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Generalized Contraction Theorems in $\mathfrak{M}$-Fuzzy Cone Metric Spaces

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Abstract. This work defines $\mathfrak{M}$-Fuzzy Cone Metric Space, as a new metric space. It also analyzes possible forms of contractive conditions and groups them accordingly to set up generalized contractive conditions for self-mappings defined over $\mathfrak{M}$-fuzzy cone metric spaces. We prove the existence of fixed points of these mappings and exhibit the same through a suitable example.
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1 Introduction

A self-mapping f, defined on a metric space (M,d), is said to be a contraction if for some $k \in [0,1)$, it fulfills the condition $d(f(x), f(y)) \leq kd(x,y)$, for all $x, y \in M$. Stefan Banach, a Polish mathematician, used these contractions to bring out his fixed point theorem, a remarkable finding, known as Banach Contraction Principle.

Theorem 1.1. *(Banach [1])*
(X,d) is a complete metric space and $f : X \to X$ is a contraction mapping. Thus, there exists a constant $r < 1$ such that $d(f(x), f(y)) \leq rd(x,y)$ for each $x, y \in X$. From this, one draws three conclusions:
(i) f has a unique fixed point, say x_0;
(ii) For each $x \in X$ the Picard sequence $\{f^n(x)\}$ converges to x_0;
(iii) The convergence is uniform if X is bounded.

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This principle has made a great impact in the domain of research. Since then, it has been the origin of numerous findings as all these findings are its modifications. These modifications are made to the contractive conditions and the settings of the domain. One of the remarkable extensions is due to Hardy and Rogers in the year 1973. His work is an extension of Reich's fixed point theorem.

Theorem 1.2. *(Hardy and Rogers) [10]*
Let (M, d) be a metric space and T a self-mapping of M satisfying the condition for $x, y \in M$

$$(1)\ d(Tx, Ty) \leq ad(x, Tx) + bd(y, Ty) + cd(x, Ty) + d(y, Tx) + fd(x, y),$$

where a, b, c, d, e, f are nonnegative and we set $\alpha = a + b + c + d + e + f$. Then
(a) If M is complete and $\alpha < 1$, T has a unique fixed point.
(b) If (1) is modified to the condition

$$(1')\ d(Tx, Ty) \leq ad(x, Tx) + bd(y, Ty) + cd(x, Ty) + d(y, Tx) + fd(x, y),$$

and in this case, we assume M is compact, T is continuous and $\alpha = 1$, then T has a unique fixed point.

Likewise, the Banach Contraction Principle has seen numerous extensions and generalizations. Besides, in the year 1965, Zadeh [19] made a great contribution to the field of mathematics by introducing the definition of fuzzy set, an idea to handle uncertainties well. Since then, new metrics are being discovered over fuzzy sets. A few fuzzy metrics, that were found at the initial stage, can be found in [2, 4, 12, 13]. Making a slight change in the definition of Kramosil and Michalek [13], George and Veeramani [5] present a fuzzy metric space which is more adaptable due to its topological structure. In the year 2000, Gregori and Sapena [7] defined fuzzy contractive mappings and proved fixed point theorems on both of these fuzzy metric spaces

Sedghi and Shobe [16] presented $\mathfrak{M}$-fuzzy metric spaces in the year 2006. Huang and Zhang [11] defined cone metric spaces as a generalization of metric spaces in the year 2007. Combining the concept of cone metric spaces and fuzzy metric spaces [5], Oner et. al. [14] came up with the concept of fuzzy cone metric spaces.

Here, we aim to present $\mathfrak{M}$-fuzzy cone metric spaces in the sense of [16] and [14]. We also define generalized fuzzy cone contractive conditions and prove some fixed point theorems for self-mappings in the settings of $\mathfrak{M}$-fuzzy cone metric spaces.

2 Preliminaries

Definition 2.1. [14] Let E be a real Banach space and $\mathscr{C}$ be a subset of E. $\mathscr{C}$ is called a cone if and only if:
[C1] $\mathscr{C}$ is closed, nonempty, and $\mathscr{C}$ is not equal to $\{0\}$,
[C2] $a, b \in \mathbb{R}$, $a, b \geq 0$, $\mathbf{c}_1, \mathbf{c}_2 \in \mathscr{C}$ imply $a\mathbf{c}_1 + b\mathbf{c}_2 \in \mathscr{C}$,
[C3] $\mathbf{c} \in \mathscr{C}$ and $-\mathbf{c} \in \mathscr{C}$ imply $\mathbf{c} = 0$.
The cones considered here are subsets of a real Banach space E and are with nonempty interiors.

Definition 2.2. An $\mathfrak{M}$-Fuzzy Cone Metric Space (briefly, $\mathfrak{M}$-FCM Space) is a 3-tuple $(\mathcal{Z}, \mathfrak{M}, *)$ where $\mathcal{Z}$ is an arbitrary set, $*$ is a continuous t-norm, $\mathscr{C}$ is a cone and $\mathfrak{M}$ a fuzzy set in $\mathcal{Z}^3 \times int(\mathscr{C})$ satisfying the following conditions: For all $\zeta, \eta, \omega, \mathsf{u} \in \mathcal{Z}$ and $\mathbf{c}, \mathbf{c}' \in int(\mathscr{C})$,
[MFC1] $\mathfrak{M}(\zeta, \eta, \omega, \mathbf{c}) > 0$,
[MFC2] $\mathfrak{M}(\zeta, \eta, \omega, \mathbf{c}) = 1$ if and only if $\zeta = \eta = \omega$,
[MFC3] $\mathfrak{M}(\zeta, \eta, \omega, \mathbf{c}) = \mathfrak{M}(p\{\zeta, \eta, \omega\}, \mathbf{c})$, where p is a permutation,
[MFC4] $\mathfrak{M}(\zeta, \eta, \omega, \mathbf{c} + \mathbf{c}') \geq \mathfrak{M}(\zeta, \eta, \mathsf{u}, \mathbf{c}) * \mathfrak{M}(\mathsf{u}, \omega, \omega, \mathbf{c}')$,
[MFC5] $\mathfrak{M}(\zeta, \eta, \omega, \cdot) : int(\mathscr{C}) \to [0, 1]$ is continuous.
Here, $\mathfrak{M}$ is called $\mathfrak{M}$-Fuzzy Cone Metric on $\mathcal{Z}$. The function $\mathfrak{M}(\zeta, \eta, \omega, \mathbf{c})$ denote the degree of nearness between ζ, η and ω with respect to $\mathbf{c}$.

Example 2.3. Let $E = \mathbb{R}^2$ and consider the cone $\mathscr{C} = \{(\mathbf{c}_1, \mathbf{c}_2) \in \mathbb{R}^2 : \mathbf{c}_1 \geq 0, \mathbf{c}_2 \geq 0\}$ in E. Let the t-norm $*$ be defined by $\mathfrak{a} * \mathfrak{b} = \mathfrak{a}\mathfrak{b}$. Define the function $\mathfrak{M} : \mathbb{R}^3 \times int(\mathscr{C}) \to [0, 1]$ by $\mathfrak{M}(\zeta, \eta, \omega, \mathbf{c}) = \frac{1}{e^{\frac{|\zeta-\eta|+|\eta-\omega|+|\omega-\zeta|}{\|c\|}}}$, for all $\zeta, \eta, \omega \in \mathbb{R}$ and $\mathbf{c} \in int(\mathscr{C})$. Then $(\mathbb{R}, \mathfrak{M}, *)$ is an $\mathfrak{M}$-Fuzzy Cone Metric Space.

Definition 2.4. A symmetric $\mathfrak{M}$-FCM Space is an $\mathfrak{M}$-FCM Space $(\mathcal{Z}, \mathfrak{M}, *)$ satisfying

$$\mathfrak{M}(\eta, \omega, \omega, \mathbf{c}) = \mathfrak{M}(\omega, \eta, \eta, \mathbf{c}), \text{ for all } \eta, \omega \in \mathcal{Z} \text{ and } \mathbf{c} \in int(\mathscr{C}).$$

Definition 2.5. Let $(\mathcal{Z}, \mathfrak{M}, *)$ be an $\mathfrak{M}$-FCM Space. A self-mapping $\mathcal{K} : \mathcal{Z} \to \mathcal{Z}$ is said to be $\mathfrak{M}$-fuzzy cone contractive if there exists $k \in (0, 1)$ such that

$$\left(\frac{1}{\mathfrak{M}(\mathcal{K}(\zeta), \mathcal{K}(\eta), \mathcal{K}(\omega), \mathbf{c})} - 1\right) \leq k \left(\frac{1}{\mathfrak{M}(\zeta, \eta, \omega, \mathbf{c})} - 1\right),$$

for all $\zeta, \eta, \omega \in \mathcal{Z}$ and $\mathbf{c} \in int(\mathscr{C})$.

Remark 2.6. In the above definition, k excludes the value zero, for if $k = 0$, then it is possible to have

$$\left(\frac{1}{\mathfrak{M}(\mathcal{K}(\zeta), \mathcal{K}(\eta), \mathcal{K}(\omega), \mathbf{c})} - 1\right) > \left(\frac{1}{\mathfrak{M}(\zeta, \eta, \omega, \mathbf{c})} - 1\right),$$

for all distinct $\zeta, \eta, \omega \in \mathcal{Z}$ and $\mathbf{c} \in \mathrm{int}(\mathscr{C})$, and $\mathcal{K}$ cannot have any fixed point.

Definition 2.7. In an $\mathfrak{M}$-FCM Space $(\mathcal{Z}, \mathfrak{M}, *)$, $\mathfrak{M}$ is said to be triangular if, for all $\zeta, \eta, \omega, \mathsf{u} \in \mathcal{Z}$ and $\mathbf{c} \in \mathrm{int}(\mathscr{C})$,

$$\left(\frac{1}{\mathfrak{M}(\zeta, \eta, \omega, \mathbf{c})} - 1\right) \leq \left(\frac{1}{\mathfrak{M}(\zeta, \eta, \mathsf{u}, \mathbf{c})} - 1\right) + \left(\frac{1}{\mathfrak{M}(\mathsf{u}, \omega, \omega, \mathbf{c})} - 1\right).$$

Definition 2.8. Let $(\mathcal{Z}, \mathfrak{M}, *)$ be an $\mathfrak{M}$-FCM Space. For $\mathsf{u} \in \mathcal{Z}, r > 0$ and $\mathbf{c} \in int(\mathscr{C})$, the open ball $B_{\mathscr{C}}(\mathsf{u}, r, \mathbf{c})$, with center at u and radius r, is defined by

$$B_{\mathscr{C}}(\mathsf{u}, r, \mathbf{c}) = \{\mathsf{w} \in \mathcal{Z} : \mathfrak{M}(\mathsf{u}, \mathsf{w}, \mathsf{w}, \mathbf{c}) > 1 - r\}.$$

Lemma 2.9. *[18] For each $\mathbf{c}_1 \in int(\mathscr{C})$ and $\mathbf{c}_2 \in int(\mathscr{C})$, there exists $\mathbf{c} \in int(\mathscr{C})$ such that $\mathbf{c}_1 - \mathbf{c} \in int(\mathscr{C})$ and $\mathbf{c}_2 - \mathbf{c} \in int(\mathscr{C})$.*

Theorem 2.10. *Let $(\mathcal{Z}, \mathfrak{M}, *)$ is an $\mathfrak{M}$-FCM Space. Then $\tau_{\mathscr{C}}$, defined hereunder, is a topology:*

$$\tau_{\mathscr{C}} = \left\{ \begin{array}{c} \mathscr{D} \subseteq \mathcal{Z} : a \in \mathscr{D} \quad \text{if and only if there exists} \quad r \in (0, 1) \\ \text{and} \quad \mathbf{c} \in int(\mathscr{C}) \quad \text{such that} \quad \mathscr{B}_{\mathscr{C}}(a, r, \mathbf{c}) \subset \mathscr{D} \end{array} \right\}.$$

Proof. **(i)** It is obvious that $\emptyset \in \tau_{\mathscr{C}}$ and $\mathcal{Z} \in \tau_{\mathscr{C}}$.
(ii) Suppose $\mathscr{D}_1 \in \tau_{\mathscr{C}}$ and $\mathscr{D}_2 \in \tau_{\mathscr{C}}$ and $a \in \mathscr{D}_1 \cap \mathscr{D}_2$. Then $a \in \mathscr{D}_1$ and $a \in \mathscr{D}_2$.
Then, there exists $r_1, r_2 \in (0,1)$ and $\mathbf{c}_1, \mathbf{c}_2 \in int(\mathscr{C})$ such that

$$\mathscr{B}_{\mathscr{C}}(a, r_1, \mathbf{c}_1) \subset \mathscr{D}_1 \quad \text{and} \quad \mathscr{B}_{\mathscr{C}}(a, r_2, \mathbf{c}_2) \subset \mathscr{D}_2.$$

By Lemma 2.9, there exists $\mathbf{c} \in int(\mathscr{C})$ such that $\mathbf{c}_1 - \mathbf{c} \in int(\mathscr{C}), \mathbf{c}_2 - \mathbf{c} \in int(\mathscr{C})$.
Let $r = \min\{r_1, r_2\}$. Then $\mathscr{B}_{\mathscr{C}}(a, r, \mathbf{c}) \subset \mathscr{B}_{\mathscr{C}}(a, r_1, \mathbf{c}_1) \cap \mathscr{B}_{\mathscr{C}}(a, r_2, \mathbf{c}_2) \subset \mathscr{D}_1 \cap \mathscr{D}_2$.
Hence, $\mathscr{D}_1 \cap \mathscr{D}_2 \in \tau_{\mathscr{C}}$.
(iii) Let $\mathscr{D}_j \in \tau_{\mathscr{C}}$ for each $j \in J$, an index set, and let $a \in U_{j \in J} \mathscr{D}_j$. Then $a \in \mathscr{D}_{j_0}$ for some $j_0 \in J$.
Hence, there exists $r \in (0,1)$ and $\mathbf{c} \in int(\mathscr{C})$ such that $\mathscr{B}_{\mathscr{C}}(a, r, \mathbf{c}) \subset D_{j_0}$.
As $D_{j_0} \subset U_{j \in J} \mathscr{D}_j$, we have that $\mathscr{B}_{\mathscr{C}}(a, r, \mathbf{c}) \subset U_{j \in J} \mathscr{D}_j$.
Thus, $U_{j \in J} \mathscr{D}_j \in \tau_{\mathscr{C}}$.
From (i), (ii) and (iii), $\tau_{\mathscr{C}}$ is a topology. □

Remark 2.11. [5] For any $r_1 > r_2$, there exists r_3 such that $r_1 * r_3 \geq r_2$ and for any r_4 there exists $r_5 \in (0,1)$ such that $r_5 * r_5 \geq r_4$, where $r_1, r_2, r_3, r_4, r_5 \in (0,1)$.

Theorem 2.12. *Let $(\mathcal{Z}, \mathfrak{M}, *)$ be an $\mathfrak{M}$-FCM Space. Then $(\mathcal{Z}, \tau_{\mathscr{C}})$ is Hausdorff.*

Proof. Let $\zeta, \omega \in \mathcal{Z}$ be distinct. Then $0 < \mathfrak{M}(\zeta, \omega, \omega, \mathbf{c}) < 1$ for all $\mathbf{c} \in int(\mathscr{C})$.
Let $\mathfrak{M}(\zeta, \omega, \omega, \mathbf{c}) = r$.
Now, for each $r_0 \in (r, 1)$, there exists $r_1 \in (0,1)$ such that $r_1 * r_1 \geq r_0$.
Suppose $\mathscr{B}_{\mathscr{C}}\left(\zeta, 1 - r_1, \frac{\mathbf{c}}{2}\right) \cap \mathscr{B}_{\mathscr{C}}\left(\omega, 1 - r_1, \frac{\mathbf{c}}{2}\right)$ is nonempty.
Then there exists $z \in \mathscr{B}_{\mathscr{C}}\left(\zeta, 1 - r_1, \frac{\mathbf{c}}{2}\right) \cap \mathscr{B}_{\mathscr{C}}\left(\omega, 1 - r_1, \frac{\mathbf{c}}{2}\right)$ and we have that

$$r = \mathfrak{M}(\zeta, \omega, \omega, \mathbf{c}) \geq \mathfrak{M}\left(\zeta, \omega, \zeta, \frac{\mathbf{c}}{2}\right) * \mathfrak{M}\left(\zeta, \omega, \omega, \frac{\mathbf{c}}{2}\right) \geq r_1 * r_1 \geq r_0 > r.$$

This is a contradiction. Hence, $\mathscr{B}_{\mathscr{C}}\left(\zeta, 1 - r_1, \frac{\mathbf{c}}{2}\right) \cap \mathscr{B}_{\mathscr{C}}\left(\omega, 1 - r_2, \frac{\mathbf{c}}{2}\right)$ is empty.
Therefore, $(\zeta, \tau_{\mathscr{C}})$ is Hausdorff. □

Definition 2.13. Let $(\mathcal{Z}, \mathfrak{M}, *)$ be an $\mathfrak{M}$-FCM Space, $\zeta' \in \mathcal{Z}$ and $\{\zeta_n\}$ be a sequence in $\mathcal{Z}$.

(i) $\{\zeta_n\}$ is said to converge to ζ' if for all $\mathbf{c} \in int(\mathscr{C}), \lim_{n \to \infty}\left(\frac{1}{M(\zeta_n, \zeta', \zeta', \mathbf{c})} - 1\right) = 0$. It is denoted by $\lim_{n \to \infty} \zeta_n = \zeta'$ or by $\zeta_n \to \zeta'$ as $n \to \infty$.

(ii) $\{\zeta_n\}$ is said to be a Cauchy sequence if for all $\mathbf{c} \in int(\mathscr{C})$ and $m \in \mathbb{N}$, we have that $\lim_{n \to \infty}\left(\frac{1}{\mathfrak{M}(\zeta_{n+m}, \zeta_n, \zeta_n, \mathbf{c})} - 1\right) = 0$.

(iii) $(\mathcal{Z}, \mathfrak{M}, *)$ is called a complete $\mathfrak{M}$-FCM space if every Cauchy sequence in $\mathcal{Z}$ converges.

Definition 2.14. Let $(\mathcal{Z}, M, *)$ be an $\mathfrak{M}$-FCM Space. A sequence $\{\zeta_n\}$ in $\mathcal{Z}$ is $\mathfrak{M}$-fuzzy cone contractive if there exists $k \in (0,1)$ such that

$$\left(\frac{1}{\mathfrak{M}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}, \mathbf{c})} - 1\right) \leq k\left(\frac{1}{\mathfrak{M}(\zeta_{n-1}, \zeta_n, \zeta_n, \mathbf{c})} - 1\right), \text{ for all } \mathbf{c} \in int(\mathscr{C}).$$

Lemma 2.15. *An $\mathfrak{M}$-FCM Space $(\mathcal{Z}, \mathfrak{M}, *)$ is symmetric.*

Proof. Let $\eta, w \in \mathcal{Z}$ and $\mathbf{c} \in int(\mathscr{C})$. Then,

$$\lim_{r\to 0} \mathfrak{M}(\eta,\eta,\omega,\mathbf{c}+r) \geq \lim_{r\to 0} \left(\mathfrak{M}(\eta,\eta,\eta,r) * \mathfrak{M}(\eta,\omega,\omega,\mathbf{c})\right),$$
$$\lim_{r\to 0} \mathfrak{M}(\omega,\omega,\eta,\mathbf{c}+r) \geq \lim_{r\to 0} \left(\mathfrak{M}(\omega,\omega,\omega,r) * \mathfrak{M}(\omega,\eta,\eta,\mathbf{c})\right).$$

These inequalities imply that

$$\mathfrak{M}(\eta,\eta,\omega,\mathbf{c}) \geq \mathfrak{M}(\eta,\omega,\omega,\mathbf{c}) \quad \text{and} \quad \mathfrak{M}(\omega,\omega,\eta,\mathbf{c}) \geq \mathfrak{M}(\omega,\eta,\eta,\mathbf{c}).$$

Hence, $\mathfrak{M}(\eta,\omega,\omega,\mathbf{c}) = \mathfrak{M}(\omega,\eta,\eta,\mathbf{c})$. □

Lemma 2.16. *Let $(\mathcal{Z},\mathfrak{M},*)$ be an $\mathfrak{M}$-FCM Space, where M is triangular. Then any $\mathfrak{M}$-fuzzy cone contractive sequence in $\mathcal{Z}$ is a Cauchy sequence.*

Proof. Let the sequence $\{\zeta_n\}$ be $\mathfrak{M}$-fuzzy cone contractive in $\mathcal{Z}$. Then, there exists $k \in (0,1)$, such that

$$\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},\mathbf{c})} - 1\right) \leq k \left(\frac{1}{\mathfrak{M}(\zeta_{n-1},\zeta_n,\zeta_n,\mathbf{c})} - 1\right). \tag{2.1}$$

Since $\mathfrak{M}$ is triangular, by Lemma(2.15), for $m > n$,

$$\begin{aligned}
\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_n,\zeta_m,\mathbf{c})} - 1\right) &\leq \left(\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_n,\zeta_{n+1},\mathbf{c})} - 1\right) + \left(\frac{1}{\mathfrak{M}(\zeta_{n+1},\zeta_{n+1},\zeta_m,\mathbf{c})} - 1\right)\right) \\
&\leq \left(\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_n,\zeta_{n+1},\mathbf{c})} - 1\right) + \left(\frac{1}{\mathfrak{M}(\zeta_{n+1},\zeta_{n+1},\zeta_{n+2},\mathbf{c})} - 1\right)\right. \\
&\quad \left. + \left(\frac{1}{\mathfrak{M}(\zeta_{n+2},\zeta_{n+2},\zeta_m,\mathbf{c})} - 1\right)\right).
\end{aligned}$$

Continuing the process, and using (2.1), we finally arrive at

$$\begin{aligned}
\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_n,\zeta_m,\mathbf{c})} - 1\right) &\leq \begin{pmatrix} \left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_n,\zeta_{n+1},\mathbf{c})} - 1\right) + \left(\frac{1}{\mathfrak{M}(\zeta_{n+1},\zeta_{n+1},\zeta_{n+2},\mathbf{c})} - 1\right) \\ + \cdots + \left(\frac{1}{\mathfrak{M}(\zeta_{m-1},\zeta_{m-1},\zeta_m,\mathbf{c})} - 1\right) \end{pmatrix} \\
&\leq k^n \left(\frac{1}{\mathfrak{M}(\zeta_0,\zeta_0,\zeta_1,\mathbf{c})} - 1\right) + \cdots + k^{m-1} \left(\frac{1}{\mathfrak{M}(\zeta_0,\zeta_0,\zeta_1,\mathbf{c})} - 1\right) \\
&= (k^n + \cdots + k^{m-1}) \left(\frac{1}{\mathfrak{M}(\zeta_0,\zeta_0,\zeta_1,\mathbf{c})} - 1\right) \\
&\leq \frac{k^n}{1-k} \left(\frac{1}{\mathfrak{M}(\zeta_0,\zeta_0,\zeta_1,\mathbf{c})} - 1\right).
\end{aligned} \tag{2.2}$$

From (2.2), we have that $\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_n,\zeta_m,\mathbf{c})} - 1\right) \to 0$ as $n \to \infty$.
Hence, $\{\zeta_n\}$ is a Cauchy sequence. □

3 Main Results

This section aims to prove the existence of fixed points of self-mappings under generalized $\mathfrak{M}$-fuzzy cone contractive conditions in a complete $\mathfrak{M}$-FCM Space.

Theorem 3.1. *Let $(\mathcal{Z},\mathfrak{M},*)$ be a complete $\mathfrak{M}$-FCM Space where $\mathfrak{M}$ is triangular. If $\mathcal{K}:\mathcal{Z}\to\mathcal{Z}$ is such that, for all $\zeta,\eta,\omega\in\mathcal{Z}$ and $\mathbf{c}\in int(\mathscr{C})$,*

$$\left(\frac{1}{\mathfrak{M}(\mathcal{K}\zeta,\mathcal{K}\eta,\mathcal{K}\omega,\mathbf{c})}-1\right)\leq\left\{\begin{array}{l} k_1\left(\frac{1}{\mathfrak{M}(\zeta,\eta,\omega,\mathbf{c})}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta,\mathcal{K}\zeta,\mathcal{K}\zeta,\mathbf{c})}-1\right)+\\ k_3\left(\frac{1}{\mathfrak{M}(\eta,\mathcal{K}\omega,\mathcal{K}\omega,\mathbf{c})}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\eta,\mathcal{K}\eta,\mathcal{K}\eta,\mathbf{c})}-1\right)+\\ k_5\left(\frac{1}{\mathfrak{M}(\omega,\mathcal{K}\omega,\mathcal{K}\omega,\mathbf{c})}-1\right)+k_6\left(\frac{1}{\mathfrak{M}(\omega,\mathcal{K}\eta,\mathcal{K}\eta,\mathbf{c})}-1\right)+\\ k_7\left(\frac{1}{\mathfrak{M}(\eta,\mathcal{K}\zeta,\omega,\mathbf{c})}-1\right)\end{array}\right\}, \tag{3.1}$$

where $k_i\in[0,\infty],i=1,\ldots,7$ and $\sum_{i=1}^{6}k_i<1$. Then $\mathcal{K}$ has a fixed point and such a point is unique if $k_1+k_7<1$.

Proof. Let $\zeta_0\in\mathcal{Z}$ be arbitrary. Generate a sequence $\{\zeta_n\}$ with $\zeta_n=\mathcal{K}\zeta_{n-1}$ for $n\in\mathbb{N}$. If there exists a non-negative integer m such that $\zeta_{m+1}=\zeta_m$, then $\mathcal{K}\zeta_m=\zeta_m$ and ζ_m becomes a fixed point of $\mathcal{K}$.
Suppose $\zeta_n\neq\zeta_{n-1}$ for any $n\in\mathbb{N}$.
From (3.1),

$$\begin{aligned}\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},\mathbf{c})}-1\right)&\leq\left(\frac{1}{\mathfrak{M}(\mathcal{K}\zeta_{n-1},\mathcal{K}\zeta_n,\mathcal{K}\zeta_n,\mathbf{c})}-1\right)\\ &\leq\left\{\begin{array}{l} k_1\left(\frac{1}{\mathfrak{M}(\zeta_{n-1},\zeta_n,\zeta_n,\mathbf{c})}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta_{n-1},\mathcal{K}\zeta_{n-1},\mathcal{K}\zeta_{n-1},\mathbf{c})}-1\right)\\ +k_3\left(\frac{1}{\mathfrak{M}(\zeta_n,\mathcal{K}\zeta_n,\mathcal{K}\zeta_n,\mathbf{c})}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\zeta_n,\mathcal{K}\zeta_n,\mathcal{K}\zeta_n,\mathbf{c})}-1\right)\\ +k_5\left(\frac{1}{\mathfrak{M}(\zeta_n,\mathcal{K}\zeta_n,\mathcal{K}\zeta_n,\mathbf{c})}-1\right)+k_6\left(\frac{1}{\mathfrak{M}(\zeta_n,\mathcal{K}\zeta_n,\mathcal{K}\zeta_n,\mathbf{c})}-1\right)\\ +k_7\left(\frac{1}{\mathfrak{M}(\zeta_n,\mathcal{K}\zeta_{n-1},\mathcal{K}\zeta_n,\mathbf{c})}-1\right)\end{array}\right\}\\ &=\left\{\begin{array}{l} k_1\left(\frac{1}{\mathfrak{M}(\zeta_{n-1},\zeta_n,\zeta_n,\mathbf{c})}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta_{n-1},\zeta_n,\zeta_n,\mathbf{c})}-1\right)\\ +k_3\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},\mathbf{c})}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},\mathbf{c})}-1\right)\\ +k_5\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},\mathbf{c})}-1\right)+k_6\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},\mathbf{c})}-1\right)\\ +k_7\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_n,\zeta_n,\mathbf{c})}-1\right)\end{array}\right\}\\ &=\left\{\begin{array}{l}(k_1+k_2)\left(\frac{1}{\mathfrak{M}(\zeta_{n-1},\zeta_n,\zeta_n,\mathbf{c})}-1\right)\\ +(k_3+k_4+k_5+k_6)\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},\mathbf{c})}-1\right).\end{array}\right\}.\end{aligned}$$

Hence, we have that

$$\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},\mathbf{c})}-1\right)\leq\frac{k_1+k_2}{1-(k_3+k_4+k_5+k_6)}\left(\frac{1}{\mathfrak{M}(\zeta_{n-1},\zeta_n,\zeta_n,\mathbf{c})}-1\right). \tag{3.2}$$

Put $k=\frac{k_1+k_2}{1-(k_3+k_4+k_5+k_6)}$. Then, $k<1$ and (3.2) becomes

$$\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},\mathbf{c})}-1\right)\leq k\left(\frac{1}{\mathfrak{M}(\zeta_{n-1},\zeta_n,\zeta_n,\mathbf{c})}-1\right). \tag{3.3}$$

(3.3) makes the sequence $\{\zeta_n\}$ $\mathfrak{M}$-fuzzy cone contractive. Hence, by Lemma(2.16), $\{\zeta_n\}$ is Cauchy in $\mathcal{Z}$. As $\mathcal{Z}$ is complete, there exists $\zeta'\in\mathcal{Z}$ such that

$$\lim_{n\to\infty}\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta',\zeta',\mathbf{c})}-1\right)=0. \tag{3.4}$$

By repeated application of (3.3), we obtain that

$$\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},\mathbf{c})}-1\right)\leq k^n\left(\frac{1}{\mathfrak{M}(\zeta_0,\zeta_1,\zeta_1,\mathbf{c})}-1\right).$$

Therefore, we have that

$$\lim_{n\to\infty}\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},\mathbf{c})}-1\right)=0. \tag{3.5}$$

Now,

$$\begin{aligned}\left(\frac{1}{\mathfrak{M}(\zeta_{n+1},\mathcal{K}\zeta',\mathcal{K}\zeta',\mathbf{c})}-1\right) &= \left(\frac{1}{\mathfrak{M}(\mathcal{K}\zeta_n,\mathcal{K}\zeta',\mathcal{K}\zeta',\mathbf{c})}-1\right)\\
&\leq\left\{\begin{array}{l} k_1\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta',\zeta',\mathbf{c})}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta_n,\mathcal{K}\zeta_n,\mathcal{K}\zeta_n,\mathbf{c})}-1\right)\\ +k_3\left(\frac{1}{\mathfrak{M}(\zeta',\mathcal{K}\zeta',\mathcal{K}\zeta',\mathbf{c})}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\zeta',\mathcal{K}\zeta',\mathcal{K}\zeta',\mathbf{c})}-1\right)\\ +k_5\left(\frac{1}{\mathfrak{M}(\zeta',\mathcal{K}\zeta',\mathcal{K}\zeta',\mathbf{c})}-1\right)+k_6\left(\frac{1}{\mathfrak{M}(\zeta',\mathcal{K}\zeta',\mathcal{K}\zeta',\mathbf{c})}-1\right)\\ +k_7\left(\frac{1}{\mathfrak{M}(\zeta',\mathcal{K}\zeta_n,\zeta',\mathbf{c})}-1\right)\end{array}\right\}\\
&=\left\{\begin{array}{l} k_1\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta',\zeta',\mathbf{c})}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},\mathbf{c})}-1\right)\\ +k_3\left(\frac{1}{\mathfrak{M}(\zeta',\mathcal{K}\zeta',\mathcal{K}\zeta',\mathbf{c})}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\zeta',\mathcal{K}\zeta',\mathcal{K}\zeta',\mathbf{c})}-1\right)\\ +k_5\left(\frac{1}{\mathfrak{M}(\zeta',\mathcal{K}\zeta',\mathcal{K}\zeta',\mathbf{c})}-1\right)+k_6\left(\frac{1}{\mathfrak{M}(\zeta',\mathcal{K}\zeta',\mathcal{K}\zeta',\mathbf{c})}-1\right)\\ +k_7\left(\frac{1}{\mathfrak{M}(\zeta',\zeta_{n+1},\zeta',\mathbf{c})}-1\right)\end{array}\right\}\\
&\to k'\left(\frac{1}{\mathfrak{M}(\zeta',\mathcal{K}\zeta',\mathcal{K}\zeta',\mathbf{c})}-1\right)\text{ as } n\to\infty,\\
&\text{where } k'=k_3+k_4+k_5+k_6, \text{since by (3.4) and (3.5)}.\end{aligned}$$

Hence,

$$\limsup_{n\to\infty}\left(\frac{1}{\mathfrak{M}(\zeta_{n+1},\mathcal{K}\zeta',\mathcal{K}\zeta',\mathbf{c})}-1\right)\leq k'\left(\frac{1}{\mathfrak{M}(\zeta',\mathcal{K}\zeta',\mathcal{K}\zeta',\mathbf{c})}-1\right). \tag{3.6}$$

As $\mathfrak{M}$ is triangular,

$$\left(\frac{1}{\mathfrak{M}(\mathcal{K}\zeta',\mathcal{K}\zeta',\zeta',\mathbf{c})}-1\right)\leq\left(\frac{1}{\mathfrak{M}(\mathcal{K}\zeta',\mathcal{K}\zeta',\zeta_{n+1},\mathbf{c})}-1\right)+\left(\frac{1}{\mathfrak{M}(\zeta_{n+1},\zeta',\zeta',\mathbf{c})}-1\right). \tag{3.7}$$

From (3.5) to (3.7), we can bring that

$$\left(\frac{1}{\mathfrak{M}(\mathcal{K}\zeta',\mathcal{K}\zeta',\zeta',\mathbf{c})}-1\right)\leq k'\left(\frac{1}{\mathfrak{M}(\zeta',\mathcal{K}\zeta',\mathcal{K}\zeta',\mathbf{c})}-1\right).$$

This gives, $\left(\frac{1}{\mathfrak{M}(\mathcal{K}\zeta',\mathcal{K}\zeta',\zeta',t)}-1\right)=0$ since $k'<1$, and, hence we have

$$\mathcal{K}\zeta'=\zeta'.$$

Thus, we can conclude that ζ' is a fixed point of $\mathcal{K}$. Suppose $\mathcal{K}\zeta''=\zeta''$. From (3.1),

$$\left(\frac{1}{\mathfrak{M}(\mathcal{K}\zeta',\mathcal{K}\zeta'',\mathcal{K}\zeta'',\mathbf{c})}-1\right)\leq\left\{\begin{array}{l} k_1\left(\frac{1}{\mathfrak{M}(\zeta',\zeta'',\zeta'',\mathbf{c})}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta',\mathcal{K}\zeta',\mathcal{K}\zeta',\mathbf{c})}-1\right)\\ +k_3\left(\frac{1}{\mathfrak{M}(\zeta'',\mathcal{K}\zeta'',\mathcal{K}\zeta'',\mathbf{c})}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\zeta'',\mathcal{K}\zeta'',\mathcal{K}\zeta'',\mathbf{c})}-1\right)\\ +k_5\left(\frac{1}{\mathfrak{M}(\zeta'',\mathcal{K}\zeta'',\mathcal{K}\zeta'',\mathbf{c})}-1\right)+k_6\left(\frac{1}{\mathfrak{M}(\zeta'',\mathcal{K}\zeta'',\mathcal{K}\zeta'',\mathbf{c})}-1\right)\\ +k_7\left(\frac{1}{\mathfrak{M}(\zeta'',\mathcal{K}\zeta',\zeta'',\mathbf{c})}-1\right)\end{array}\right\}.$$

This gives that

$$\left(\frac{1}{\mathfrak{M}(\zeta',\zeta'',\zeta'',\mathbf{c})}-1\right)\leq(k_1+k_7)\left(\frac{1}{\mathfrak{M}(\zeta',\zeta'',\zeta'',\mathbf{c})}-1\right).$$

Therefore, $\left(\frac{1}{\mathfrak{M}(\zeta',\zeta'',\zeta'',\mathbf{c})}-1\right)=0,\ \text{ if } k_1+k_7<1.$
Hence, we can conclude that $\mathcal{K}$ has a unique fixed point if $k_1+k_7<1$. □

Example 3.2. Let $\mathcal{Z}=[0,\infty)$ with metric d defined by $d(\zeta,\eta)=|\zeta-\eta|$ for all $\zeta,\eta\in\mathcal{Z}$ and let $\mathscr{C}=\mathbb{R}^+$. Define the t-norm $*$ by $i*j=\min\{i,j\}$. Define $\mathfrak{M}$ by

$$\mathfrak{M}(\zeta,\eta,\omega,\mathbf{c})=\frac{c}{c+(|\zeta-\eta|+|\eta-\omega|+|\omega-\zeta|)},$$

for all $\zeta,\eta,\omega\in\mathcal{Z}$ and $t\in int(\mathscr{C})$.
Then, it is clear that $(\mathcal{Z},\mathfrak{M},*)$ is a complete $\mathfrak{M}$-FCM Space, and, that $\mathfrak{M}$ is triangular.
Consider the self-mapping, $\mathcal{K}:\mathcal{Z}\to\mathcal{Z}$, given by $\mathcal{K}u=\begin{cases}\frac{5}{4}u+3, & u\in[0,1],\\ \frac{3}{4}u+\frac{7}{2}, & u\in[1,\infty).\end{cases}$
Then,

$$\left(\frac{1}{\mathfrak{M}(\mathcal{K}\zeta,\mathcal{K}\eta,\mathcal{K}\omega,\mathbf{c})}-1\right)=\frac{5}{4}\left(\frac{1}{\mathfrak{M}(\zeta,\eta,\omega,\mathbf{c})}-1\right),$$

where $\zeta,\eta,\omega\in[0,1]$. Hence $\mathcal{K}$ is not $\mathfrak{M}$-fuzzy cone contractive. Therefore, we cannot assure the existence of fixed points using the contraction theorem. But, here $\mathcal{K}$ satisfies the condition (3.1) with

$$k_1=\frac{3}{80},k_2=\frac{17}{80},k_3=k_4=k_5=\frac{1}{20},k_6=0,k_7=\frac{1}{20}.$$

Therefore, $\mathcal{K}$ has a unique fixed point and this point is $u=14$.

Corollary 3.3. *Let $(\mathcal{Z},\mathfrak{M},*)$ be a complete $\mathfrak{M}$-FCM Space where $\mathfrak{M}$ is triangular. If $\mathcal{K}:\mathcal{Z}\to\mathcal{Z}$ is such that for all $\zeta,\eta,\omega\in\mathcal{Z},\mathbf{c}\in int(\mathscr{C})$,*

$$\left(\frac{1}{\mathfrak{M}(\mathcal{K}\zeta,\mathcal{K}\eta,\mathcal{K}\omega,\mathbf{c})}-1\right)\leq\left\{\begin{array}{l}k_1\left(\frac{1}{\mathfrak{M}(\zeta,\eta,\omega,\mathbf{c})}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta,\mathcal{K}\zeta,\mathcal{K}\zeta,\mathbf{c})}-1\right)\\ +k_3\left(\frac{1}{\mathfrak{M}(\eta,\mathcal{K}\omega,\mathcal{K}\omega,\mathbf{c})}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\eta,\mathcal{K}\zeta,\omega,\mathbf{c})}-1\right)\end{array}\right\},$$

where $k_i\in[0,\infty],i=1,\ldots,4$ and $k_1+k_2+k_3<1$. Then $\mathcal{K}$ has a fixed point and such a point is unique if $k_1+k_4<1$.

Corollary 3.4. *Let $(\mathcal{Z},\mathfrak{M},*)$ be a complete $\mathfrak{M}$-FCM Space, where $\mathfrak{M}$ is triangular. If $\mathcal{K}:\mathcal{Z}\to\mathcal{Z}$ is such that for all $\zeta,\eta,\omega\in\mathcal{Z},\mathbf{c}\in int(\mathscr{C})$,*

$$\left(\frac{1}{\mathfrak{M}(\mathcal{K}\zeta,\mathcal{K}\eta,\mathcal{K}\omega,\mathbf{c})}-1\right)\leq\left\{\begin{array}{l}k_1\left(\frac{1}{\mathfrak{M}(\zeta,\eta,\omega,\mathbf{c})}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta,\mathcal{K}\zeta,\mathcal{K}\zeta,\mathbf{c})}-1\right)\\ +k_3\left(\frac{1}{\mathfrak{M}(\eta,\mathcal{K}\omega,\mathcal{K}\omega,\mathbf{c})}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\eta,\mathcal{K}\eta,\mathcal{K}\eta,\mathbf{c})}-1\right)\\ +k_5\left(\frac{1}{\mathfrak{M}(\omega,\mathcal{K}\omega,\mathcal{K}\omega,\mathbf{c})}-1\right)+k_6\left(\frac{1}{\mathfrak{M}(\omega,\mathcal{K}\eta,\mathcal{K}\eta,\mathbf{c})}-1\right)\end{array}\right\},$$

where $k_i\in[0,\infty],i=1,\ldots,6$ and $\sum_{i=1}^{6}k_i<1$. Then $\mathcal{K}$ has a unique fixed point.

Corollary 3.5. *Let $(\mathcal{Z},\mathfrak{M},*)$ be a complete $\mathfrak{M}$-FCM Space, where $\mathfrak{M}$ is triangular. If $\mathcal{K}:\mathcal{Z}\to\mathcal{Z}$ satisfies (3.1) with $\sum_{i=1}^{7}k_i<1$, then $\mathcal{K}$ has a unique fixed point.*

The following theorem gives a more generalized contractive condition which considers almost all forms of possible restrictions.

Theorem 3.6. *Let $(\mathcal{Z},\mathfrak{M},*)$ be a complete $\mathfrak{M}$-FCM Space, where $\mathfrak{M}$ is triangular. If $\mathcal{K}:\mathcal{Z}\to\mathcal{Z}$ is such that for all $\zeta,\eta,\omega\in\mathcal{Z},\mathbf{c}\in int(\mathscr{C})$,*

$$\left(\frac{1}{\mathfrak{M}(\mathcal{K}\zeta,\mathcal{K}\eta,\mathcal{K}\omega,\mathbf{c})}-1\right)\leq\left\{\begin{array}{l} k_1\left(\frac{1}{\mathfrak{M}(\zeta,\eta,\omega,\mathbf{c})}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta,\mathcal{K}\zeta,\omega,\mathbf{c})}-1\right)\\ +k_3\left(\frac{1}{\mathfrak{M}(\zeta,\zeta,\mathcal{K}\zeta,\mathbf{c})}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\zeta,\mathcal{K}\zeta,\mathcal{K}\zeta,\mathbf{c})}-1\right)\\ +k_5\left(\frac{1}{\mathfrak{M}(\eta,\mathcal{K}\eta,\omega,\mathbf{c})}-1\right)+k_6\left(\frac{1}{\mathfrak{M}(\eta,\mathcal{K}\omega,\omega,\mathbf{c})}-1\right)\\ +k_7\left(\frac{1}{\mathfrak{M}(\mathcal{K}\zeta,\mathcal{K}\eta,\omega,\mathbf{c})}-1\right)+k_8\left(\frac{1}{\mathfrak{M}(\mathcal{K}\zeta,\mathcal{K}\omega,\eta,\mathbf{c})}-1\right)\\ +k_9\left(\frac{1}{\mathfrak{M}(\eta,\eta,\mathcal{K}\eta,\mathbf{c})}-1\right)+k_{10}\left(\frac{1}{\mathfrak{M}(\omega,\omega,\mathcal{K}\omega,\mathbf{c})}-1\right)\\ +k_{11}\left(\frac{1}{\mathfrak{M}(\eta,\mathcal{K}\eta,\mathcal{K}\eta,\mathbf{c})}-1\right)+k_{12}\left(\frac{1}{\mathfrak{M}(\omega,\mathcal{K}\omega,\mathcal{K}\omega,\mathbf{c})}-1\right)\\ +k_{13}\left(\frac{1}{\mathfrak{M}(\omega,\mathcal{K}\eta,\mathcal{K}\eta,\mathbf{c})}-1\right)+k_{14}\left(\frac{1}{\mathfrak{M}(\eta,\mathcal{K}\omega,\mathcal{K}\omega,\mathbf{c})}-1\right)\\ +k_{15}\left(\frac{1}{\mathfrak{M}(\mathcal{K}\eta,\mathcal{K}\omega,\zeta,\mathbf{c})}-1\right)+k_{16}\left(\frac{1}{\mathfrak{M}(\zeta,\mathcal{K}\eta,\omega,\mathbf{c})}-1\right)\end{array}\right\}, \tag{3.8}$$

where $k_i\in[0,\infty],i=1,\ldots,16$ and $k_1+\cdots+k_{14}+2(k_{15}+k_{16})<1$. Then $\mathcal{K}$ has a unique fixed point.

Proof. Let $\zeta_0\in\mathcal{Z}$ be arbitrary. Generate a sequence $\{\zeta_n\}$ with $\zeta_n=\mathcal{K}\zeta_{n-1}$ for $n\in\mathbb{N}$. If there exists a non-negative integer m such that $\zeta_{m+1}=\zeta_m$, then $\mathcal{K}\zeta_m=\zeta_m$ and ζ_m becomes a fixed point of $\mathcal{K}$.
Suppose $\zeta_n\neq\zeta_{n-1}$ for any $n\in\mathbb{N}$.
As $\mathfrak{M}$ is triangular,

$$\left(\frac{1}{\mathfrak{M}(\zeta_{n+1},\zeta_{n+1},\zeta_{n-1},\mathbf{c})}-1\right)\leq\left(\frac{1}{\mathfrak{M}(\zeta_{n-1},\zeta_{n-1},\zeta_n,\mathbf{c})}-1\right)+\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},\mathbf{c})}-1\right), \tag{3.9}$$

$$\left(\frac{1}{\mathfrak{M}(\zeta_{n-1},\zeta_n,\zeta_{n+1},\mathbf{c})}-1\right)\leq\left(\frac{1}{\mathfrak{M}(\zeta_{n-1},\zeta_n,\zeta_n,\mathbf{c})}-1\right)+\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},\mathbf{c})}-1\right). \tag{3.10}$$

Using (3.8) as in Theorem(3.1), together with (3.9) to (3.10), we arrive at

$$\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},\mathbf{c})}-1\right)\leq\frac{k_1+\cdots+k_4+k_{15}+k_{16}}{1-(k_5+\cdots+k_{16})}\left(\frac{1}{\mathfrak{M}(\zeta_{n-1},\zeta_n,\zeta_n,\mathbf{c})}-1\right).$$

Putting $k=\frac{k_1+\cdots+k_4+k_{15}+k_{16}}{1-(k_5+\cdots+k_{16})}$, the above inequality becomes

$$\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},\mathbf{c})}-1\right)\leq k\left(\frac{1}{\mathfrak{M}(\zeta_{n-1},\zeta_n,\zeta_n,\mathbf{c})}-1\right). \tag{3.11}$$

And, this makes the sequence $\{\zeta_n\}$ $\mathfrak{M}$-fuzzy cone contractive. Hence, by Lemma 2.16, $\{\zeta_n\}$ is Cauchy in $\mathcal{Z}$. As $\mathcal{Z}$ is complete, there exists $\zeta'\in\mathcal{Z}$ such that

$$\lim_{n\to\infty}\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta',\zeta',\mathbf{c})}-1\right)=0. \tag{3.12}$$

By repeated application of (3.11), we obtain that

$$\begin{aligned} \left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},\mathbf{c})}-1\right) &\leq k^n\left(\frac{1}{\mathfrak{M}(\zeta_0,\zeta_1,\zeta_1,\mathbf{c})}-1\right), \\ \lim_{n\to\infty}\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},\mathbf{c})}-1\right) &= 0. \end{aligned} \tag{3.13}$$

From (3.8),

$$\left(\frac{1}{\mathfrak{M}(\zeta_{n+1},\mathcal{K}\zeta',\mathcal{K}\zeta',\mathbf{c})}-1\right)=\left(\frac{1}{\mathfrak{M}(\mathcal{K}\zeta_n,\mathcal{K}\zeta',\zeta',\mathbf{c})}-1\right)\leq k'\left(\frac{1}{\mathfrak{M}(\mathcal{K}\zeta',\mathcal{K}\zeta',\zeta',\mathbf{c})}-1\right),$$

where $k'=k_5+\cdots+k_{16}$.
Hence,

$$\lim_{n\to\infty}\sup\left(\frac{1}{\mathfrak{M}(\zeta_{n+1},\mathcal{K}\zeta',\mathcal{K}\zeta',\mathbf{c})}-1\right)\leq k'\left(\frac{1}{\mathfrak{M}(\mathcal{K}\zeta',\mathcal{K}\zeta',\zeta',\mathbf{c})}-1\right).$$

As $\mathfrak{M}$ is triangular,

$$\left(\frac{1}{\mathfrak{M}(\zeta',\mathcal{K}\zeta',\mathcal{K}\zeta',\mathbf{c})}-1\right)\leq\left(\frac{1}{\mathfrak{M}(\mathcal{K}\zeta',\mathcal{K}\zeta',\zeta_{n+1},\mathbf{c})}-1\right)+\left(\frac{1}{\mathfrak{M}(\zeta_{n+1},\zeta',\zeta',\mathbf{c})}-1\right). \tag{3.14}$$

From (3.13) and (3.14), we can bring that

$$\left(\frac{1}{\mathfrak{M}(\zeta',\mathcal{K}\zeta',\mathcal{K}\zeta',\mathbf{c})}-1\right)\leq k'\left(\frac{1}{\mathfrak{M}(\zeta',\mathcal{K}\zeta',\mathcal{K}\zeta',\mathbf{c})}-1\right).$$

This gives that $\left(\frac{1}{\mathfrak{M}(\zeta',\mathcal{K}\zeta',\mathcal{K}\zeta',\mathbf{c})}-1\right)=0$, as $k'<1$, and, hence we have $\mathcal{K}\zeta'=\zeta'$. Thus, we can conclude that ζ' is a fixed point of $\mathcal{K}$.
Suppose $\mathcal{K}\zeta''=\zeta''$. Then from (3.8) and by Lemma 2.15,

$$\left(\frac{1}{\mathfrak{M}(\zeta',\zeta'',\zeta'',\mathbf{c})}-1\right)\leq k''\left(\frac{1}{\mathfrak{M}(\zeta',\zeta'',\zeta'',\mathbf{c})}-1\right),$$

where $k''=k_1+k_2+k_7+k_8+k_{15}+k_{16}$.
This implies $\left(\frac{1}{\mathfrak{M}(\zeta',\zeta'',\zeta'',\mathbf{c})}-1\right)=0$, as $k''<1$, and hence we have $\zeta'=\zeta''$.
Thus, we can conclude that $\mathcal{K}$ has a unique fixed point. □

Corollary 3.7. *Let $(\mathcal{Z},\mathfrak{M},*)$ be a complete $\mathfrak{M}$-FCM Space, where $\mathfrak{M}$ is triangular. If $\mathcal{K}:\mathcal{Z}\to\mathcal{Z}$ is such that for all $\zeta,\eta,\omega\in\mathcal{Z}$, $\mathbf{c}\in int(c)$,*

$$\left(\frac{1}{\mathfrak{M}(\mathcal{K}\zeta,\mathcal{K}\eta,\mathcal{K}\omega,\mathbf{c})}-1\right)\leq\left\{\begin{array}{l} k_1\left(\frac{1}{\mathfrak{M}(\zeta,\eta,\omega,\mathbf{c})}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta,\mathcal{K}\zeta,\omega,\mathbf{c})}-1\right)+ \\ k_3\left(\frac{1}{\mathfrak{M}(\mathcal{K}\zeta,\mathcal{K}\eta,\omega,\mathbf{c})}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\eta,\mathcal{K}\omega,\mathcal{K}\omega,\mathbf{c})}-1\right)+ \\ k_5\left(\frac{1}{\mathfrak{M}(\mathcal{K}\eta,\mathcal{K}\omega,\zeta,\mathbf{c})}-1\right)+k_6\left(\frac{1}{\mathfrak{M}(\zeta,\mathcal{K}\eta,\omega,\mathbf{c})}-1\right) \end{array}\right\},$$

where $k_i\in[0,\infty],i=1,\ldots,6$ and $k_1+k_2+k_3+k_4+2(k_5+k_6)<1$. Then $\mathcal{K}$ has a unique fixed point.

Conclusion:

We constructed some fixed point theorems as an extension of Banach contraction theorem by giving a general form of contractive conditions for self-mappings and proved the existence of fixed points for these self-mappings.

Conflict of Interest

The authors have no conflicts of interest to declare.

References

[1] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales,* Fund. Math., **3** (1922), 133-181. DOI

[2] Z. Deng, *Fuzzy pseudo-metric spaces,* J. Math. Anal. Appl., **86**(1) (1982), 74-95. DOI

[3] P. Diamond and P. Kloeden, *Metric spaces of fuzzy sets*, Fuzzy Sets and Systems, 35(2)(1990), 241-249. DOI

[4] M. A. Erceg, *Metric spaces in fuzzy set theory*, J. Math. Anal. Appl., **69**(1) (1979), 205-230. DOI

[5] A. George and P. Veeramani, *On some results in fuzzy metric spaces,* Fuzzy Sets and Systems, **64** (1994), 395-399. DOI

[6] A. George and P. Veeramani, *On some results of analysis for fuzzy metric spaces,* Fuzzy Sets and Systems, **90**(3)(1997), 365-368. DOI

[7] V. Gregori and A. Sapena, *On fixed point theorems in fuzzy metric spaces,* Fuzzy sets and Systems, **125** (2002), 245-252. DOI

[8] V. Gupta, S. S. Chauhan and I. K. Sandhu, *Banach Contraction Theorem on Extended Fuzzy Cone b-metric Space,* Thai J. Math., 20(1)(2022), 177-194. URL

[9] V. Gupta, A. Kaushik and M. Verma, *Some new fixed point results on $V-\psi$-fuzzy contraction endowed with graph,* Journal of Intelligent & Fuzzy Systems, **36**(6) (2019), 6549-6554. DOI

[10] G. E. Hardy and T. D. Rogers, *A generalization of a fixed point theorem of Reich,* Canad. Math. Bull., **16**(2) (1973), 201-206. DOI

[11] L.-G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., **332**(2) (2007), 1468-1476. DOI

[12] O. Kaleva and S. Seikkala, *On fuzzy metric spaces,* Fuzzy Sets and Systems, **12**(3) (1984), 215-229. DOI

[13] I. Kramosil and J. Michálek, *Fuzzy metric and statistical metric spaces,* Kybernetica, **11**(5) (1975), 326-334. URL

[14] T. Öner, M. B. Kandemir and B. Tanay, *Fuzzy cone metric spaces*, J. Nonlinear Sci. Appl., **5** (2015), 610-616. URL

[15] U. R. Saif and L. Hong-Xu, *Fixed point theorems in fuzzy cone metric spaces,* J. Nonlinear Sci. Appl., **10** (2017), 5763-5769. DOI

[16] S. Sedghi and N. Shobe, *Fixed point theorem in $\mathfrak{M}$-fuzzy metric spaces with property (E)*, Advances in Fuzzy Mathematics, **1**(1) (2006), 55-65. URL

[17] T. Turkoglu and M. Abuloha, *Cone metric spaces and fixed point theorems in diametrically contractive mappings,* Acta Math. Sin. (Engl. Ser.), **26** (2010), 489-496. DOI

[18] C. S. Wong, *Generalized contractions and fixed point theorems*, Proc. Amer. Math. Soc., **42**(2) (1974), 409-417. DOI

[19] L. A. Zadeh, *Fuzzy sets*, Information and Control, 8 (1965), 338-353. DOI