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The expressions and behavior of solutions for nonlinear systems of rational difference equations

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Abstract. In this paper, we investigate the form of the solutions of the following systems of difference equations of second order

$$\begin{aligned} x_{n+1} &= \frac{x_n y_{n-1}}{x_n + y_n}, \quad y_{n+1} = \frac{x_{n-1} y_n}{x_n + y_n}, \\ x_{n+1} &= \frac{x_n y_{n-1}}{x_n - y_n}, \quad y_{n+1} = \frac{x_{n-1} y_n}{x_n - y_n}, \quad n = 0, 1, ..., \end{aligned}$$

where the initial conditions $x_{-1},\ x_0,\ y_{-1}$ and y_0 are arbitrary nonzero real numbers.

Keywords: Periodic solution, boundedness, systems of difference equations.
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1 Introduction

In this paper, we deal with the behavior of the solution of the following system of difference equation

$$\begin{aligned} x_{n+1} &= \frac{x_n y_{n-1}}{x_n + y_n}, \quad y_{n+1} = \frac{x_{n-1} y_n}{x_n + y_n}, \\ x_{n+1} &= \frac{x_n y_{n-1}}{x_n - y_n}, \quad y_{n+1} = \frac{x_{n-1} y_n}{x_n - y_n}, \quad n = 0, 1, ..., \end{aligned}$$

where the initial conditions $x_{-1},\ x_0,\ y_{-1}$ and y_0 are arbitrary nonzero real numbers.

The hypothesis of difference equations involves a focal position in applicable analysis. There is no uncertainty that the hypothesis of difference equations will keep on playing a vital part in science overall.

Nonlinear difference equations of order greater than one are of principal significance in applications. Such equations likewise normally seem like discrete analogs and numerical arrangements of differential and defer differential equations, showing several assorted wonders in science, biology, physics, physiology, engineering, and economics.

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As of late, there has been incredible enthusiasm for examining difference equation systems. One reason for this is the need for a few strategies to investigate equations emerging in mathematical models portraying genuine. For instance, there are many papers related to the system of difference equations.

Clark and Kulenović [4] have been investigated the positive solutions behavior of the following system

$$x_{n+1} = \frac{x_n}{a + cy_n}, \quad y_{n+1} = \frac{y_n}{b + dx_n}.$$

The authors in [20] have obtained the solutions form of the following system of difference equations

$$x_{n+1} = \frac{Ax_n + y_n}{x_{n-p}}, \quad y_{n+1} = \frac{A + x_n}{y_{n-q}}.$$

Touafek and Elsayed [25] investigated the periodic nature and gave the form of the solutions to the following systems of rational difference equations

$$x_{n+1} = \frac{y_n}{x_{n-1}(\pm 1 \pm y_n)}, \quad y_{n+1} = \frac{x_n}{y_{n-1}(\pm 1 \pm x_n)}.$$

Din et al. [5] dealt with the behavior of the solutions of the following fourth-order system of rational difference equations of the form

$$x_{n+1} = \frac{\alpha x_{n-3}}{\beta + \gamma y_n y_{n-1} y_{n-2} y_{n-3}}, \quad y_{n+1} = \frac{\alpha_1 x_{n-3}}{\beta_1 + \gamma_1 x_n x_{n-1} x_{n-2} x_{n-3}}.$$

The persistence and the asymptotic behavior of positive solutions of the system of two difference equations of exponential form

$$x_{n+1} = a + bx_{n-1}e^{-y_n}, \quad y_{n+1} = c + dy_{n-1}e^{-x_n},$$

have been studied by Papaschinopoulos et al. [21].

Yalçınkaya [27] obtained the sufficient conditions for the global asymptotic stability of the system of two nonlinear difference equations

$$x_{n+1} = \frac{x_n + y_{n-1}}{x_n y_{n-1} - 1}, \quad y_{n+1} = \frac{y_n + x_{n-1}}{y_n x_{n-1} - 1}.$$

Elsayed [9] investigated the expressions of solutions and the periodic nature of the following systems of rational difference equations

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm y_n x_{n-2} x_{n-3}}, \quad y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm x_n y_{n-2} y_{n-3}}.$$

Yang et al. [29] studied the global behavior of the system of the two nonlinear difference equations

$$x_{n+1} = \frac{Ax_n}{1 + y_n^p}, \quad y_{n+1} = \frac{By_n}{1 + x_n^p}.$$

Camouzis and Papaschinopoulos [2] studied the dynamics of a system of the rational third-order difference equation

$$x_{n+1} = 1 + \frac{x_n}{y_{n-m}}, \quad y_{n+1} = 1 + \frac{y_n}{x_{n-m}}.$$

The expression of solutions to the following system of nonlinear difference equations

$$x_{n+1}=\frac{f(z_n)}{y_{n-1}},\quad y_{n+1}=\frac{f(x_n)}{z_{n-1}},\quad z_{n+1}=\frac{f(y_n)}{x_{n-1}},$$

has been studied by Williams [26].

Definition (*Periodicity*).

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p}=x_n$ for all $n\geq -k$.

2 Form of the solutions

Consider the system

$$x_{n+1}=\frac{x_n y_{n-1}}{x_n+y_n},\quad y_{n+1}=\frac{x_{n-1}y_n}{x_n+y_n},\qquad n=0,1,...,\tag{2.1}$$

with the initial values are arbitrary nonzero real numbers with $x_0\neq -y_0$, $x_{-1}\neq -y_{-1}$ and $x_0y_{-1}\neq -x_{-1}y_0$.

In the following result, we realize the form of the solutions of System (2.1).

Theorem 2.1. *Let $\{x_n,y_n\}_{n=-1}^{\infty}$ be the solutions of System (2.1). Then for $n=0,1,2,...$*

$$\begin{aligned}
x_{6n} &= \tfrac{a^{n+1}b^nc^nd^n}{(a+c)^n(b+d)^n(ad+bc)^n}, & y_{6n} &= \tfrac{a^nb^nc^{n+1}d^n}{(a+c)^n(b+d)^n(ad+bc)^n},\\
x_{6n+1} &= \tfrac{a^{n+1}b^nc^nd^{n+1}}{(a+c)^{n+1}(b+d)^n(ad+bc)^n}, & y_{6n+1} &= \tfrac{a^nb^{n+1}c^{n+1}d^n}{(a+c)^{n+1}(b+d)^n(ad+bc)^n},\\
x_{6n+2} &= \tfrac{a^{n+1}b^nc^{n+1}d^{n+1}}{(a+c)^n(b+d)^n(ad+bc)^{n+1}}, & y_{6n+2} &= \tfrac{a^{n+1}b^{n+1}c^{n+1}d^n}{(a+c)^n(b+d)^n(ad+bc)^{n+1}},\\
x_{6n+3} &= \tfrac{a^nb^{n+1}c^{n+1}d^{n+1}}{(a+c)^{n+1}(b+d)^{n+1}(ad+bc)^n}, & y_{6n+3} &= \tfrac{a^nb^{n+1}c^{n+1}d^{n+1}}{(a+c)^{n+1}(b+d)^{n+1}(ad+bc)^n},
\end{aligned}$$

$$\begin{aligned}
x_{6n+4} &= \tfrac{a^{n+1}b^{n+1}c^{n+2}d^n}{(a+c)^{n+1}(b+d)^n(ad+bc)^{n+1}}, & y_{6n+4} &= \tfrac{a^{n+2}b^nc^{n+1}d^{n+1}}{(a+c)^{n+1}(b+d)^n(ad+bc)^{n+1}},\\
x_{6n+5} &= \tfrac{a^{n+1}b^{n+2}c^{n+1}d^{n+1}}{(a+c)^{n+1}(b+d)^{n+1}(ad+bc)^{n+1}}, & y_{6n+5} &= \tfrac{a^{n+1}b^{n+1}c^{n+1}d^{n+2}}{(a+c)^{n+1}(b+d)^{n+1}(ad+bc)^{n+1}},
\end{aligned}$$

where $x_0=a,\ x_{-1}=b,\ y_0=c,\ y_{-1}=d.$

Proof. For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$. That is;

$$\begin{aligned}
x_{6n-6} &= \tfrac{a^nb^{n-1}c^{n-1}d^{n-1}}{(a+c)^{n-1}(b+d)^{n-1}(ad+bc)^{n-1}}, & y_{6n-6} &= \tfrac{a^{n-1}b^{n-1}c^nd^{n-1}}{(a+c)^{n-1}(b+d)^{n-1}(ad+bc)^{n-1}},\\
x_{6n-5} &= \tfrac{a^nb^{n-1}c^{n-1}d^n}{(a+c)^n(b+d)^{n-1}(ad+bc)^{n-1}}, & y_{6n-5} &= \tfrac{a^{n-1}b^nc^nd^{n-1}}{(a+c)^n(b+d)^{n-1}(ad+bc)^{n-1}},\\
x_{6n-4} &= \tfrac{a^nb^{n-1}c^nd^n}{(a+c)^{n-1}(b+d)^{n-1}(ad+bc)^n}, & y_{6n-4} &= \tfrac{a^nb^nc^nd^{n-1}}{(a+c)^{n-1}(b+d)^{n-1}(ad+bc)^n},\\
x_{6n-3} &= \tfrac{a^{n-1}b^nc^nd^n}{(a+c)^n(b+d)^n(ad+bc)^{n-1}}, & y_{6n-3} &= \tfrac{a^{n-1}b^nc^nd^n}{(a+c)^n(b+d)^n(ad+bc)^{n-1}},\\
x_{6n-2} &= \tfrac{a^nb^nc^{n+1}d^{n-1}}{(a+c)^n(b+d)^{n-1}(ad+bc)^n}, & y_{6n-2} &= \tfrac{a^{n+1}b^{n-1}c^nd^n}{(a+c)^n(b+d)^{n-1}(ad+bc)^n},\\
x_{6n-1} &= \tfrac{a^nb^{n+1}c^nd^n}{(a+c)^n(b+d)^n(ad+bc)^n}, & y_{6n-1} &= \tfrac{a^nb^nc^nd^{n+1}}{(a+c)^n(b+d)^n(ad+bc)^n}.
\end{aligned}$$

Now, it follows from System (2.1) that

$$\begin{aligned}
x_{6n} &= \frac{x_{6n-1}y_{6n-2}}{x_{6n-1}+y_{6n-1}} \\
&= \frac{\left(\frac{a^n b^{n+1} c^n d^n}{(a+c)^n (b+d)^n (ad+bc)^n}\right)\left(\frac{a^{n+1} b^{n-1} c^n d^n}{(a+c)^n (b+d)^{n-1} (ad+bc)^n}\right)}{\left(\frac{a^n b^{n+1} c^n d^n}{(a+c)^n (b+d)^n (ad+bc)^n}\right)+\left(\frac{a^n b^n c^n d^{n+1}}{(a+c)^n (b+d)^n (ad+bc)^n}\right)} \\
&= \frac{\left(\frac{a^{2n+1} b^{2n} c^{2n} d^{2n}}{(a+c)^{2n} (b+d)^{2n-1} (ad+bc)^{2n}}\right)(a+c)^n (b+d)^n (ad+bc)^n}{a^n b^n c^n d^n (b+d)} \\
&= \frac{a^{2n+1} b^{2n} c^{2n} d^{2n}}{(a+c)^n (b+d)^{n-1} (ad+bc)^n a^n b^n c^n d^n (b+d)} \\
&= \frac{a^{n+1} b^n c^n d^n}{(a+c)^n (b+d)^n (ad+bc)^n},
\end{aligned}$$

$$\begin{aligned}
y_{6n} &= \frac{y_{6n-1}x_{6n-2}}{x_{6n-1}+y_{6n-1}} \\
&= \frac{\left(\frac{a^n b^n c^n d^{n+1}}{(a+c)^n (b+d)^n (ad+bc)^n}\right)\left(\frac{a^n b^n c^{n+1} d^{n-1}}{(a+c)^n (b+d)^{n-1} (ad+bc)^n}\right)}{\left(\frac{a^n b^{n+1} c^n d^n}{(a+c)^n (b+d)^n (ad+bc)^n}\right)+\left(\frac{a^n b^n c^n d^{n+1}}{(a+c)^n (b+d)^n (ad+bc)^n}\right)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\left(\frac{a^{2n} b2^n c^{2n+1} d^{2n}}{(a+c)^{2n} (b+d)^{2n-1} (ad+bc)^{2n}}\right)(a+c)^n (b+d)^n (ad+bc)^n}{a^n b^n c^n d^n (b+d)} \\
&= \frac{a^{2n} b^{2n} c^{2n+1} d^{2n}}{a^n b^n c^n d^n (a+c)^n (b+d)^{n-1} (ad+bc)^n} \\
&= \frac{a^n b^n c^{n+1} d^n}{(a+c)^n (b+d)^n (ad+bc)^n}.
\end{aligned}$$

Similarly

$$\begin{aligned}
x_{6n+1} &= \frac{x_{6n}y_{6n-1}}{x_{6n}+y_{6n}} \\
&= \frac{\left(\frac{a^{n+1} b^n c^n d^n}{(a+c)^n (b+d)^n (ad+bc)^n}\right)\left(\frac{a^n b^n c^n d^{n+1}}{(a+c)^n (b+d)^n (ad+bc)^n}\right)}{\left(\frac{a^{n+1} b^n c^n d^n}{(a+c)^n (b+d)^n (ad+bc)^n}\right)+\left(\frac{a^n b^n c^{n+1} d^n}{(a+c)^n (b+d)^n (ad+bc)^n}\right)} \\
&= \frac{\left(\frac{a^{2n+1} b^{2n} c^{2n} d^{2n+1}}{(a+c)^{2n} (b+d)^{2n} (ad+bc)^{2n}}\right)(a+c)^n (b+d)^n (ad+bc)^n}{a^n b^n c^n d^n (a+c)} \\
&= \frac{a^{2n+1} b^{2n} c^{2n} d^{2n+1}}{a^n b^n c^n d^n (a+c)^{n+1} (b+d)^n (ad+bc)^n} \\
&= \frac{a^{n+1} b^n c^n d^{n+1}}{(a+c)^{n+1} (b+d)^n (ad+bc)^n},
\end{aligned}$$

$$\begin{aligned}
y_{6n+1} &= \frac{y_{6n}x_{6n-1}}{x_{6n}+y_{6n}} \\
&= \frac{\left(\frac{a^n b^n c^{n+1} d^n}{(a+c)^n(b+d)^n(ad+bc)^n}\right)\left(\frac{a^n b^{n+1} c^n d^n}{(a+c)^n(b+d)^n(ad+bc)^n}\right)}{\left(\frac{a^{n+1} b^n c^n d^n}{(a+c)^n(b+d)^n(ad+bc)^n}\right)+\left(\frac{a^n b^n c^{n+1} d^n}{(a+c)^n(b+d)^n(ad+bc)^n}\right)} \\
&= \frac{\left(\frac{a^{2n} b^{2n+1} c^{2n+1} d^{2n}}{(a+c)^{2n}(b+d)^{2n}(ad+bc)^{2n}}\right)(a+c)^n(b+d)^n(ad+bc)^n}{a^n b^n c^n d^n (a+c)} \\
&= \frac{a^{2n} b^{2n+1} c^{2n+1} d^{2n}}{a^n b^n c^n d^n (a+c)^{n+1}(b+d)^n(ad+bc)^n} \\
&= \frac{a^n b^{n+1} c^{n+1} d^n}{(a+c)^{n+1}(b+d)^n(ad+bc)^n}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
x_{6n+2} &= \frac{x_{6n+1}y_{6n}}{x_{6n+1}+y_{6n+1}} \\
&= \frac{\left(\frac{a^{n+1} b^n c^n d^{n+1}}{(a+c)^{n+1}(b+d)^n(ad+bc)^n}\right)\left(\frac{a^n b^n c^{n+1} d^n}{(a+c)^n(b+d)^n(ad+bc)^n}\right)}{\left(\frac{a^{n+1} b^n c^n d^{n+1}}{(a+c)^{n+1}(b+d)^n(ad+bc)^n}\right)+\left(\frac{a^n b^{n+1} c^{n+1} d^n}{(a+c)^{n+1}(b+d)^n(ad+bc)^n}\right)} \\
&= \frac{\left(\frac{a^{2n+1} b^{2n} c^{2n+1} d^{2n+1}}{(a+c)^{2n+1}(b+d)^{2n}(ad+bc)^{2n}}\right)(a+c)^{n+1}(b+d)^n(ad+bc)^n}{a^n b^n c^n d^n (ad+bc)} \\
&= \frac{a^{2n+1} b^{2n} c^{2n+1} d^{2n+1}}{a^n b^n c^n d^n (a+c)^n(b+d)^n(ad+bc)^{n+1}} \\
&= \frac{a^{n+1} b^n c^{n+1} d^{n+1}}{(a+c)^n(b+d)^n(ad+bc)^{n+1}},
\end{aligned}$$

$$\begin{aligned}
y_{6n+2} &= \frac{y_{6n+1}x_{6n}}{x_{6n+1}+y_{6n+1}} \\
&= \frac{\left(\frac{a^n b^{n+1} c^{n+1} d^n}{(a+c)^{n+1}(b+d)^n(ad+bc)^n}\right)\left(\frac{a^{n+1} b^n c^n d^n}{(a+c)^n(b+d)^n(ad+bc)^n}\right)}{\left(\frac{a^{n+1} b^n c^n d^{n+1}}{(a+c)^{n+1}(b+d)^n(ad+bc)^n}\right)+\left(\frac{a^n b^{n+1} c^{n+1} d^n}{(a+c)^{n+1}(b+d)^n(ad+bc)^n}\right)} \\
&= \frac{\left(\frac{a^{2n+1} b^{2n+1} c^{2n+1} d^{2n}}{(a+c)^{2n+1}(b+d)^{2n}(ad+bc)^{2n}}\right)(a+c)^{n+1}(b+d)^n(ad+bc)^n}{a^n b^n c^n d^n (ad+bc)} \\
&= \frac{a^{2n+1} b^{2n+1} c^{2n+1} d^{2n}}{a^n b^n c^n d^n (a+c)^n(b+d)^n(ad+bc)^{n+1}} \\
&= \frac{a^{n+1} b^{n+1} c^{n+1} d^n}{(a+c)^n(b+d)^n(ad+bc)^{n+1}}.
\end{aligned}$$

We have

$$\begin{aligned}
x_{6n+3} &= \frac{x_{6n+2}y_{6n+1}}{x_{6n+2}+y_{6n+2}} \\
&= \frac{\left(\frac{a^{n+1}b^{n}c^{n+1}d^{n+1}}{(a+c)^{n}(b+d)^{n}(ad+bc)^{n+1}}\right)\left(\frac{a^{n}b^{n+1}c^{n+1}d^{n}}{(a+c)^{n+1}(b+d)^{n}(ad+bc)^{n}}\right)}{\left(\frac{a^{n+1}b^{n}c^{n+1}d^{n+1}}{(a+c)^{n}(b+d)^{n}(ad+bc)^{n+1}}\right)+\left(\frac{a^{n+1}b^{n+1}c^{n+1}d^{n}}{(a+c)^{n}(b+d)^{n}(ad+bc)^{n+1}}\right)} \\
&= \frac{\left(\frac{a^{2n+1}b^{2n+1}c^{2n+2}d^{2n+1}}{(a+c)^{2n+1}(b+d)^{2n}(ad+bc)^{2n+1}}\right)(a+c)^{n}(b+d)^{n}(ad+bc)^{n+1}}{a^{n+1}b^{n}c^{n+1}d^{n}(b+d)} \\
&= \frac{a^{2n+1}b^{2n+1}c^{2n+2}d^{2n+1}}{a^{n+1}b^{n}c^{n+1}d^{n}(a+c)^{n+1}(b+d)^{n+1}(ad+bc)^{n}} \\
&= \frac{a^{n}b^{n+1}c^{n+1}d^{n+1}}{(a+c)^{n+1}(b+d)^{n+1}(ad+bc)^{n}}.
\end{aligned}$$

$$\begin{aligned}
y_{6n+3} &= \frac{y_{6n+2}x_{6n+1}}{x_{6n+2}+y_{6n+2}} \\
&= \frac{\left(\frac{a^{n+1}b^{n+1}c^{n+1}d^{n}}{(a+c)^{n}(b+d)^{n}(ad+bc)^{n+1}}\right)\left(\frac{a^{n+1}b^{n}c^{n}d^{n+1}}{(a+c)^{n+1}(b+d)^{n}(ad+bc)^{n}}\right)}{\left(\frac{a^{n+1}b^{n}c^{n+1}d^{n+1}}{(a+c)^{n}(b+d)^{n}(ad+bc)^{n+1}}\right)+\left(\frac{a^{n+1}b^{n+1}c^{n+1}d^{n}}{(a+c)^{n}(b+d)^{n}(ad+bc)^{n+1}}\right)} \\
&= \frac{\left(\frac{a^{2n+2}b^{2n+1}c^{2n+1}d^{2n+1}}{(a+c)^{2n+1}(b+d)^{2n}(ad+bc)^{2n+1}}\right)(a+c)^{n}(b+d)^{n}(ad+bc)^{n+1}}{a^{n+1}b^{n}c^{n+1}d^{n}(d+b)} \\
&= \frac{a^{2n+2}b^{2n+1}c^{2n+1}d^{2n+1}}{a^{n+1}b^{n}c^{n+1}d^{n}(a+c)^{n+1}(b+d)^{n+1}(ad+bc)^{n}} \\
&= \frac{a^{n+1}b^{n+1}c^{n}d^{n+1}}{(a+c)^{n+1}(b+d)^{n+1}(ad+bc)^{n}}.
\end{aligned}$$

Similarly

$$\begin{aligned}
x_{6n+4} &= \frac{x_{6n+3}y_{6n+2}}{x_{6n+3}+y_{6n+3}} \\
&= \frac{\left(\frac{a^{n}b^{n+1}c^{n}d^{n+1}}{(a+c)^{n+1}(b+d)^{n+1}(ad+bc)^{n}}\right)\left(\frac{a^{n+1}b^{n+1}c^{n+1}d^{n}}{(a+c)^{n}(b+d)^{n}(ad+bc)^{n+1}}\right)}{\left(\frac{a^{n}b^{n+1}c^{n+1}d^{n+1}}{(a+c)^{n+1}(b+d)^{n+1}(ad+bc)^{n}}\right)+\left(\frac{a^{n+1}b^{n+1}c^{n}d^{n+1}}{(a+c)^{n+1}(b+d)^{n+1}(ad+bc)^{n}}\right)} \\
&= \frac{\left(\frac{a^{2n+1}b^{2n+2}c^{2n+1}d^{2n+1}}{(a+c)^{2n+1}(b+d)^{2n+1}(ad+bc)^{2n+1}}\right)(a+c)^{n+1}(b+d)^{n+1}(ad+bc)^{n}}{a^{n}b^{n+1}c^{n}d^{n+1}(c+a)} \\
&= \frac{a^{2n+1}b^{2n+2}c^{2n+2}d^{2n+1}}{a^{n}b^{n+1}c^{n}d^{n+1}(a+c)^{n+1}(b+d)^{n}(ad+bc)^{n+1}} \\
&= \frac{a^{n+1}b^{n+1}c^{n+2}d^{n}}{(a+c)^{n+1}(b+d)^{n}(ad+bc)^{n+1}},
\end{aligned}$$

$$
\begin{aligned}
y_{6n+4} &= \frac{y_{6n+3}x_{6n+2}}{x_{6n+3}+y_{6n+3}} \\
&= \frac{\left(\frac{a^{n+1}b^{n+1}c^{n}d^{n+1}}{(a+c)^{n+1}(b+d)^{n+1}(ad+bc)^{n}}\right)\left(\frac{a^{n+1}b^{n}c^{n+1}d^{n+1}}{(a+c)^{n}(b+d)^{n}(ad+bc)^{n+1}}\right)}{\left(\frac{a^{n}b^{n+1}c^{n+1}d^{n+1}}{(a+c)^{n+1}(b+d)^{n+1}(ad+bc)^{n}}\right)+\left(\frac{a^{n+1}b^{n+1}c^{n}d^{n+1}}{(a+c)^{n+1}(b+d)^{n+1}(ad+bc)^{n}}\right)} \\
&= \frac{\left(\frac{a^{2n+2}b^{2n+1}c^{2n+1}d^{2n+2}}{(a+c)^{2n+1}(b+d)^{2n+1}(ad+bc)^{2n+1}}\right)(a+c)^{n+1}(b+d)^{n+1}(ad+bc)^{n}}{a^{n}b^{n+1}c^{n}d^{n+1}(c+a)} \\
&= \frac{a^{2n+2}b^{2n+1}c^{2n+1}d^{2n+2}}{a^{n}b^{n+1}c^{n}d^{n+1}(a+c)^{n+1}(b+d)^{n}(ad+bc)^{n+1}} \\
&= \frac{a^{n+2}b^{n}c^{n+1}d^{n+1}}{(a+c)^{n+1}(b+d)^{n}(ad+bc)^{n+1}}.
\end{aligned}
$$

We have,

$$
\begin{aligned}
x_{6n+5} &= \frac{x_{6n+4}y_{6n+3}}{x_{6n+4}+y_{6n+4}} \\
&= \frac{\left(\frac{a^{n+1}b^{n+1}c^{n+2}d^{n}}{(a+c)^{n+1}(b+d)^{n}(ad+bc)^{n+1}}\right)\left(\frac{a^{n+1}b^{n+1}c^{n}d^{n+1}}{(a+c)^{n+1}(b+d)^{n+1}(ad+bc)^{n}}\right)}{\left(\frac{a^{n+1}b^{n+1}c^{n+2}d^{n}}{(a+c)^{n+1}(b+d)^{n}(ad+bc)^{n+1}}\right)+\left(\frac{a^{n+2}b^{n}c^{n+1}d^{n+1}}{(a+c)^{n+1}(b+d)^{n}(ad+bc)^{n+1}}\right)} \\
&= \frac{\left(\frac{a^{2n+2}b^{2n+2}c^{2n+2}d^{2n+1}}{(a+c)^{2n+2}(b+d)^{2n+1}(ad+bc)^{2n+1}}\right)(a+c)^{n+1}(b+d)^{n}(ad+bc)^{n+1}}{a^{n+1}b^{n}c^{n+1}d^{n}(bc+ad)} \\
&= \frac{a^{2n+2}b^{2n+2}c^{2n+2}d^{2n+1}}{a^{n+1}b^{n}c^{n+1}d^{n}(a+c)^{n+1}(b+d)^{n+1}(ad+bc)^{n+1}} \\
&= \frac{a^{n}+1b^{n+2}c^{n+1}d^{n+1}}{(a+c)^{n+1}(b+d)^{n+1}(ad+bc)^{n+1}},
\end{aligned}
$$

$$
\begin{aligned}
y_{6n+5} &= \frac{y_{6n+4}x_{6n+3}}{x_{6n+4}+y_{6n+4}} \\
&= \frac{\left(\frac{a^{n+2}b^{n}c^{n+1}d^{n+1}}{(a+c)^{n+1}(b+d)^{n}(ad+bc)^{n+1}}\right)\left(\frac{a^{n}b^{n+1}c^{n+1}d^{n+1}}{(a+c)^{n+1}(b+d)^{n+1}(ad+bc)^{n}}\right)}{\left(\frac{a^{n+1}b^{n+1}c^{n+2}d^{n}}{(a+c)^{n+1}(b+d)^{n}(ad+bc)^{n+1}}\right)+\left(\frac{a^{n+2}b^{n}c^{n+1}d^{n+1}}{(a+c)^{n+1}(b+d)^{n}(ad+bc)^{n+1}}\right)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\left(\frac{a^{2n+2}b^{2n+1}c^{2n+2}d^{2n+2}}{(a+c)^{2n+2}(b+d)^{2n+1}(ad+bc)^{2n+1}}\right)(a+c)^{n+1}(b+d)^{n}(ad+bc)^{n+1}}{a^{n+1}b^{n}c^{n+1}d^{n}(bc+ad)} \\
&= \frac{a^{2n+2}b^{2n+1}c^{2n+2}d^{2n+2}}{a^{n+1}b^{n}c^{n+1}d^{n}(a+c)^{n+1}(b+d)^{n+1}(ad+bc)^{n+1}} \\
&= \frac{a^{n+1}b^{n+1}c^{n+1}d^{n+2}}{(a+c)^{n+1}(b+d)^{n+1}(ad+bc)^{n+1}}.
\end{aligned}
$$

The proof is complete. □

The following two theorems are devoted to the existence of prime periodic solutions of period twelve.

Theorem 2.2. *Suppose that $a = -2c$, $b = d$. Then the System (2.1) has a periodic positive solution of period twelve, taking the following form*

$$\begin{aligned}\{x_n\} &= \left\{a, 2d, -a, \frac{-d}{2}, a, -b, -a, -2b, a, \frac{b}{2}, -a, b, a, 2d, ...\right\},\\ \{y_n\} &= \left\{c, -b, -a, \frac{-d}{2}, -2a, -d, -c, b, a, \frac{b}{2}, 2a, b, c, -b, ...\right\}.\end{aligned}$$

Proof. Assume that $a = -2c$ and $b = d$, then we see the solution of System (2.1) as follows:

$$\begin{aligned}x_{12n} &= a, \ \ x_{12n+1} = 2d, \ \ x_{12n+2} = -a, \ \ x_{12n+3} = \frac{-d}{2}, \ \ x_{12n+4} = a, \ \ x_{12n+5} = -b,\\ x_{12n+6} &= -a, \ \ x_{12n+7} = -2b, \ \ x_{12n+8} = a, \ \ x_{12n+9} = \frac{b}{2}, \ \ x_{12n+10} = -a, \ \ x_{12n+11} = b,\\ y_{12n} &= c, \ \ y_{12n+1} = -b, \ \ y_{12n+2} = -a, \ \ y_{12n+3} = \frac{-d}{2}, \ \ y_{12n+4} = -2a, \ \ y_{12n+5} = -d,\\ y_{12n+6} &= -c, \ \ y_{12n+7} = b, \ \ y_{12n+8} = a, \ \ y_{12n+9} = \frac{b}{2}, \ \ y_{12n+10} = 2a, \ \ y_{12n+11} = b.\end{aligned}$$

Thus we have a periodic solution of period twelve and the proof is complete. □

Theorem 2.3. *Suppose that $a = c, b = -2d$. Then the System (2.1) has a periodic positive solution of period twelve, it will be taken the following form*

$$\begin{aligned}\{x_n\} &= \left\{a, \frac{d}{2}, -c, \frac{-b}{2}, a, 2d, -c, \frac{b}{4}, a, -d, -a, -2d, a, \frac{d}{2}, ...\right\},\\ \{y_n\} &= \left\{c, \frac{b}{2}, 2c, \frac{-b}{2}, \frac{-a}{2}, \frac{b}{2}, -a, d, -2a, -d, \frac{a}{2}, d, ...\right\}.\end{aligned}$$

Proof. Assume that $a = c$ and $b = -2d$ then we see the solution of System (2.1)

$$\begin{aligned}x_{12n} &= a, \ \ x_{12n+1} = \frac{d}{2}, \ \ x_{12n+2} = -c, \ \ x_{12n+3} = \frac{-b}{2}, \ \ x_{12n+4} = a, \ \ x_{12n+5} = 2d,\\ x_{12n+6} &= -c, \ \ x_{12n+7} = \frac{b}{4}, \ \ x_{12n+8} = a, \ \ x_{12n+9} = -d, \ \ x_{12n+10} = -a, \ \ x_{12n+11} = -2d,\\ y_{12n} &= c, \ \ y_{12n+1} = \frac{b}{2}, \ \ y_{12n+2} = 2c, \ \ y_{12n+3} = \frac{-b}{2}, \ \ y_{12n+4} = \frac{-a}{2}, \ \ y_{12n+5} = \frac{b}{2},\\ y_{12n+6} &= -a, \ \ y_{12n+7} = d, \ \ y_{12n+8} = -2a, \ \ y_{12n+9} = -d, \ \ y_{12n+10} = \frac{a}{2}, \ \ y_{12n+11} = d.\end{aligned}$$

Thus we have a periodic solution of period twelve and the proof is complete. □

Lemma 2.4. *Let $\{x_n, y_n\}_{n=-1}^{\infty}$ be a positive solution of System (2.1), that is $x_n, y_n > 0$, $n = -1, 0, ...$, then*

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0.$$

Proof. Using the fact that

$$(a+c)(b+d)(ad+bc) = 2abcd + d(b+d)a^2 + c(d^2+b^2)a + b(b+d)c^2,$$

we get

$$\frac{abcd}{(a+c)(b+d)(ad+bc)} < \frac{1}{2},$$

from which it follows that

$$\left(\frac{abcd}{(a+c)(b+d)(ad+bc)}\right)^n < \frac{1}{2^n}.$$

So, it follows that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0.$$

Consider the system

$$x_{n+1} = \frac{x_n y_{n-1}}{x_n - y_n}, \quad y_{n+1} = \frac{x_{n-1} y_n}{x_n - y_n}, \qquad n = 0, 1, ..., \tag{2.2}$$

with the initial values are arbitrary nonzero real numbers and $x_0 \neq y_0,\ x_{-1} \neq y_{-1}$ and $x_0 y_{-1} \neq x_{-1} y_0$.

In the following result, we realize the form of the solutions of System (2.2). □

Theorem 2.5. *Let $\{x_n, y_n\}$ be the solutions of System (2.2). Assume that x_0, x_{-1}, y_0 and y_{-1} are arbitrary nonzero real numbers with $a \neq c$, $b \neq d$ and $ad \neq bc$, then the solutions of System (2.2) are given by the following formulas for $n = 0, 1, 2, ...$*

$$\begin{aligned}
x_{6n} &= \frac{(-1)^n a^{n+1} b^n c^n d^n}{(a-c)^n (b-d)^n (ad-bc)^n}, & y_{6n} &= \frac{(-1)^n a^n b^n c^{n+1} d^n}{(a-c)^n (b-d)^n (ad-bc)^n}, \\
x_{6n+1} &= \frac{a^{n+1} b^n c^n d^{n+1}}{(a-c)^{n+1} (b-d)^n (ad-bc)^n}, & y_{6n+1} &= \frac{a^n b^{n+1} c^{n+1} d^n}{(a-c)^{n+1} (b-d)^n (ad-bc)^n}, \\
x_{6n+2} &= \frac{(-1)^n a^{n+1} b^n c^{n+1} d^{n+1}}{(a-c)^n (b-d)^n (ad-bc)^{n+1}}, & y_{6n+2} &= \frac{(-1)^n a^{n+1} b^{n+1} c^{n+1} d^n}{(a-c)^n (b-d)^n (ad-bc)^{n+1}}, \\
x_{6n+3} &= -\frac{a^n b^{n+1} c^{n+1} d^{n+1}}{(a-c)^{n+1} (b-d)^{n+1} (ad-bc)^n}, & y_{6n+3} &= -\frac{a^{n+1} b^{n+1} c^n d^{n+1}}{(a-c)^{n+1} (b-d)^{n+1} (ad-bc)^n}, \\
x_{6n+4} &= \frac{(-1)^{n+1} a^{n+1} b^{n+1} c^{n+2} d^n}{(a-c)^{n+1} (b-d)^n (ad-bc)^{n+1}}, & y_{6n+4} &= \frac{(-1)^{n+1} a^{n+2} b^n c^{n+1} d^{n+1}}{(a-c)^{n+1} (b-d)^n (ad-bc)^{n+1}}, \\
x_{6n+5} &= \frac{a^{n+1} b^{n+2} c^{n+1} d^{n+1}}{(a-c)^{n+1} (b-d)^{n+1} (ad-bc)^{n+1}}, & y_{6n+5} &= \frac{a^{n+1} b^{n+1} c^{n+1} d^{n+2}}{(a-c)^{n+1} (b-d)^{n+1} (ad-bc)^{n+1}},
\end{aligned}$$

such that $x_0 = a,\ x_{-1} = b,\ y_0 = c,\ y_{-1} = d$.

Proof. The proof is similar to that of Theorem 2.1 so it will left to the reader. □

3 Numerical examples

In this section, we shows some numerical examples that confirm the results obtained for System (2.1) and System (2.2).

Example 3.1. Consider System (2.1) with initial conditions $x_{-1} = 0.52,\ x_0 = 0.9,\ y_{-1} = -0.4,\ y_0 = 0.2,$ then the solution are unbounded and goes to infinity. (See eqreffig1).

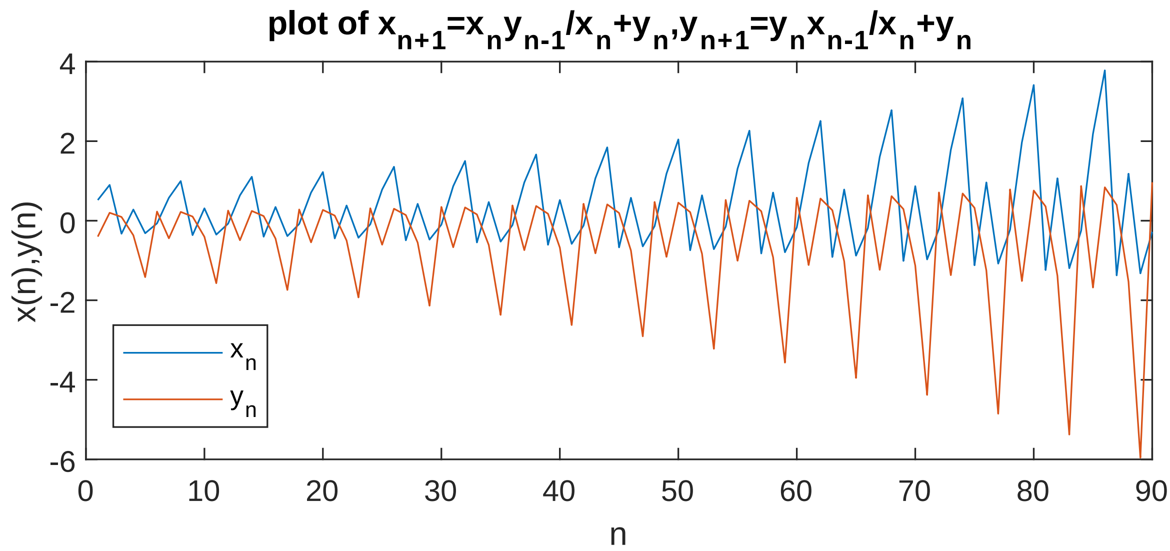

Figure 3.1: This figure displays the behavior of the solution of the System (2.1) when $x_{-1} = 0.52,\ x_0 = 0.9,\ y_{-1} = -0.4,\ y_0 = 0.2$

Example 3.2. Consider the System (2.1) with $x_{-1} = 0.5, x_0 = -1.8, y_{-1} = 0.5,\ y_0 = 0.9$, then the solution is periodic with period twelve and takes the form
$\left\{ \begin{array}{c} (0.5, 0.5),\ (-1.8, 0.9),\ (1, -0.5),\ (1.8, 1.8),\ (-0.25, -0.25),\ (-1.8, -1), \\ (-0.5, -0.5),\ (1.8, -0.9),\ (-1, .5),\ (-1.8, -1.8),\ (0.25, 0.25),\ (1.8, -3.6), ... \end{array} \right\}$. (See Figure (3.2)).

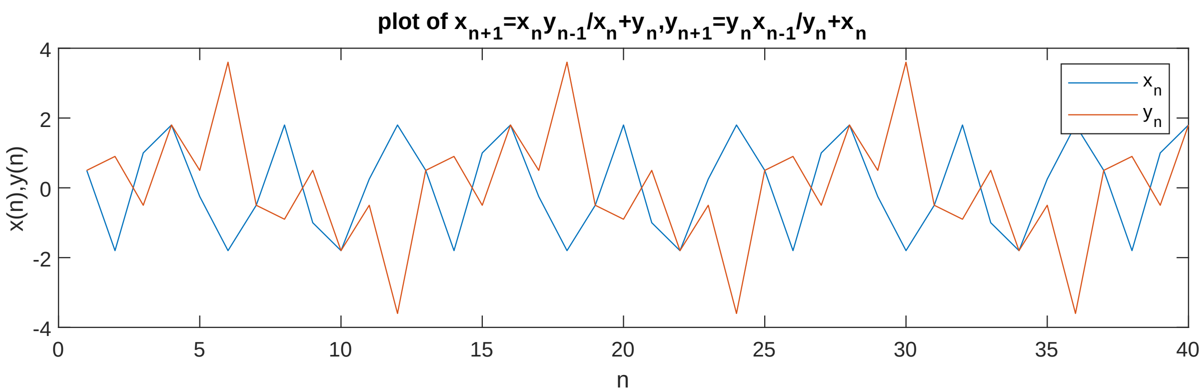

Figure 3.2: This figure shows the periodicity of the solution of the System (2.1) with $x_{-1} = 0.5, x_0 = -1.8, y_{-1} = 0.5,\ y_0 = 0.9$.

Example 3.3. Consider the System (2.1) when we put the initial conditions $x_{-1} = 10,\ x_0 = -1.8,\ y_{-1} = -5,\ y_0 = -1.8$, then the solution is periodic with period twelve and takes the form
$\left\{ \begin{array}{c} (10, -5),\ (-1.8, -1.8),\ (-2, 5),\ (0.5, -3.6),\ (1.8, -5),\ (-5, 0.9), \\ (-1.8, 5),\ (-10, 1.8),\ (1.8, -5),\ (2.5, 3.6),\ (-1.8, 5),\ (5, -0.9), ... \end{array} \right\}$. (See Figure (3.3)).

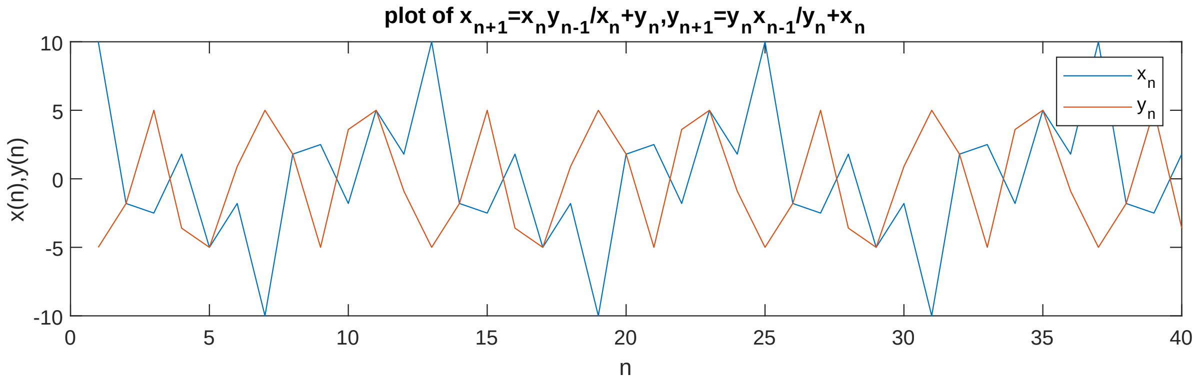

Figure 3.3: This figure shows the periodic solution of period twelve of the system $x_{n+1} = x_n y_{n-1}/(x_n + y_n), y_{n+1} = x_{n-1} y_n/(x_n + y_n)$, when $x_{-1} = 10,\ x_0 = -1.8,\ y_{-1} = -5,\ y_0 = -1.8$.

Example 3.4. Suppose the difference equations System (2.1) with the positive initial conditions $x_{-1} = 0.5,\ x_0 = 1.18,\ y_{-1} = 0.96,\ y_0 = 0.9$. Then the solutions are bounded and converges to zero (See Figure (3.4)).

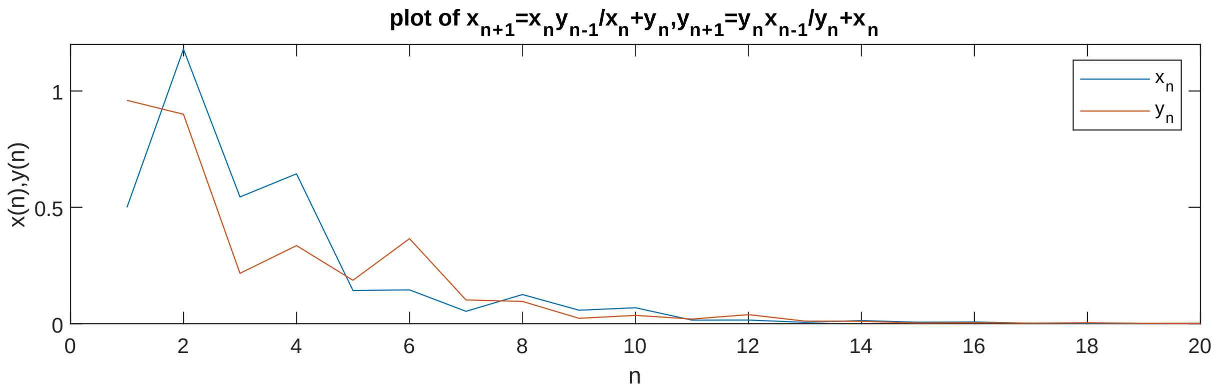

Figure 3.4: his figure shows the boundedness of the solution of the system $x_{n+1} = x_n y_{n-1}/(x_n + y_n), y_{n+1} = x_{n-1} y_n/(x_n + y_n)$, when $x_{-1} = 0.5,\ x_0 = 1.18,\ y_{-1} = 0.96,\ y_0 = 0.9$.

Example 3.5. Consider the System (2.2) when we choose the initial conditions $x_{-1} = 7,\ x_0 = 5,\ y_{-1} = 9,\ y_0 = 0.9,$ then the solution is bounded (See Figure (3.5)).

Example 3.6. Consider the System (2.2) when we take $x_{-1} = -0.3,\ x_0 = -0.4,\ y_{-1} = -0.4,\ y_0 = -1.4,$ then the solution is unbounded and goes to infinity (See Figure (3.6)).

Declarations

Availability of data and materials

Data sharing not applicable to this article.

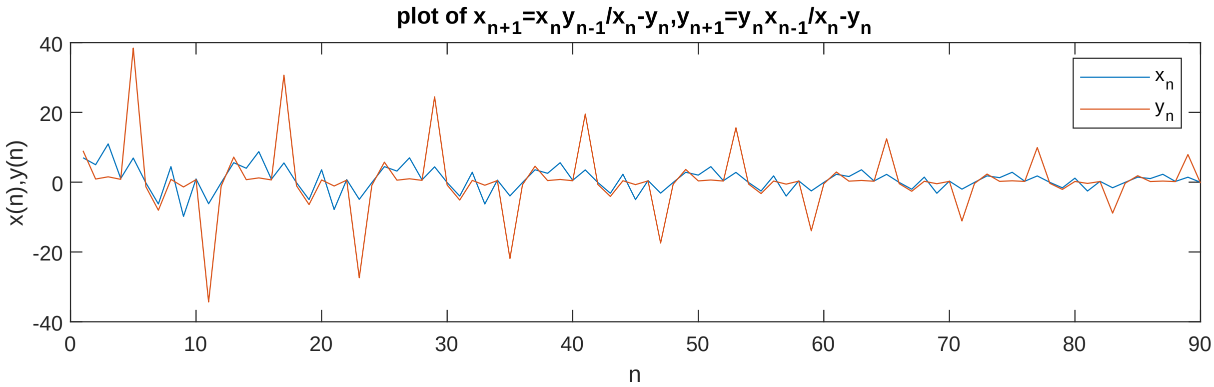

Figure 3.5: This figure shows the boundedness of the solution of the System (2.2), with $x_{-1} = 7,\ x_0 = 5,\ y_{-1} = 9,\ y_0 = 0.9$.

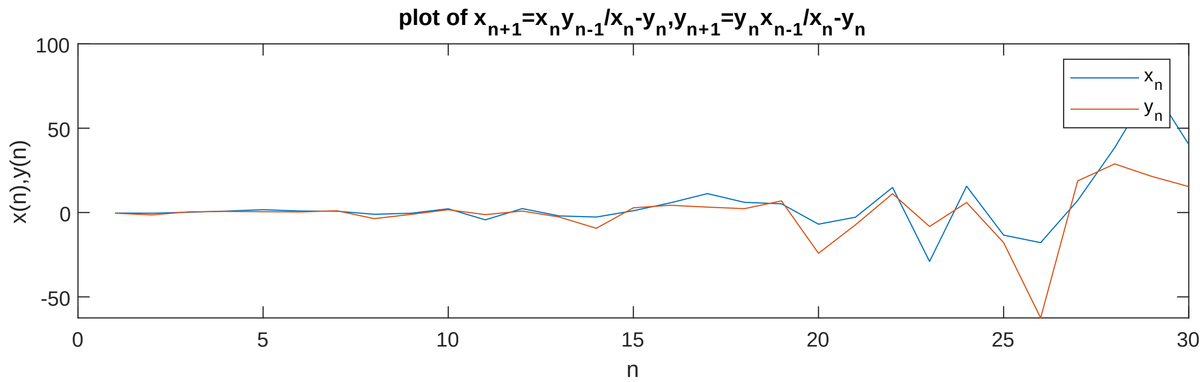

Figure 3.6: This figure displays the unboundedness of the solution of the system $x_{n+1} = x_n y_{n-1}/(x_n - y_n), y_{n+1} = x_{n-1}y_n/(x_n - y_n)$, when $x_{-1} = -0.3,\ x_0 = -0.4,\ y_{-1} = -0.4,\ y_0 = -1.4$.

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Authors' contributions

The authors declare that the study was realized in collaboration with equal responsibility.

Conflict of interest

The authors have no conflicts of interest to declare.

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