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On some radicals and proper classes associated to simple modules

Research Article

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Abstract: For a unitary right module M, there are two known partitions of simple modules in the category $\sigma[M]$: the first one divides them into M-injective modules and M-small modules, while the second one divides them into M-projective modules and M-singular modules. We study inclusions between the first two and the last two classes of simple modules in terms of some associated radicals and proper classes.

1. Introduction

For an associative ring R with identity, let M be a unitary right R-module. Then the category $\sigma[M]$ is a Grothendieck category which consists of all right R-modules subgenerated by M, that is, submodules of M-generated right R-modules. An object X of $\sigma[M]$ is called:

1. *M-injective* if the functor $\operatorname{Hom}_R(-, X)$ preserves exactness of all short exact sequences $0 \to K \to M \to N \to 0$.

2. *M-projective* if the functor $\operatorname{Hom}_R(X, -)$ preserves exactness of all short exact sequences $0 \to K \to M \to N \to 0$.

3. *M-small* if X is superfluous in some module $Y \in \sigma[M]$.

4. *M-singular* if $X \cong Y/Z$ for some essential submodule Z of a module $Y \in \sigma[M]$.

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There are two classical dichotomies concerning the simple modules in $\sigma[M]$. The first one partitions them into the pair of classes $(\mathcal{S}(\mathscr{I}), \mathcal{S}(\mathscr{M}))$, where $\mathcal{S}(\mathscr{I})$ is the class of simple M-injective modules in $\sigma[M]$ and $\mathcal{S}(\mathscr{M})$ is the class of simple M-small modules in $\sigma[M]$ [4, 4.2]. The second one partitions them into the pair of classes $(\mathcal{S}(\mathscr{P}), \mathcal{S}(\mathscr{S}))$, where $\mathcal{S}(\mathscr{P})$ is the class of simple M-projective modules in $\sigma[M]$ and $\mathcal{S}(\mathscr{S})$ is the class of simple M-singular modules in $\sigma[M]$ [3, 8.2].

In the present note we give equivalent conditions under which the above two pairs of classes of simple modules may be linked in terms of some associated radicals and proper classes. More precisely, we are interested in such characterizations for possible inclusions between the four classes of simple modules $\mathcal{S}(\mathscr{I})$, $\mathcal{S}(\mathscr{M})$, $\mathcal{S}(\mathscr{P})$ and $\mathcal{S}(\mathscr{S})$. They are inspired by and complete results by Preisser Montaño [5].

2. Preliminaries

In this paper R will be an associative ring with identity and any module M will be a unitary right R-module.

2.1. Radicals

Let $\mathcal{A}$ be an abelian category. We first recall the definitions of preradicals and radicals from [6, Chapter 6, §1]. A *preradical* τ of $\mathcal{A}$ is a subfunctor of the identity functor on $\mathcal{A}$, that is, a functor $\tau : \mathcal{A} \longrightarrow \mathcal{A}$ such that:

1. $\tau(A) \subseteq A$ for every object A of $\mathcal{A}$.

2. $\tau(f) = f \mid_{\tau(A)}: \tau(A) \longrightarrow \tau(B)$ for every morphism $f : A \longrightarrow B$ in $\mathcal{A}$.

A preradical τ of $\mathcal{A}$ is *idempotent* if $\tau\tau = \tau$. Preradicals of $\mathcal{A}$ are partially ordered by the relation defined by $\tau \leq \tau'$ if and only if $\tau(A) \subseteq \tau'(A)$ for every object A of $\mathcal{A}$. If the abelian category $\mathcal{A}$ is complete and cocomplete, then $\tau(\oplus_{i \in I} A_i) = \oplus_{i \in I} \tau(A_i)$ and $\tau(\prod_{i \in I} A_i) \subseteq \prod_{i \in I} \tau(A_i)$ for any family $\{A_i\}_{i \in I}$ of objects of $\mathcal{A}$ and any preradical τ of $\mathcal{A}$. Note that $\tau(P) = P\tau(R_R)$ for all projective modules P in the category Mod-R of right R-modules and any preradical τ of Mod-R.

A preradical τ of $\mathcal{A}$ is called *radical* if $\tau(A/\tau(A)) = 0$, for all objects A of $\mathcal{A}$. In particular, one has the Jacobson radical Rad on a Grothendieck category, where $\mathrm{Rad}(X)$ is the intersection of maximal subobjects of X for every object X.

Let τ be a radical of $\sigma[M]$. Let L be a submodule of a module $N \in \sigma[M]$. Then L is called a τ-*supplement* in N if there exists a submodule X of N such that $N = L+X$ and $L \cap X \subseteq \tau(L)$, equivalently, $N = L + X$ and $L \cap X = \tau(L)$ (see [1, 1.11] or [5, 4.9]).

A similar concept may also be given with respect to a class $\mathcal{C}$ of modules in $\sigma[M]$. A submodule K of a module $N \in \sigma[M]$ is called $\mathcal{C}$-*small* in N if for every submodule $X \leq N$, the equality $K + X = N$ and $N/X \in \mathcal{C}$ imply $X = N$. We denote this by $K \ll_{\mathcal{C}} N$ (see [5, 8.1]). For the class $\mathscr{S}$ of M-singular modules, note that $\mathscr{S}$-small submodules are the δ-small submodules in $\sigma[M]$ as defined by Zhou in [7] (see [5, 8.2(i)]). A submodule L of a module $N \in \sigma[M]$ is called a $\mathcal{C}$-*supplement* in N if there exists a submodule L' of N such that $L + L' = N$ and $L \cap L' \ll_{\mathcal{C}} L$ (see [5, 9.1]).

Denote by $\mathcal{S}(\mathcal{C})$ the class of simple modules in $\mathcal{C}$, and let $N \in \sigma[M]$. Then denote

$$\mathrm{rad}_{\mathcal{S}(\mathcal{C})}(N) = \mathrm{Rej}(N, \mathcal{S}(\mathcal{C})) = \bigcap \{\mathrm{Ker}(f) \mid f : N \longrightarrow S, S \in \mathcal{S}(\mathcal{C})\}$$

(see [5, 10.1]). Note that $\mathrm{rad}_{\mathcal{S}(\mathcal{C})}$ is a radical, $\mathrm{Rad} \leq \mathrm{rad}_{\mathcal{S}(\mathcal{C})}$ and $\mathrm{rad}_{\mathcal{S}(\mathcal{C})}(N) = N$ if and only if N has no nonzero simple factor modules in $\mathcal{C}$ (see [5, 10.2]).

On the other hand, if $\mathcal{C}$ is closed under submodules and factor modules, then we also have

$$\mathrm{rad}_{\mathcal{S}(\mathcal{C})}(N) = \sum \{L \leq N \mid L \ll_{\mathcal{C}} N\}$$

(see [5, 10.3]). Then $\mathrm{rad}_{\mathcal{S}(\mathcal{C})}(N) = N$ if and only if every finitely generated submodule of N is $\mathcal{C}$-small in N (see [5, 10.2 and 10.3]).

2.2. Proper classes

Let $\mathcal{A}$ be an abelian category. We now recall the definition of proper classes from [2] (also see [5, Chapter 2, Section 3]). Let $\mathcal{P}$ be a class of short exact sequences in $\mathcal{A}$. If

$$0 \longrightarrow A \stackrel{f}{\longrightarrow} B \stackrel{g}{\longrightarrow} C \longrightarrow 0$$

belongs to $\mathcal{P}$, then f is called a $\mathcal{P}$-*monomorphism* and g a $\mathcal{P}$-*epimorphism.*

A class $\mathcal{P}$ of short exact sequences in $\mathcal{A}$ is called a *proper class* if it satisfies the following axioms:

(P1) $\mathcal{P}$ is closed under isomorphisms.

(P2) $\mathcal{P}$ contains all splitting short exact sequences of $\mathcal{A}$.

(P3) If f and f' are $\mathcal{P}$-monomorphisms and the composition $f'f$ is defined, then $f'f$ is a $\mathcal{P}$-monomorphism.

(P4) If f and f' are monomorphisms such that the composition $f'f$ is defined and is a $\mathcal{P}$-monomorphism, then f is a $\mathcal{P}$-monomorphism.

(P5) If g and g' are $\mathcal{P}$-epimorphisms and the composition gg' is defined, then gg' is a $\mathcal{P}$-epimorphism.

(P6) If g and g' are epimorphisms such that the composition gg' is defined and is a $\mathcal{P}$-epimorphism, then g is a $\mathcal{P}$-epimorphism.

Proper classes in $\mathcal{A}$ are partially ordered by the relation defined by $\mathcal{P} \subseteq \mathcal{P}'$ if and only if every short exact sequence in $\mathcal{P}$ belongs to $\mathcal{P}'$.

The class of all short exact sequences in $\sigma[M]$

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

such that A is a $\mathcal{C}$-supplement in B, is a proper class and it is denoted by $\mathcal{C}$-Suppl (see [5, 9.2]). The class of all short exact sequences in $\sigma[M]$

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

such that A is a $\mathrm{rad}_{\mathcal{S}(\mathcal{C})}$-supplement in B, is a proper class and it is denoted by $\mathrm{rad}_{\mathcal{S}(\mathcal{C})}$-Suppl (see [5, 10.4]). If $\mathcal{C} = \sigma[M]$, then $\mathrm{rad}_{\mathcal{S}(\mathcal{C})} = \mathrm{Rad}$. And if $\mathcal{C}$ is a class in $\sigma[M]$ closed under submodules and factor modules, then $\mathcal{C}$-Suppl $\subseteq \mathrm{rad}_{\mathcal{S}(\mathcal{C})}$-Suppl (see [5, 10.7]).

3. Class inclusions

Following [5, 10.20] and keeping the notation from the introduction, we consider the radicals

$$\alpha = \mathrm{rad}_{\mathcal{S}(\mathscr{I})},$$

$$\beta = \mathrm{rad}_{\mathcal{S}(\mathscr{P})},$$

$$\gamma = \mathrm{rad}_{\mathcal{S}(\mathscr{M})},$$

$$\delta = \mathrm{rad}_{\mathcal{S}(\mathscr{S})}$$

of $\sigma[M]$ associated to the corresponding classes of simple modules which are M-injective, M-projective, M-small and M-singular respectively. Note that $\mathrm{Rad} = \alpha \cap \gamma = \beta \cap \delta$ and α is an idempotent radical (see [5, 10.22]).

Then we also have the proper classes α-Suppl, β-Suppl, γ-Suppl and δ-Suppl associated to the radicals α, β, γ and δ.

We begin with four class inclusions for which we have equivalent conditions in terms of both associated radicals and associated proper classes. We first recall the following result, which motivated our study.

Proposition 3.1. *[5, Proposition 10.26] Let M be a module. The following conditions are equivalent:*

1. $\mathcal{S}(\mathscr{S}) \subseteq \mathcal{S}(\mathscr{I})$, i.e. every M-singular simple module is M-injective.

2. $\alpha \leq \delta$.

3. α-Suppl $\subseteq \delta$-Suppl.

Proposition 3.2. *Let M be a module. The following are equivalent:*

1. $\mathcal{S}(\mathscr{M}) \subseteq \mathcal{S}(\mathscr{P})$, i.e. every M-small simple module is M-projective.

2. $\beta \leq \gamma$.

3. β-Suppl $\subseteq \gamma$-Suppl.

Proof. We only need to prove (3) $\Rightarrow$ (1). Let S be an M-small simple module. Then $\gamma(S) = 0$. Since S is simple, we have $\beta(S) = 0$ or $\beta(S) = S$. Assume that $\beta(S) = S$. Then the short exact sequence

$$0 \longrightarrow S \longrightarrow E_M(S) \longrightarrow E_M(S)/S \longrightarrow 0$$

belongs to β-Suppl, and hence it belongs to γ-Suppl by hypothesis, where $E_M(S)$ is the M-injective hull of S in $\sigma[M]$. So, there exists a submodule T of $E_M(S)$ such that $E_M(S) = S + T$ and $S \cap T = \gamma(S) = 0$. This means that S is M-injective, a contradiction. Therefore $\beta(S) = 0$. Now there exists a nonzero homomorphism $f : S \longrightarrow S^{'}$ with $S^{'}$ M-projective simple. Thus S is M-projective. $\square$

One obtains the following two propositions in a similar way.

Proposition 3.3. *Let M be a module. The following are equivalent:*

1. $\mathcal{S}(\mathscr{P}) \subseteq \mathcal{S}(\mathscr{I})$, i.e. every M-projective simple module is M-injective.

2. $\alpha \leq \beta$.

3. α-Suppl $\subseteq \beta$-Suppl.

Proposition 3.4. *Let M be a module. The following are equivalent:*

1. $\mathcal{S}(\mathscr{M}) \subseteq \mathcal{S}(\mathscr{S})$, i.e. every M-small simple module is M-singular.

2. $\delta \leq \gamma$.

3. δ-Suppl $\subseteq \gamma$-Suppl.

The remaining four possible class inclusions may only be partially given in general as above.

Proposition 3.5. *Let M be a module. Consider the following conditions:*

1. $\mathcal{S}(\mathscr{I}) \subseteq \mathcal{S}(\mathscr{S})$, *i.e. every M-injective simple module is M-singular.*

2. $\delta \leq \alpha$.

3. δ-*Suppl* $\subseteq \alpha$-*Suppl.*

Then (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3).

Proof. For (1) $\Rightarrow$ (2) $\Rightarrow$ (3) see [5, Proposition 10.27].

(2) $\Rightarrow$ (1) Let S be an M-injective simple module. Then $\alpha(S) = 0$. Since $\delta \leq \alpha$, we have $\delta(S) = 0$. Now there exists a nonzero homomorphism $f : S \longrightarrow S'$ with S' simple M-singular, because $\delta(S) \neq S$. Therefore S is M-singular. $\qquad\square$

One obtains the following three propositions in a similar way.

Proposition 3.6. *Let M be a module. Consider the following conditions:*

1. $\mathcal{S}(\mathscr{P}) \subseteq \mathcal{S}(\mathscr{M})$, *i.e. every M-projective simple module is M-small.*

2. $\gamma \leq \beta$.

3. γ-*Suppl* $\subseteq \beta$-*Suppl.*

Then (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3).

Proposition 3.7. *Let M be a module. Consider the following conditions:*

1. $\mathcal{S}(\mathscr{I}) \subseteq \mathcal{S}(\mathscr{P})$, *i.e. every M-injective simple module is M-projective.*

2. $\beta \leq \alpha$.

3. β-*Suppl* $\subseteq \alpha$-*Suppl.*

Then (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3).

Proposition 3.8. *Let M be a module. Consider the following conditions:*

1. $\mathcal{S}(\mathscr{S}) \subseteq \mathcal{S}(\mathscr{M})$, *i.e. every M-singular simple module is M-small.*

2. $\gamma \leq \delta$.

3. γ-*Suppl* $\subseteq \delta$-*Suppl.*

Then (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3).

Obviously, the equalities $\mathcal{S}(\mathscr{I}) = \mathcal{S}(\mathscr{P})$ and $\mathcal{S}(\mathscr{M}) = \mathcal{S}(\mathscr{S})$ hold if and only if the associated radicals are equal if and only if the associated proper classes coincide.

In what follows we are interested in conditions under which we have equivalences in Propositions 3.5, 3.6, 3.7 and 3.8.

Let τ be a preradical of $\sigma[M]$. Recall that an epimorphism $f : P \longrightarrow L$ is called a *projective τ-cover* of L in $\sigma[M]$ if $\text{Ker}(f) \subseteq \tau(P)$ and P is projective in $\sigma[M]$ (see [1, 2.11]).

Proposition 3.9. *Let M be a module. Assume that the class*

$$\mathbb{T}_\gamma = \{X \in \sigma[M] \mid \gamma(X) = X\}$$

is closed under submodules and $\delta\gamma(P) = 0$ for every projective module in $\sigma[M]$. If every simple module has a projective γ-cover in $\sigma[M]$, then the implication (3) $\Rightarrow$ (1) holds in Proposition 3.8.

Proof.   Assume that γ-Suppl $\subseteq \delta$-Suppl. Let S be an M-singular simple module. So, we have $\delta(S) = 0$. If $\gamma(S) = 0$, then S will be M-small. Suppose the contrary. Then $\gamma(S) = S$, because S is simple. Since every simple module has a projective γ-cover in $\sigma[M]$, there exists an epimorphism $f : P \longrightarrow S$ such that $\mathrm{Ker}(f) \subseteq \gamma(P)$ and P is projective in $\sigma[M]$. By [1, Lemma 2.12], $f(\gamma(P)) = \gamma(S) = S$. Now we have $P = \gamma(P) + \mathrm{Ker}(f) = \gamma(P)$. Since $\mathbb{T}_\gamma$ is closed under submodules, we have $\gamma(\mathrm{Ker}(f)) = \mathrm{Ker}(f)$, and hence the short exact sequence

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow P \longrightarrow S \longrightarrow 0$$

belongs to γ-Suppl. By hypothesis, there exists a submodule L of P such that $P = \mathrm{Ker}(f) + L$ and $L \cap \mathrm{Ker}(f) = \delta(\mathrm{Ker}(f))$. On the other hand, $\delta(P) = \delta\gamma(P) = 0$ and hence $\delta(\mathrm{Ker}(f)) = 0$. Therefore $P = \mathrm{Ker}(f) \oplus L$. This implies that S is M-projective, a contradiction. Therefore $\gamma(S) = 0$, and hence S is M-small. $\square$

In a similar manner one obtains the following three propositions.

Proposition 3.10. *Let M be a module. Assume that the class*

$$\mathbb{T}_\beta = \{X \in \sigma[M] \mid \beta(X) = X\}$$

is closed under submodules and $\alpha\beta(P) = 0$ for every projective module in $\sigma[M]$. If every simple module has a projective β-cover in $\sigma[M]$, then the implication (3) $\Rightarrow$ (1) holds in Proposition 3.7.

Proposition 3.11. *Let M be a module. Assume that the class*

$$\mathbb{T}_\gamma = \{X \in \sigma[M] \mid \gamma(X) = X\}$$

is closed under submodules and $\beta\gamma(P) = 0$ for every projective module in $\sigma[M]$. If every simple module has a projective γ-cover in $\sigma[M]$, then the implication (3) $\Rightarrow$ (1) holds in Proposition 3.6.

Proposition 3.12. *Let M be a module. Assume that the class*

$$\mathbb{T}_\delta = \{X \in \sigma[M] \mid \delta(X) = X\}$$

is closed under submodules and $\alpha\delta(P) = 0$ for every projective module in $\sigma[M]$. If every simple module has a projective δ-cover in $\sigma[M]$, then the implication (3) $\Rightarrow$ (1) holds in Proposition 3.5.

Example 3.13. (see [7, Example 4.1]) Consider the category Mod-R. Let $Q = \prod_{i=1}^{\infty} F_i$, where each $F_i = \mathbb{Z}_2$. Let R be the subring of Q generated by $\oplus_{i=1}^{\infty} F_i$ and 1_Q. By [7, Example 4.1], every simple R-module has a projective δ-cover, while

$$F_0 = R/(\oplus_{i=1}^{\infty} F_i), F_1, F_2, \ldots$$

are the only simple R-modules and F_0 is the only singular simple R-module. Therefore $F_1, F_2, \ldots$ are the only projective simple R-modules. This means that every simple R-module has a projective β-cover except for F_0. For, assume that F_0 has a projective β-cover $f : P \longrightarrow F_0$. Since F_0 is simple singular, $\beta(F_0) = F_0$. Therefore $\beta(P) = P = P\beta(R_R)$. Note that

$$\beta(Q_R) = \beta \left(\prod_{i=1}^{\infty} F_i \right) \subseteq \prod_{i=1}^{\infty} \beta(F_i) = 0$$

implies that $\beta(R_R) = 0$. So $P = 0$, which is a contradiction. Therefore F_0 does not have a projective β-cover.

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