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Exponential growth of solution for a couple of semi-linear pseudo-parabolic equations with memory and source terms

Amar Ouaoua [1] and Wissem Boughamsa [2]

[1]August 20, 1955 University, Skikda, Faculty of the sciences and technology, (LAMAHIS) Laboratory, Algeria

[2]August 20, 1955 University, Skikda, Faculty of the sciences, (LAMAHIS) Laboratory, Algeria

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Abstract. This work is concerned with coupled semi-linear pseudo-parabolic equations with memory terms in both equations, associated with the homogeneous Dirichlet boundary condition. We show that the solution grows exponentially under specific conditions regarding the relaxation functions and initial energy. In order to prove the result, we use the energy method based on the construction of a suitable Lyapunov function. The most important behavior of the evolution system is the exponential growth phenomena because of its wide range of applications in modern science, such as chemistry, biology, ecology, and other areas of engineering and physical sciences.

Keywords: Memory terms, parabolic equation, exponential growth, source terms.

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1 Introduction

We consider the following boundary value problem:

$$\begin{cases} u_t - \Delta u - \Delta u_t + \int_0^t g(t-s)\,\Delta u(s)\,ds + |u|^{m-2} u_t = f_1(u,v), & \text{in } \Omega\times(0,T) \\ v_t - \Delta v - \Delta v_t + \int_0^t h(t-s)\,\Delta v(s)\,ds + |v|^{k-2} v_t = f_2(u,v), & \text{in } \Omega\times(0,T) \\ u(x,t) = 0,\, v(x,t) = 0, & \text{in } \partial\Omega\times(0,T) \\ u(x,0) = u_0,\ v(x,0) = v_0, & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$ with smooth boundary $\partial\Omega$, m and k are real positive constants. The relaxation functions g and h satisfying some conditions we suppose later, and the two functions $f_1(u,v)$ and $f_2(u,v)$ are given by

$$\begin{cases} f_1(u,v) = |u+v|^{2(r+1)}(u+v) + |u|^r u\,|v|^{r+2}. \\ f_2(u,v) = |u+v|^{2(r+1)}(u+v) + |u|^{r+2} v\,|v|^r. \end{cases} \tag{1.2}$$

✉ Corresponding author. Email: ouaouaama21@gmail.com

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Recently, Pişkin and Ekinci in [15] treated the following system

$$\begin{cases} u_t - \Delta u + |u|^{q-2} u_t = f_1(u,v), \\ v_t - \Delta v + |v|^{q-2} v_t = f_2(u,v). \end{cases} \tag{1.3}$$

They proved the exponential growth of the solution with initial negative energy.
In the absence of $|u|^{q-2} u_t$ and $|v|^{q-2} v_t$ terms the system (1.3) becomes

$$\begin{cases} u_t - \Delta u = f_1(u,v), \\ v_t - \Delta v = f_2(u,v). \end{cases}$$

This type of equation is naturally found in physics, chemistry, biology, ecology, and other areas of engineering and physical sciences (see [4–6]). Korpusov in [9] has determined sufficient conditions for the blow-up of a finite time and the solvability of the following generalized Boussinesq equation

$$\frac{\partial}{\partial t}\left(\Delta u - u - |u|^p u\right) + u(u+\alpha)(u-\beta) = 0, \tag{1.4}$$

with initial boundary value, for $\alpha, \beta > 0$ in $\mathbb{R}^3$, by concativity method [7, 10]. The result is extended by [20, 21]. Based on idea in [3] for the equation (1.5) the authors of [20] proved that the L^p-norm of the solution for (1.4) grows as an exponential function.

$$u_t - \Delta u - \Delta u_t + |u|^{m-2} u_t = |u|^{p-2} u. \tag{1.5}$$

In [16] Polat established a blow up result of the solution with vanishing initial energy in a bounded domain of $\mathbb{R}$ with the absence of the term $-\Delta u_t$ in equation (1.5). Many authors [1, 2, 8, 17] have considered the following initial value problem for the generalized Boussinesq equation with nonlinear Newmann condition

$$\begin{cases} u_t - \Delta u - \Delta u_t + |u|^{m-2} u_t = f(u), \\ u = 0, \ x \in \Gamma_0, \\ \frac{\partial u}{\partial v} + g(u) = 0, \ x \in \Gamma_1, \\ u(x,0) = u_0(x), \ x \in \Omega. \end{cases} \tag{1.6}$$

In [2] the authors have established both the existence of the solution and a generalisation of the energy functions under some restrictions on the initial data. They have also proved a blow-up result. Ouaoua et al in [13] considered the following nonlinear Kirchhoff type reaction-diffusion equation

$$u_t - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u + |u|^{m-2} u_t = |u|^{r-2} u, \quad (x,t) \in \Omega \times (0,T), \tag{1.7}$$

where $M(s) = a + bs^\gamma$, a, b and γ are positive constants. Under suitable assumptions on the initial data, they obtained global existence and stability of solutions with positive initial energy. In the case of the variable exponents, also, Ouaoua and Maouni in [14] considered the following equation

$$u_t - div\left(|\nabla u|^{p(x)-2} \nabla u\right) + \omega |u|^{m(x)-2} u_t = b|u|^{r(x)-2} u \quad \text{in } \Omega \times (0,T). \tag{1.8}$$

They proved blow-up, exponential growth of solution with negative initial energy in the both case of equation. Messaoudi in [11] proved under suitable conditions on g and p, a blow-up result for solutions of the semilinear viscoelastic equation

$$u_t - \Delta u + \int_0^t g(t-s)\Delta u(s,x)\,dx = |u|^{p-2}u \text{ in } \Omega \times (0,T), \tag{1.9}$$

with positive initial energy.
The paper is organized as follows. In section 2, we present the necessary assumptions and lemmas, while in section 3, we prove the main result.

2 Preliminaries and assumptions

This section will give some notations and statements of assumptions for the relaxation functions. We denote $L^p(\Omega)$ by L^p and $H_0^1(\Omega)$ by H_0^1. The norm and inner product of $L^p(\Omega)$ are denoted by $\|.\|_p$ and $(a,b) = \int a(x)\,b(x)\,dx$ respectively.

For the relaxation functions g and h we assume that
(A_1) $g, h : \mathbb{R}^+ \to \mathbb{R}^+$ are of class C^1 and satisfying

$$\begin{cases} g(s) \geq 0,\ 1 - \int_0^\infty g(s)\,ds = l > 0, \\ h(s) \geq 0,\ 1 - \int_0^\infty h(s)\,ds = k > 0, \end{cases} \tag{2.1}$$

and

$$g'(s), h'(s) \leq 0,\ \forall s \geq 0. \tag{2.2}$$

(A_2)

$$\begin{cases} -1 < r \text{ if } n = 1, 2, \\ -1 < r \leq \frac{3-n}{n-2} \text{ if } n \geq 3. \end{cases} \tag{2.3}$$

Throughout this paper, we use the following notations

$$(\phi \circ \psi)(t) = \int_0^t \phi(t-s) \int_\Omega |\psi(t) - \psi(s)|^2\,dxds.$$

We can easily verify that

$$uf_1(u,v) + vf_2(u,v) = 2(r+2)F(u,v),\ \forall (u,v) \in \mathbb{R}^2,$$

where

$$F(u,v) = \frac{1}{2(r+2)}\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right].$$

Lemma 2.1. *[12]: There exist two positive constants c_0 and c_1 such that*

$$\frac{c_0}{2(r+2)}\left(|u|^{2(r+2)} + |v|^{2(r+2)}\right) \leq F(u,v) \leq \frac{c_1}{2(r+2)}\left(|u|^{2(r+2)} + |v|^{2(r+2)}\right). \tag{2.4}$$

3 Main result and proof

In this section we state and prove our main result. First we state the definition for the strong solution of the problem (1.1).

Definition 3.1. A strong solution of the problem (1.1) on $\Omega \times [0,T]$, is a pair of (u,v) satisfying

$$(u,v) \in \left(C\left([0,T];H_0^1\right) \cap C^1[0,T];L^2\right)^2, \left(|u|^{m-2}u_t, |v|^{k-2}v_t\right) \in \left[L^2(\Omega \times [0,T])\right]^2$$

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)),$$

$$\int_0^t \int_\Omega u_t \psi dxds - \int_0^t \int_\Omega \int_0^s g(t-\tau) \nabla u(\tau) \nabla \psi(s) d\tau dxds$$
$$+ \int_0^t \int_\Omega \nabla u \nabla \psi dxds - \int_0^t \int_\Omega f_1(u,v) \psi dxds + \int_0^t \int_\Omega |u|^{m-2} u_t \psi dxds = 0.$$

and

$$\int_0^t \int_\Omega v_t \phi dxds - \int_0^t \int_\Omega \int_0^s h(t-\tau) \nabla v(\tau) \nabla \phi(s) d\tau dxds$$
$$+ \int_0^t \int_\Omega \nabla v \nabla \phi dxds - \int_0^t \int_\Omega f_2(u,v) \phi dxds + \int_0^t \int_\Omega |v|^{k-2} v_t \phi dxds = 0.$$

for all $t \in [0,T]$ and all $\psi, \phi \in C\left([0,T], H_0^1\right)$.

The energy functional $E(t)$ associated with a solution (u,v) of (1.1) is given by

$$E(t) = \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right) \|\nabla u\|_2^2 + \frac{1}{2}\left(1 - \int_0^t h(s)\,ds\right) \|\nabla v\|_2^2 + \frac{1}{2}(g \circ \nabla u)$$
$$+ \frac{1}{2}(h \circ \nabla v) - \frac{1}{2(r+2)} \int_\Omega [u f_1(u,v) + v f_2(u,v)]\,dx. \tag{3.1}$$

Lemma 3.2. *Let (u,v) be a solution of (1.1), then $E(t)$ is a nonincreasing function for $t > 0$ satisfying*

$$E'(t) = \frac{1}{2}\left((g' \circ \nabla u)(t) + (h' \circ \nabla v)(t) - g(t)\|\nabla u(t)\|_2^2 - h(t)\|\nabla v(t)\|_2^2\right)$$
$$- \|u_t\|_2^2 - \|v_t\|_2^2 - \int_\Omega |u|^{m-2} u_t^2 dx - \int_\Omega |v|^{k-2} v_t^2 dx \le 0, \tag{3.2}$$

and

$$E(t) \le E(0) \le 0, \quad \forall t \ge 0.$$

Proof. Multiplying first equation of (1.1) by u_t and second by v_t, integrating over Ω and using integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\left[\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + \left(1 - \int_0^t h(s)\,ds\right)\|\nabla v\|_2^2\right.$$
$$- \frac{1}{2(r+2)} \int_\Omega [u f_1(u,v) + v f_2(u,v)]\,dx + \frac{1}{2}\int_\Omega \int_0^t g(t-s)|\nabla u(t) - \nabla u(s)|^2 dsdx$$
$$- \frac{1}{2}\int_\Omega \int_0^t g(t-s)|\nabla u(t)|^2 dsdx + \frac{1}{2}\int_\Omega \int_0^t h(t-s)|\nabla v(t) - \nabla v(s)|^2 dsdx$$
$$\left. - \frac{1}{2}\int_\Omega \int_0^t h(t-s)|\nabla v(t)|^2 dsdx\right]$$
$$- \frac{1}{2}\int_\Omega \int_0^t g'(t-s)|\nabla u(t) - \nabla u(s)|^2 dsdx + \frac{1}{2}\int_0^t g(t-s)\,ds\,\|\nabla u(t)\|_2^2$$
$$- \frac{1}{2}\int_\Omega \int_0^t h'(t-s)|\nabla v(t) - \nabla v(s)|^2 dsdx + \frac{1}{2}\int_0^t h(t-s)\,ds\,\|\nabla v(t)\|_2^2$$
$$- \int_\Omega |u|^{m-2} u_t^2 dx - \int_\Omega |v|^{k-2} v_t^2 dx - \|u_t\|_2^2 - \|v_t\|_2^2,$$

then by definition of $E(t)$, we get

$$E'(t) = \frac{1}{2}\left((g' \circ \nabla u)(t) + (h' \circ \nabla v)(t) - g(t)\|\nabla u(t)\|_2^2 - h(t)\|\nabla v(t)\|_2^2\right)$$
$$- \|u_t\|_2^2 - \|v_t\|_2^2 - \int_\Omega |u|^{m-2} u_t^2 dx - \int_\Omega |v|^{k-2} v_t^2 dx. \tag{3.3}$$

This infer by (2.2) that $E'(t) \le 0, \forall t \ge 0$ □

Theorem 3.3. *Suppose that assumptions* (A_1) *and* (A_2) *hold,* $(u_0, v_0) \in (H_0^1)^2$ *and* (u, v) *is a local strong solution of the system (1.1), and* $E(0) < 0$.

Furthermore, we assume that $\max\left(\int_0^\infty g(s)\,ds, \int_0^\infty h(s)\,ds\right) < p/(p+1)$ *and*

$$2(r+2) > \max(m, k). \tag{3.4}$$

Then the solution of the system (1.1) exponentially grows.

Proof. We set

$$H(t) = -E(t). \tag{3.5}$$

By the definition of $H(t)$ and (3.5)

$$H(t) = -E(t) \ge 0. \tag{3.6}$$

By using (3.1) and (3.5) we get

$$H(t) - \frac{1}{2(r+2)}\int_\Omega [u f_1(u,v) + v f_2(u,v)]\,dx \le 0. \tag{3.7}$$

So that

$$0 < H(0) \le H(t) \le \frac{1}{2(r+2)}\int_\Omega [u f_1(u,v) + v f_2(u,v)]\,dx. \tag{3.8}$$

We define a Lyaponov function

$$L(t) = H(t) + \frac{\varepsilon}{2}\left(\|u\|_2^2 + \|v\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2\right), \tag{3.9}$$

for ε small to be chosen later.

By taking the time derivative of (3.9) and (1.1), we obtain

$$L'(t) = H'(t) + \varepsilon\int_\Omega u_t u dx + \varepsilon\int_\Omega v_t v dx + \varepsilon\int_\Omega \nabla u_t \nabla u dx + \varepsilon\int_\Omega \nabla v_t \nabla v dx$$
$$\ge H'(t) - \varepsilon\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + \varepsilon\int_\Omega\int_0^t g(t-s)\nabla u(t)\nabla u(s)\,dsdx$$
$$+ \varepsilon\int_\Omega\int_0^t h(t-s)\nabla v(t)\nabla v(s)\,dsdx - \varepsilon\int_\Omega |u|^{m-2} u_t u dx - \varepsilon\int_\Omega |v|^{k-2} v_t v dx$$
$$+ \varepsilon\int_\Omega [u f_1(u,v) + v f_2(u,v)]\,dx. \tag{3.10}$$

Consequently, by (3.6) inequality (3.10) can be rewritten as

$$L'(t) \ge -\varepsilon\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + \int_\Omega |u|^{m-2} u_t^2 dx + \int_\Omega |v|^{k-2} v_t^2 dx$$
$$- \varepsilon\int_\Omega |u|^{m-2} u_t u dx - \varepsilon\int_\Omega |v|^{k-2} v_t v dx + \varepsilon\int_\Omega [u f_1(u,v) + v f_2(u,v)]\,dx$$
$$+ \varepsilon\int_\Omega\int_0^t g(t-s)\nabla u(t)\nabla u(s)\,dsdx + \varepsilon\int_\Omega\int_0^t h(t-s)\nabla v(t)\nabla v(s)\,dsdx. \tag{3.11}$$

Therefore, combining with

$$\int_{\Omega}\int_{0}^{t} g(t-s)\nabla u(t)\nabla u(s)\,dsdx = \int_{0}^{t} g(s)\,ds\,\|u(t)\|_{2}^{2} + \int_{0}^{t} g(t-s)\int_{\Omega}\nabla u(t)(\nabla u(s)-\nabla u(t))\,dxds. \tag{3.12}$$

The Cauchy-Schwartz and Young's inequalities allow us to estimate the last term in the right side of (3.12) as follows

$$\int_{0}^{t} g(t-s)\int_{\Omega}\nabla u(t)[\nabla u(s)-\nabla u(t)]\,dxds \leq \frac{1}{2}(g\circ\nabla u)+\frac{1}{2}\left(\int_{0}^{t} g(s)\,ds\right)\|\nabla u\|_{2}^{2}. \tag{3.13}$$

Similarly

$$\int_{\Omega}\int_{0}^{t} h(t-s)\nabla v(t)\nabla v(s)\,dsdx = \int_{0}^{t} h(s)\,ds\,\|v(t)\|_{2}^{2} + \int_{0}^{t} h(t-s)\int_{\Omega}\nabla v(t)(\nabla v(s)-\nabla v(t))\,dxds, \tag{3.14}$$

and

$$\int_{0}^{t} h(t-s)\int_{\Omega}\nabla v(t)[\nabla v(s)-\nabla v(t)]\,dxds \leq \frac{1}{2}(h\circ\nabla v)+\frac{1}{2}\left(\int_{0}^{t} h(s)\,ds\right)\|\nabla v\|_{2}^{2}. \tag{3.15}$$

Inserting (3.12) and (3.14) into (3.11), using (3.13) and (3.15), lead to

$$\begin{aligned} L'(t) \geq\; & \varepsilon\left(\frac{1}{2}\int_{0}^{t} g(s)\,ds-1\right)\|\nabla u\|_{2}^{2}+\varepsilon\left(\frac{1}{2}\int_{0}^{t} h(s)\,ds-1\right)\|\nabla v\|_{2}^{2} \\ & +\varepsilon\int_{\Omega}|u|^{m-2}u_{t}^{2}dx+\varepsilon\int_{\Omega}|v|^{k-2}v_{t}^{2}dx \\ & -\varepsilon\int_{\Omega}[uf_{1}(u,v)+vf_{2}(u,v)]\,dx-\frac{1}{2}\varepsilon(g\circ\nabla u)-\frac{1}{2}\varepsilon(h\circ\nabla v) \\ & -\varepsilon\int_{\Omega}|u|^{m-2}u_{t}udx-\varepsilon\int_{\Omega}|v|^{k-2}v_{t}vdx. \end{aligned} \tag{3.16}$$

To estimate the two last terms in the right hand-side of (3.16), by using the following Young inequality

$$XY \leq \delta X^{2}+\delta^{-1}Y^{2},\ X,\ Y \geq 0,\ \delta > 0.$$

We have

$$\int_{\Omega}|u|^{m-2}u_{t}udx \leq \delta\int_{\Omega}|u|^{m-2}u_{t}^{2}dx+\delta^{-1}\|u\|_{m}^{m}. \tag{3.17}$$

Similarly

$$\int_{\Omega}|v|^{k-2}v_{t}vdx \leq \delta\int_{\Omega}|v|^{k-2}v_{t}^{2}dx+\delta^{-1}\|v\|_{k}^{k}. \tag{3.18}$$

Inserting the last two estimates into (3.16), we obtain

$$\begin{aligned} L'(t) \geq\ & \varepsilon\left(\frac{1}{2}\int_0^t g(s)\,ds - 1\right)\|\nabla u\|_2^2 + \varepsilon\left(\frac{1}{2}\int_0^t h(s)\,ds - 1\right)\|\nabla v\|_2^2 \\ & + (1-\varepsilon\delta)\int_\Omega |u|^{m-2}u_t^2 dx + (1-\varepsilon\delta)\int_\Omega |v|^{k-2}v_t^2 dx \\ & - \varepsilon\delta^{-1}\|u\|_m^m - \varepsilon\delta^{-1}\|v\|_k^k + \varepsilon\int_\Omega [uf_1(u,v) + vf_2(u,v)]\,dx \\ & - \frac{1}{2}\varepsilon(g\circ\nabla u) - \frac{1}{2}\varepsilon(h\circ\nabla v). \end{aligned} \tag{3.19}$$

By using (3.1), we get

$$\begin{aligned} L'(t) \geq\ & \varepsilon\left(\frac{1}{2}\int_0^t g(s)\,ds - 1\right)\|\nabla u\|_2^2 + \varepsilon\left(\frac{1}{2}\int_0^t h(s)\,ds - 1\right)\|\nabla v\|_2^2 \\ & - \varepsilon\delta^{-1}\|u\|_m^m - \varepsilon\delta^{-1}\|v\|_k^k + (1-\varepsilon\delta)\int_\Omega |u|^{m-2}u_t^2 dx + (1-\varepsilon\delta)\int_\Omega |v|^{k-2}v_t^2 dx \\ & + \left[2\varepsilon(r+2)H(t) + \frac{2(r+1)}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2\right. \\ & + \frac{2(r+1)}{2}\left(1 - \int_0^t h(s)\,ds\right)\|\nabla v\|_2^2 \\ & \left. + \frac{2(r+1)}{2}(g\circ\nabla u) + \frac{2(r+1)}{2}(h\circ\nabla v)\right] - \frac{\varepsilon}{2}(g\circ\nabla u) - \frac{\varepsilon}{2}(h\circ\nabla v). \end{aligned} \tag{3.20}$$

So that

$$\begin{aligned} L'(t) \geq\ & (1-\varepsilon\delta)\int_\Omega |u|^{m-2}u_t^2 dx + (1-\varepsilon\delta)\int_\Omega |v|^{k-2}v_t^2 dx \\ & - \varepsilon\delta^{-1}\|u\|_m^m - \varepsilon\delta^{-1}\|v\|_k^k + \varepsilon\left(\frac{2(r+2)}{2} - \frac{1}{2}\right)(g\circ\nabla u) \\ & + \varepsilon\left(\frac{2(r+2)}{2} - \frac{1}{2}\right)(h\circ\nabla v) + 2(r+2)H(t) \\ & + \varepsilon\left(\frac{2(r+2)-2}{2} - \frac{1-2(r+2)}{2}\int_0^\infty g(s)\,ds\right)\|\nabla u\|_2^2 \\ & + \varepsilon\left(\frac{2(r+2)-2}{2} - \frac{1-2(r+2)}{2}\int_0^\infty h(s)\,ds\right)\|\nabla v\|_2^2. \end{aligned} \tag{3.21}$$

By (3.4), the embedding $L^m(\Omega) \hookrightarrow L^{2(r+2)}(\Omega), L^k(\Omega) \hookrightarrow L^{2(r+2)}(\Omega)$, and using the algebraic inequality

$$z^\nu \leq (z+1) \leq \left(1 + \frac{1}{a}\right)(z+a),\ \forall z > 0,\ 0 < \nu \leq 1,\ a \geq 0, \tag{3.22}$$

we have

$$\begin{aligned} \|u\|_m^m &\leq c\|u\|_{2(r+2)}^m \leq c\left(\|u\|_{2(r+2)}^{2(r+2)}\right)^{\frac{m}{2(r+2)}} \\ &\leq c\left(1 + \frac{1}{H(0)}\right)\left(\|u\|_{2(r+2)}^{2(r+2)} + H(0)\right) \\ &\leq c_1\left(\|u\|_{2(r+2)}^{2(r+2)} + H(t)\right). \end{aligned}$$

By the Lemma 1 and (3.7), we obtain

$$\|u\|_m^m \leq c_2\int_\Omega [uf_1(u,v) + vf_2(u,v)]\,dx, \tag{3.23}$$

Similarly

$$\|v\|_k^k \le c_3 \int_\Omega [u f_1(u,v) + v f_2(u,v)]\, dx. \tag{3.24}$$

Therefore, by (3.23) and (3.24), the inequality (3.21) becomes

$$\begin{aligned} L'(t) \ge{}& (1-\varepsilon\delta) \int_\Omega |u|^{m-2} u_t^2 dx + (1-\varepsilon\delta) \int_\Omega |v|^{k-2} v_t^2 dx \\ &+ 2\varepsilon(r+2) H(t) + \varepsilon a_1 \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right) \\ &+ \varepsilon a_2 [(g \circ \nabla u) + (h \circ \nabla v)] - \varepsilon c_4 \delta^{-1} \int_\Omega [u f_1(u,v) + v f_2(u,v)]\, dx, \end{aligned} \tag{3.25}$$

where

$$a_1 = \min\left(\frac{2(r+2)-2}{2} + \frac{1-2(r+2)}{2} \int_0^{+\infty} g(s)\, ds, \frac{2(r+2)-2}{2} + \frac{1-2(r+2)}{2} \int_0^{+\infty} h(s)\, ds \right),$$

$$a_2 = \frac{2(r+2)}{2} - \frac{1}{2} \text{ and } c_4 = c_2 + c_3.$$

Taking

$$0 < a_3 < \min(a_1, a_2),$$

and

$$\begin{aligned} 2H(t) \ge{}& -\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right) - [(g \circ \nabla u) + (h \circ \nabla v)] \\ &+ \frac{2}{2(r+2)} \int_\Omega [u f_1(u,v) + v f_2(u,v)]\, dx. \end{aligned}$$

We get

$$\begin{aligned} L'(t) \ge{}& (1-\varepsilon\delta) \left(\int_\Omega |u|^{m-2} u_t^2 dx + \int_\Omega |v|^{k-2} v_t^2 dx \right) + \varepsilon (2(r+2) - 2a_3) H(t) \\ &+ \varepsilon (a_1 - a_3) \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right) + \varepsilon (a_2 - a_3) [(g \circ \nabla u) + (h \circ \nabla v)] \\ &+ \varepsilon \left(\frac{2a_3}{2(r+2)} - c_4 \delta^{-1} \right) \int_\Omega [u f_1(u,v) + v f_2(u,v)]\, dx \\ &+ \varepsilon \left(a_3 \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + (g \circ \nabla u) + (h \circ \nabla v) - \frac{2}{2(r+2)} \int_\Omega [u f_1(u,v) + v f_2(u,v)]\, dx \right). \end{aligned} \tag{3.26}$$

At this point, and for large value of δ and ε small enough such that

$$\frac{2a_3}{2(r+2)} - c_4 \delta^{-1} > 0 \text{ and } 1 - \varepsilon\delta > 0.$$

Noting that

$$2(r+2) - 2a_3 > 0.$$

Then

$$L'(t) \ge M_1 \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 + (g \circ \nabla u) + (h \circ \nabla v) + H(t) \right). \tag{3.27}$$

On the other hand, we have

$$L(t) = H(t) + \frac{\varepsilon}{2} \int_\Omega u^2 dx + \frac{\varepsilon}{2} \int_\Omega v^2 dx + \frac{\varepsilon}{2} \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right).$$

By Poincaré's inequality, we obtain

$$\int_{\Omega} u^2 dx + \int_{\Omega} v^2 dx \leq c_5 \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right),$$

so that

$$\begin{aligned} L(t) &\leq M_2 \left(H(t) + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right) \\ &\leq M_2 \left(H(t) + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + (g \circ \nabla u) + (h \circ \nabla v) \right). \end{aligned} \tag{3.28}$$

Then by (3.27) and (3.28), we get

$$L'(t) \geq \xi L(t), \tag{3.29}$$

and

$$L(0) = H(0) + \frac{\varepsilon}{2} \left(\|u_0\|_2^2 + \|v_0\|_2^2 + \frac{\varepsilon}{2} \|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2 \right) > 0.$$

Finally, a simple integration of (3.29) gives the desired result. □

Declarations

Conflict of Interest

The authors have no conflicts of interest to declare.

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