Eq. (1.1) will be obtained.
The function y_i can now be approximated by using the non-polynomial fractional spline $\hat{S}_i$, where

$$\hat{S}_i(t) = \frac{\left(\tau^{\frac{5}{2}}\sqrt{\theta}(\alpha_0-1)\hat{y}_i+\tau^{\frac{5}{2}}\sqrt{\theta}(\alpha_0-1)\hat{y}_{i+1}+\tau^2\sqrt{h}(\alpha_0\theta-\alpha_1)\hat{\mathrm{M}}_i+\tau^2\sqrt{h}(\theta-\alpha_1)\hat{\mathrm{M}}_{i+1}+\sqrt{\tau\theta}(2\alpha_0-\theta\alpha_1-2)\hat{y}_i\right)}{\tau^2\theta^{\frac{3}{2}}(\alpha_0-1)}\sqrt{t-t_i}+\frac{(\tau\alpha_0\hat{\mathrm{M}}_i-\tau\hat{\mathrm{M}}_{i+1}+\alpha_1\hat{y}_i)}{\tau\alpha_0-\tau}(t-t_i)+$$
$$\left(\frac{\hat{\mathrm{M}}_{i+1}-\tau\hat{\mathrm{M}}_i-\alpha_1\hat{\hat{y}}_i}{\tau^2\alpha_0-\tau^2}\right)\left(\sin\left(\tau\left(t-t_i\right)\right)\right)-\left(\frac{\hat{\hat{y}}_i}{\tau^2}\right)\left(\cos\left(\tau\left(t-t_i\right)\right)\right)+\left(\frac{\tau^2\hat{y}_i+\sqrt{h\hat{y}_i}}{\tau^2\sqrt{h}}\right)+O(h^5). \tag{3.5}$$

In-consequence $\forall i = 1(1)m-1, t \in (t_i, t_{i+1})$, then we get:

$$\left|\mathrm{S_i(t)} - \hat{\mathrm{S}}_\mathrm{i}(\mathrm{t})\right| \equiv \varphi \mathrm{h}^5. \tag{3.6}$$

4 Convergence of the method

This section includes some important theorems and lemmas, as well as the study of non-polynomial fractional spline convergence.

Lemma 4.1. *[10] Let L be a square Matrix with $\|L\|_\infty < 1$, then the matrix $(I - L)$ is invertible. Furthermore, $\left\|(I-L)^{-1}\right\|_\infty \leq \frac{1}{1-\|L\|_\infty}$,*
Where I is the identity matrix and $\|L\|_\infty$ is the infinity norm of the matrix, $L = (l_{ij})$ that is described as following:

$$\|\mathrm{L}\|_\infty = \max_{1\leq i\leq n}\left(\sum_{j=0}^{n}\left|\mathrm{l_{ij}}\right|\right).$$

Lemma 4.2. *Let $S(t)$ satisfy in (2.1)-(2.4) and be the unique non-polynomial fractional spline, for a given function $y \in C^5[a,b]$. Then:$\|S^\alpha - y^\alpha\| \leq O(h^3)$, where $\alpha \in R$.*
***proof**. We investigate the continuity of sufficiently high-order derivatives of y by applying (2.1)-(2.4), and we obtain*

$$\begin{aligned}
S_i^{\left(\frac{1}{2}\right)}(t_i) &= -\gamma_0 y_i + \gamma_1 y_i^{\left(\frac{1}{2}\right)} + \gamma_2 y_i' + \gamma_3 y_i^{\left(\frac{3}{2}\right)} + \gamma_4 y_i'' + \gamma_5 y_i^{\left(\frac{5}{2}\right)} + \gamma_6 y_i^{(3)} + \gamma_7 y_i^{\left(\frac{7}{2}\right)} + \gamma_8 y_i^{(4)}(\alpha_1), \\
S_i^{\left(\frac{3}{2}\right)}(t_i) &= \gamma_9 \mathrm{y_i}^{\left(\frac{3}{2}\right)} + \gamma_{10}\mathrm{y_i}'' + \gamma_{11}\mathrm{y_i}^{\left(\frac{5}{2}\right)} + \gamma_{12}\mathrm{y_i}^{(3)} + \gamma_{13}\mathrm{y_i}^{\left(\frac{7}{2}\right)} + \gamma_{14}\mathrm{y_i}^{(4)}(\alpha_1), \\
S_i^{\left(\frac{5}{2}\right)}(t_i) &= -\gamma_{15}\mathrm{y_i}^{\left(\frac{3}{2}\right)} + \gamma_{16}\mathrm{y_i}'' + \gamma_{17}\mathrm{y_i}^{\left(\frac{5}{2}\right)} + \gamma_{18}\mathrm{y_i}^{(3)} + \gamma_{19}\mathrm{y_i}^{\left(\frac{7}{2}\right)} + \gamma_{20}\mathrm{y_i}^{(4)}(\alpha_1).
\end{aligned} \tag{4.1}$$

Where, $\gamma_0 = \frac{\sqrt{\pi}(\sin(\tau\mathrm{h})-\tau\mathrm{h})}{2\tau\sqrt{\mathrm{h}}(\cos(\tau\mathrm{h})-1)}, \gamma_1 = \frac{(\sqrt{\varepsilon\mathrm{h}}-1)(\tau\sqrt{\mathrm{h}(\cos(\tau\mathrm{h})-1)})-\sqrt{\varepsilon\mathrm{h}}(\sin(\tau\mathrm{h})-\tau\mathrm{h})}{\tau\sqrt{\mathrm{h}}(\cos(\tau\mathrm{h})-1)}$,
$\gamma_2 = \frac{\sqrt{\pi}\varepsilon h^{\frac{3}{2}}\tau(\cos(\tau h)-1)+\sqrt{\pi}\sin(\tau h)-\sqrt{h}\varepsilon(\sin(\tau h)-\tau h)-\sqrt{\pi}\tau\,\mathrm{h}\cos(\tau h)}{2\tau\sqrt{h}(\cos(\tau h)-1)}$,

$\gamma_3 = \frac{3\cdot\sqrt{2}\sqrt{\varepsilon}\sqrt{\tau}+2\sqrt{\pi}\tau\varepsilon^{\frac{3}{2}}h^2(\cos(\tau h)-1)-2\sqrt{\pi}\tau\varepsilon^2h^2(\sin(\tau h)-\tau h)}{3\tau\sqrt{\pi h}(\cos(\tau h)-1)}$,
$\gamma_4 = \frac{\varepsilon(2h)^{\frac{3}{2}}+4\sqrt{\pi}\varepsilon^2h^{\frac{3}{2}}\tau(\cos(\tau h)-1)-\sqrt{\pi}(\varepsilon h)^2(\sin(\tau h)-\tau h)+4\sqrt{\pi}(\cos(\tau h)-1)}{4\tau\sqrt{h}(\cos(\tau h)-1)}$,
$\gamma_5 = \frac{5\sqrt{\tau}(2\varepsilon)^{\frac{3}{2}}h^2+4\sqrt{\pi}\varepsilon^{\frac{5}{2}}h^3\tau-2\sqrt{\pi}(\varepsilon h)^{\frac{3}{2}}(\sin(\tau h)-\tau h)}{15\tau\sqrt{\pi h}(\cos(\tau h)-1)}$, $\gamma_6 = \frac{6\sqrt{\tau}(h)^{\frac{3}{2}}\varepsilon^2+\tau(2)^{\frac{3}{2}}\varepsilon^3h^{\frac{5}{2}}(\cos(\tau \mathrm{h})-1)-\sqrt{2\pi}(\sin(\tau \mathrm{h})-\tau \mathrm{h})}{12\tau\sqrt{2}\,\mathrm{h}(\cos(\tau \mathrm{h})-1)}$,
$\gamma_7 = \frac{(2\ \mathrm{h}\varepsilon)^{\frac{5}{2}}}{15\sqrt{\pi\tau}(\cos(\tau\mathrm{h})-1)}$, $\gamma_8 = \frac{(\varepsilon\mathrm{h})^3}{6\tau\sqrt{2\tau}(\cos(\tau\mathrm{h})-1)'}$, $\gamma_9 = \frac{\sqrt{2\tau\varepsilon\mathrm{h}}}{\sqrt{\pi}(\cos(\tau\mathrm{h})-1)}$, $\gamma_{10} = \frac{\tau\varepsilon\mathrm{h}-\sin(\tau\mathrm{h})+\sqrt{\tau}(\cos(\tau\mathrm{h})-1)}{\sqrt{\tau}(\cos(\tau\mathrm{h})-1)}$,
$\gamma_{11} = \frac{4\sqrt{\tau}(\varepsilon\mathrm{h})^{\frac{3}{2}}}{3\sqrt{\pi}(\cos(\tau\mathrm{h})-1)}$, $\gamma_{12} = \frac{\sqrt{\tau}(\varepsilon\mathrm{h})^2}{2(\cos(\tau\mathrm{h})-1)'}$, $\gamma_{13} = \frac{8\sqrt{\tau}(\varepsilon\mathrm{h})^{\frac{5}{2}}}{15\sqrt{\pi}(\cos(\tau\mathrm{h})-1)}$, $\gamma_{14} = \frac{\sqrt{\tau}(\varepsilon\mathrm{h})^3}{6(\cos(\tau\mathrm{h})-1)}$,
$\gamma_{15} = \frac{\tau\sqrt{2\varepsilon\tau h}}{\sqrt{\pi}}$, $\gamma_{16} = \frac{\varepsilon\mathrm{h}\tau^2\sqrt{2}+\sqrt{2\tau}\sin(\tau\mathrm{h})-1}{\sqrt{2\tau}}$, $\gamma_{17} = \frac{4(\varepsilon\tau\mathrm{h})^{\frac{3}{2}}}{3\sqrt{\pi}}$, $\gamma_{18} = \frac{4(\varepsilon\mathrm{h})^2(\tau)^{\frac{3}{2}}}{2}$, $\gamma_{19} = \frac{8(\varepsilon\mathrm{h})^{\frac{5}{2}}(\tau)^{\frac{3}{2}}}{15\sqrt{\pi}}$
and $\gamma_{20} = \frac{(\varepsilon\mathrm{h})^3(\tau)^{\frac{3}{2}}}{6}$.
Now, let $e(t) = S(t) - y(t)$*, then for* $0 \le t \le 1$,

$$\mathrm{e}\left(\mathrm{t_i}+\varepsilon\mathrm{h}\right) = e\left(t_i\right) + \frac{2}{\sqrt{\pi}}\varepsilon^{\frac{1}{2}}h^{\frac{1}{2}}y_i^{\left(\frac{1}{2}\right)} + \varepsilon h y_i^{(1)} + \frac{4}{3\sqrt{\pi}}\varepsilon^{\frac{3}{2}}h^{\frac{3}{2}}y_i^{\left(\frac{3}{2}\right)} + \frac{1}{2!}\varepsilon^2h^2y_i^{(2)} + \frac{8}{15\sqrt{\pi}}\varepsilon^{\frac{5}{2}}h^{\frac{5}{2}}y_i^{\left(\frac{5}{2}\right)} + \frac{1}{3!}\varepsilon^3h^3y_i^{(3)} + \frac{16}{105\sqrt{\pi}}\varepsilon^{\frac{7}{2}}h^{\frac{7}{2}}y_i^{\left(\frac{7}{2}\right)}\left(\alpha_j\right). \tag{4.2}$$

For $0 \le \varepsilon \le 1$ *Putting Eq. (4.1) in Eq. (4.2), we get:*
$\|e\left(x_i+\varepsilon h\right)\| \le \frac{\sqrt{\tau}(\varepsilon h)^3}{6}y_i^{(4)}\left(\alpha_1\right)$.

Lemma 4.3. *The matrix* $[I - A_1 - L^{-1}L_1A_2 + L^{-1}L_2L_3A_2 + A_3L_3]$ *is invertible, if* $\varphi\|k\|_\infty(b-a)\left(\frac{2}{3} - \frac{2\sqrt{h}}{3} + \sigma_1\sigma_2\sigma_3\tau^{\frac{3}{2}}\frac{h}{2} - \tau\sigma_3\sigma_4\right) < 1$.
Proof:
Clearly, for $j = 0, 1, \ldots, n$*, then:*

$$\begin{aligned}
&\|A\|_\infty = \|B\|_\infty \le \|k\|_\infty(b-a)\frac{2h^{\frac{1}{2}}}{3},\\
&\|C\|_\infty = \|D\|_\infty \le \|k\|_\infty(b-a)\frac{h}{2},\\
&\|Q\|_\infty = \|R\|_\infty \le \|k\|_\infty(b-a)|\sin(\tau h)|,\\
&\|G\|_\infty \le \|k\|_\infty(b-a),\\
&\|P\|_\infty \le \|k\|_\infty(b-a)\frac{1}{\tau}|\cos(\tau h)|,\\
&\|L_1\|_\infty \le \sigma_1\frac{|\cos(\tau h)|}{\sqrt{\tau\pi}},\\
&\|L_2\|_\infty \le \sigma_2\tau^{\frac{3}{2}},\\
&\|A_1\|_\infty \le \|k\|_\infty(b-a)\frac{2(1-\sqrt{h})}{3},\\
&\|A_2\|_\infty \le \|k\|_\infty(b-a)\frac{h}{2},\\
&\|A_3\|_\infty \le \|k\|_\infty(b-a)\tau\sigma_4.
\end{aligned} \tag{4.3}$$

Where

$$\sigma_4 = \frac{2(2\cos(\tau h) - 2 + \tau\sin(\tau h))}{3\tau^2(\cos(\tau h)-1)} + \frac{h\sin(\tau h)}{2(\cos(\tau h)-1)} + \frac{\sin(\tau h)^2}{\tau(\cos(\tau h)-1)} + \frac{cos(\tau h - 1)}{\tau^3}.$$

From Eq. (3.4) of matrix representation we get:

$$\left(\|A_1\| + \left\|L^{-1}\right\| \|L_2\| \|L_3\| \|A_2\| - \|A_3\| \|L_3\|\right) < 1,$$

Then we use lemma (4.1), the matrix $\left[\mathrm{I} - \mathrm{A}_1 - \mathrm{L}^{-1}\ \mathrm{L}_1\ \mathrm{A}_2 + \mathrm{L}^{-1}\ \mathrm{L}_2\ \mathrm{L}_3\ \mathrm{A}_2 + \mathrm{A}_3\ \mathrm{L}_3\right]$ *, is invertible, if* $\left\|A_1 + L^{-1}L_1A_2 - L^{-1}L_2L_3A_2 - A_3L_3\right\|_\infty < 1$*, we get:*

$$\varphi\|k\|_\infty(b-a)\left(\frac{2}{3} - \frac{2\sqrt{h}}{3} + \sigma_1\sigma_2\sigma_3\tau^{\frac{3}{2}}\frac{h}{2} - \tau\sigma_3\sigma_4\right) < 1.$$

Theorem 4.4. *[10]. Let* $y(t) \in C^5(I), k(t,x) \in C^5(I \times I)$ *such that*

$$\varphi\|k\|_\infty(b-a)\left(\frac{2}{3} - \frac{2\sqrt{h}}{3} + \sigma_1\sigma_2\sigma_3\tau^{\frac{3}{2}}\frac{h}{2} - \tau\sigma_3\sigma_4\right) < 1.$$

As a result, consider single numerical solutions and the error obtained. $E = y - \hat{S}$ *satisfies* $\|E\| \equiv O(h^3), \quad \forall\Omega \subset I$ *Where* $\tau, \theta, h, \sigma_1, \cdots, \sigma_4$*, and* σ_5 *are constants, and* $I := [a,b]$*.*

Proof:

We use Eq. (3.4) and lemma (4.1) we get

$$\|E\| \leq \frac{\|T\|}{1 - (\|A_1\| + \|L^{-1}\| \|L_2\| \|L_3\| \|A_2\| - \|A_3\| \|L_3\|)}, \tag{4.4}$$

By substituting $\|T\| \leq \omega h^3$ *and Eq. (4.3) in Eq. (4.4) we get:* $\|E\| \equiv O\left(h^3\right)$*,*

Therefore, we have

$$\|y - \hat{S}\|_\infty \leq \varphi_1 h^3, \tag{4.5}$$

And applying Eq. (3.6) and Eq. (4.5), $\|y - \hat{S}\|_\infty \leq \|y - S\|_\infty + \|S - \hat{S}\|_\infty \leq \varphi_1 h^3 + \varphi \mathrm{h}^5 \equiv \mathrm{O}(\mathrm{h}^3)$

Thus, it is as follows: $\|E\| \to 0$ *as* $h \to 0$*, then we explained the convergence of the third order proposed method.*

See [10].

5 Results and Discussion

The proposed technique is applied to some FIE (Fredholm-integral equations) test problems in this section, with a comparison of the presented method and the exact solution to illustrate the suggested technique's correctness and effectiveness, as well as to compare it with some other existing methods for solving three integral equations test problems. We calculate the results for x = 0, 0.2, 0.4, 0.6, 0.8, 1, and n = 10, 40. Python software handles all of the calculations. The absolute error $||E||$ in theorem 4.4 is applied to compute the efficiency of the proposed technique.

Example 5.1. [5] Consider the FIE:

$$g(x) = f(x) + \int_0^1 k(x,t)g(t)dt, x \in [0,1].$$

Where $k(x,t) = \frac{1}{12}\frac{tx-1}{1+x^2}$, and $f(x)$ is chosen so that the exact solution of this equation $g(x) = sin(x) + 1$. Presented the exact and approximation solutions in Table 5.1. and Figure 5.1. and the absolute errors of the proposed method and NSI method in [5] in Table 5.2.

x	Exact solutions	Proposed method
0	1.0	0.9998809889969068
0.2	1.0998334166468282	1.0997366950959002
0.4	1.1986693307950613	1.198566011640177
0.6	1.2955202066613396	1.295421184579774
0.8	1.3894183423086506	1.3893240990734304
1	1.479425538604203	1.4793383323910736

Table 5.1: Difference between exact and approximation solutions with h=0.1, and $\tau = 10^6$.

x	Best in [5] of NSI with n=20	Presented method with n=10
0	0.46×10^{-3}	1.19×10^{-4}
0.2	0.61×10^{-3}	9.67×10^{-5}
0.4	0.68×10^{-3}	1.03×10^{-4}
0.6	0.66×10^{-3}	9.90×10^{-5}
0.8	0.60×10^{-3}	9.42×10^{-5}
1	0.51×10^{-3}	8.72×10^{-5}

Table 5.2: Absolute errors E(n) for different points.

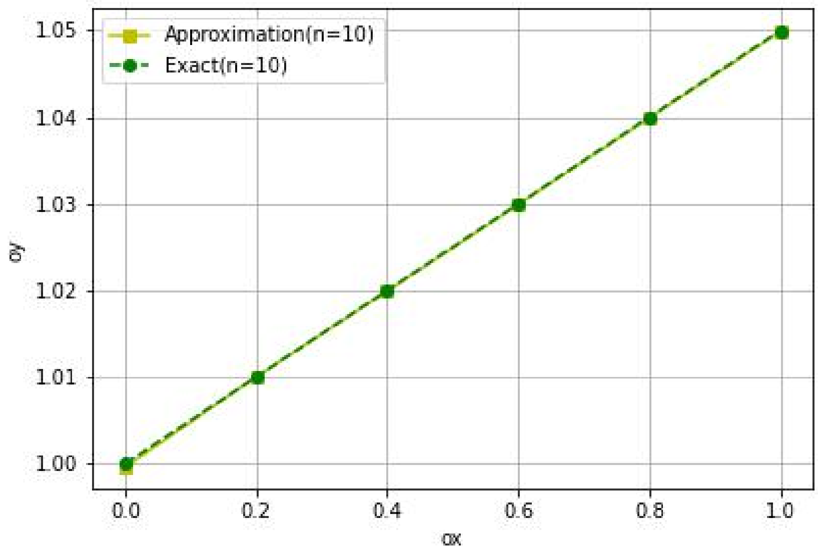

Figure 5.1: Comparison between the exact solution and approximate solution using the proposed method.

Example 5.2. [21] Consider the FIE:

$$g(x) = f(x) + \int_0^1 k(x,t)g(t)dt, x \in [0,1].$$

Where $k(x,t) = \frac{t^4}{24}x, f(x) = e^x - \frac{x^4}{24}$ and $g(x) = e^x$, is the exact solution. Presented the absolute errors of solutions in Table 5.3. and Figure 5.2.

$n=10$ and $\tau=10^5$			$n=40$ and $\tau=10^6$			$n=10^6$ and $\tau=10^{10}$		
$g(x_i)$	$S(x_i)$	$\lvert E(x_i)\rvert$	$g(x_i)$	$S(x_i)$	$\lvert E(x_i)\rvert$	$g(x_i)$	$S(x_i)$	$\lvert E(x_i)\rvert$
1.0	1.0	0.0	1.0	1.0	0.0	1.0	1.0	0.0
1.10517	1.10525	8.5×10^{-5}	1.025315	1.025312	2.1×10^{-6}	1.000001	1.000001	0.0
1.22140	1.22050	8.9×10^{-4}	1.051271	1.051262	8.2×10^{-6}	1.000002	1.000002	0.0
1.34985	1.34929	5.6×10^{-4}	1.077884	1.077849	3.4×10^{-5}	1.000003	1.000003	0.0
1.49182	1.48824	3.5×10^{-3}	1.105170	1.105097	7.3×10^{-5}	1.000004	1.000004	0.0
1.64872	1.64836	3.5×10^{-4}	1.133148	1.133007	1.4×10^{-4}	1.000005	1.000005	0.0

Table 5.3: Absolute error E(n) for different points.

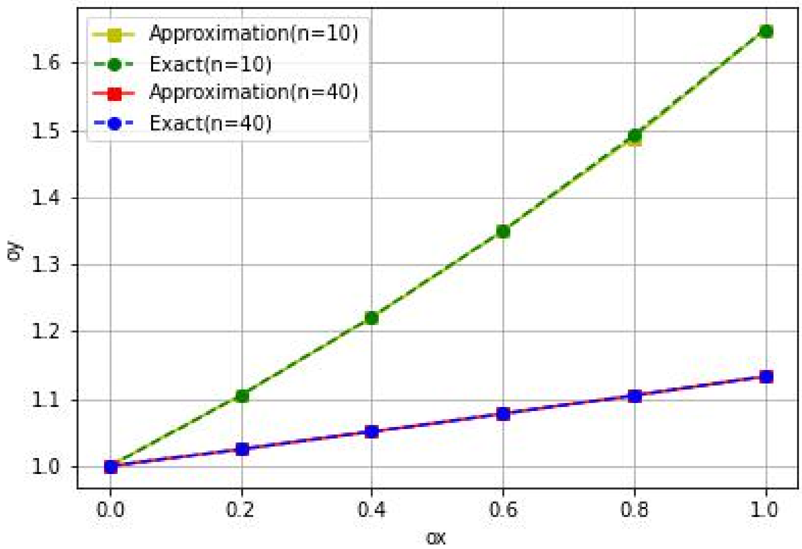

Figure 5.2: Comparison between the exact solution and approximate solution using the proposed method.

Example 5.3. [21] consider the FIE:

$$g(x)=f(x)+\int_{-1}^{1}k(x,t)g(t)dt, x\in[0,1],$$

where, $k(x,t)=\frac{x^4}{24}$, $f(x)=e^{-x}-\frac{x^2}{2}+\frac{x^3e^{-1}}{6}$ and $g(x)=e^{-x}-\frac{x^2}{2}$ is the exact solution.
Presented the absolute errors of solutions in Table 5.4 and Figure 5.3.

Example 5.4. [21] consider the IDE:

$$g(x)=f(x)+\int_{0}^{1}k(x,t)g(t)dt, x\in[0,1].$$

where, $k(x,t)=(\frac{x^4}{24}-\frac{tx^3}{6}), f(x)=xe^x+1-\frac{x^4}{24}+\frac{x^3(e-2)}{6}$ and $g(x)=xe^x+1$ is the exact solution.
Presented the absolute errors of solutions in Table 5.5 and Figure 5.4.

$n = 10$ and $\tau = 0.1$			$n = 40$ and $\tau = 0.1$			$n = 10^6$ and $\tau = 0.1$		
$g(x_i)$	$S(x_i)$	$\|E(x_i)\|$	$g(x_i)$	$S(x_i)$	$\|E(x_i)\|$	$g(x_i)$	$S(x_i)$	$\|E(x_i)\|$
1.0	1.0	0.0	1.0	1.0	0.0	1.0	1.0	0.0
0.89983	0.89989	6.04×10^{-5}	0.974997	0.974998	9.57×10^{-7}	0.99	0.99	0.0
0.79873	0.79919	4.6×10^{-4}	0.949979	0.949987	7.64×10^{-6}	0.99	0.99	0.0
0.69581	0.69730	1.48×10^{-3}	0.924930	0.924956	2.56×10^{-5}	0.99	0.99	0.0
0.59032	0.59351	3.19×10^{-3}	0.899837	0.899897	6.04×10^{-5}	0.99	0.99	4.5×10^{-25}
0.48153	0.48714	5.61×10^{-3}	0.874684	0.874801	1.17×10^{-4}	0.99	0.99	7.2×10^{-25}

Table 5.4: Absolute error E(n) for different points.

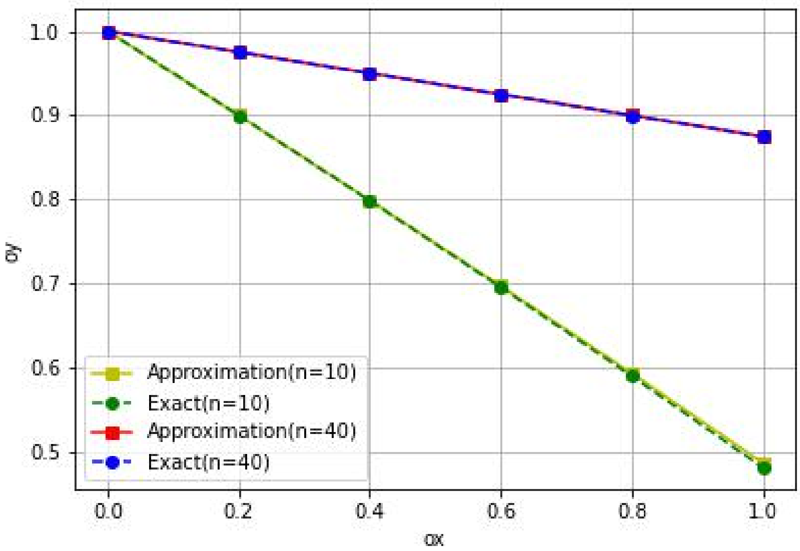

Figure 5.3: Comparison between the exact solution and approximate solution using the proposed method.

6 Conclusion

This paper presents a new general form of non-polynomial fractional spline function to approximate the Fredholm-integral equation of the second kind, and the proposed approach is innovative. The current scheme was developed by running four different examples through the Python program. The results were compared to the exact solution and show that the proposed technique is better than the method in [5]. The physical behavior of approximation and exact solutions can be evaluated in 2D for various points, and it is clear that adding step sizes ensures no error.

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$n=10$ and $\tau=10^4$			$n=40$ and $\tau=10^4$			$n=10^6$ and $\tau=10^9$		
$g(x_i)$	$S(x_i)$	$\lvert E(x_i)\rvert$	$g(x_i)$	$S(x_i)$	$\lvert E(x_i)\rvert$	$g(x_i)$	$S(x_i)$	$\lvert E(x_i)\rvert$
1.0	1.0	0.0	1.0	1.0	0.0	1.0	1.0	0.0
1.11051	1.11074	2.2×10^{-4}	1.025632	1.025637	4.9×10^{-6}	1.000001	1.000001	0.0
1.24428	1.24422	5.9×10^{-5}	1.052563	1.052553	9.9×10^{-6}	1.000002	1.000002	0.0
1.40495	1.40532	3.6×10^{-4}	1.080841	1.080870	2.8×10^{-5}	1.000003	1.000003	0.0
1.59672	1.59583	8.9×10^{-4}	1.110517	1.110525	8.8×10^{-6}	1.000004	1.000004	0.0
1.82436	1.82086	3.4×10^{-3}	1.141643	1.141698	5.4×10^{-5}	1.000005	1.000005	0.0

Table 5.5: Absolute error E(n) for different points.

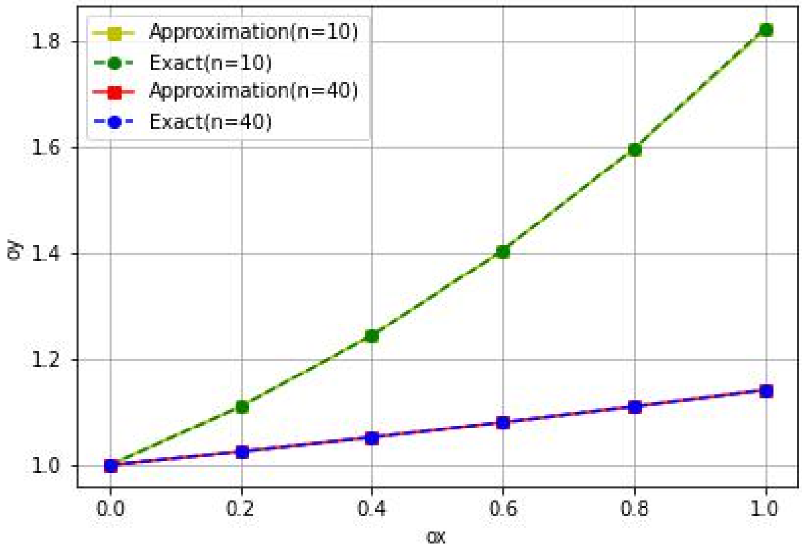

Figure 5.4: Comparison between the exact solution and approximate solution using the proposed method.

Conflict of interest

The authors have no conflicts of interest to declare.

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