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Classification of optimal quaternary Hermitian LCD codes of dimension 2

Research Article

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Abstract: Hermitian linear complementary dual codes are linear codes whose intersections with their Hermitian dual codes are trivial. The largest minimum weight among quaternary Hermitian linear complementary dual codes of dimension 2 is known for each length. We give the complete classification of optimal quaternary Hermitian linear complementary dual codes of dimension 2.

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1. Introduction

Let $\mathbb{F}_4 := \{0, 1, \omega, \omega^2\}$ be the finite field of order four, where ω satisfies $\omega^2 + \omega + 1 = 0$. The conjugate of $x \in \mathbb{F}_4$ is defined as $\overline{x} := x^2$. A quaternary $[n, k, d]$ code is a linear subspace of $\mathbb{F}_4^n$ with dimension k and minimum weight d. Throughout this paper, we consider only linear quaternary codes and omit the term "linear quaternary". Given a code C, a vector $c \in C$ is said to be a codeword of C. The weight of a codeword c is denoted by $\mathrm{wt}(c)$.

Let $u := (u_1, u_2, \dots, u_n)$, $v := (v_1, v_2, \dots, v_n)$ be vectors of $\mathbb{F}_4^n$. The Hermitian inner product is defined as $(u, v)_h := u_1\overline{v_1} + u_2\overline{v_2} + \cdots + u_n\overline{v_n}$. Given a code C, the Hermitian dual code of C is $C^{\perp h} := \{x \in \mathbb{F}_4^n \mid (x, y)_h = 0 \text{ for all } y \in C\}$. A generator matrix of the code C is any matrix whose rows form a basis of C. Moreover, a generator matrix of the Hermitian dual code $C^{\perp h}$ is said to be a parity check matrix of C. Given a matrix G, we denote the transpose of G by G^T and the conjugate of G by $\overline{G}$. Hermitian linear complementary dual codes, Hermitian LCD codes for short, are codes whose intersections with their Hermitian dual codes are trivial. The concept of LCD codes was invented by Massey [7] in 1992. LCD codes have been applied in data storage, communication systems and cryptography. For example, it is known that LCD codes can be used against side-channel attacks and

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fault injection attacks [3]. We note that a code C is a Hermitian LCD code if and only if its generator matrix G satisfies $\det G\overline{G}^T \neq 0$ [5].

Two codes C, C' are equivalent if one can be obtained from the other by a permutation of the coordinates and a multiplication of any coordinate by a nonzero scalar. We denote the equivalence of two codes C, C' by $C \simeq C'$. Let G, G' be generator matrices of two codes C, C' respectively. It is known that $C \simeq C'$ if and only if G can be obtained from G' by an elementary row operation, a permutation of the columns and multiplication of any column by a nonzero scalar.

It was shown in [6] that the upper bound of the minimum weight of the Hermitian LCD $[n, 2, d]$ codes is given as follows:

$$d \leq \begin{cases} \left\lfloor \frac{4n}{5} \right\rfloor & \text{if } \ n \equiv 1, 2, 3 \pmod 5, \\ \left\lfloor \frac{4n}{5} \right\rfloor - 1 & \text{otherwise.} \end{cases} \tag{1}$$

Also, it was proved that for all $n \neq 1$, there exists a Hermitian LCD $[n, 2, d]$ code which meets this upper bound. We say that a Hermitian LCD $[n, 2, d]$ code is optimal if it meets this upper bound. It was shown in [4] that any code over $\mathbb{F}_{q^2}$ is equivalent to some Hermitian LCD code for $q \geq 3$. Furthermore, it was proved in [6] that a Hermitian LCD code leads to a construction of a maximal-entanglement entanglement-assisted quantum error correcting code. Motivated by the results, we are concerned with the complete classification of optimal Hermitian LCD codes of dimension 2.

This paper is organized as follows. In Section 2, we present a method to construct optimal Hermitian LCD codes of dimension 2, including all inequivalent codes. Also, a method to classify optimal Hermitian LCD codes of dimension 2 is given. In Section 3, we classify optimal Hermitian LCD codes of dimension 2. Up to equivalence, the complete classification of optimal Hermitian LCD codes of dimension 2 is given. It is shown that all inequivalent codes have distinct weight enumerators, which is used for the classification.

2. Classification method

Let $\mathbf{0}_n$ be the zero vector of length n and $\mathbf{1}_n$ be the all-ones vector of length n. Let $(a_0, a_1, a_2, a_3, a_4, a_5)$ be a tuple of nonnegative integers. We introduce the following notation:

$$G(a_0, a_1, a_2, a_3, a_4, a_5) := \begin{pmatrix} 1 & 0 & \mathbf{0}_{a_0} & \mathbf{0}_{a_1} & \mathbf{1}_{a_2} & \mathbf{1}_{a_3} & \mathbf{1}_{a_4} & \mathbf{1}_{a_5} \\ 0 & 1 & \mathbf{0}_{a_0} & \mathbf{1}_{a_1} & \mathbf{0}_{a_2} & \mathbf{1}_{a_3} & \omega\mathbf{1}_{a_4} & \omega^2\mathbf{1}_{a_5} \end{pmatrix}.$$

We denote by $C(a_0, a_1, a_2, a_3, a_4, a_5)$ the code whose generator matrix is $G(a_0, a_1, a_2, a_3, a_4, a_5)$. By the same argument as in [1], we obtain the following lemma.

Lemma 2.1. *Given a code C, define $C^* := \{(x, 0) \mid x \in C\}$. Let $\mathcal{C}^*_{n,k}$ denote the set of all inequivalent Hermitian LCD $[n, k]$ codes C such that the minimum weight of $C^{\perp h}$ is 1. Then there exists a set $\mathcal{C}_{n-1,k}$ of all inequivalent Hermitian LCD $[n-1, k]$ codes such that $\mathcal{C}^*_{n,k} = \{C^* \mid C \in \mathcal{C}_{n-1,k}\}$.*

We assume $a_0 = 0$ by Lemma 2.1 and omit a_0. Furthermore, throughout this paper, we use the following notations:

$$G(\boldsymbol{a}) := G(a_1, a_2, a_3, a_4, a_5), \ C(\boldsymbol{a}) := C(a_1, a_2, a_3, a_4, a_5), \tag{2}$$

respectively, to save space.

Proposition 2.2. *Let C be an $[n, 2, d]$ code. Then there exist nonnegative integers a_1, a_2, a_3, a_4, a_5 such that $C \simeq C(\boldsymbol{a})$ and $1 + a_2 + a_3 + a_4 + a_5 = d$.*

Proof. Let G be a generator matrix of the code C. By multiplying rows by some non-zero scalars, G is changed to a generator matrix which consists only of the columns of $G(\boldsymbol{a})$. Permuting the columns,

$G(\boldsymbol{a})$ is obtained from G. Hence it holds that $C \simeq C(\boldsymbol{a})$. Since the minimum weight of C is d, we may assume that the first row of G is a codeword with weight d, which yields $1 + a_2 + a_3 + a_4 + a_5 = d$. $\square$

Given a code $C(\boldsymbol{a})$, we may assume without loss of generality that $G(\boldsymbol{a})$ satisfies

$$1 + a_2 + a_3 + a_4 + a_5 = d, \tag{3}$$

by Proposition 2.2. This assumption on a generator matrix reduces computations later.

Lemma 2.3. *Let C be an $[n, 2, d]$ code $C(\boldsymbol{a})$. Then C is a Hermitian LCD code if and only if C satisfies the following conditions:*

$$a_1 = n - d - 1, \tag{4}$$
$$a_2 \leq n - d - 1, \tag{5}$$
$$a_3, a_4, a_5 \leq n - d, \tag{6}$$
$$\begin{cases} a_3 + a_4 + a_5 + a_3a_4 + a_4a_5 + a_5a_3 \not\equiv 0 \pmod 2 & \text{if } d \text{ is even}, \\ a_3a_4 + a_4a_5 + a_5a_3 \not\equiv n - d \pmod 2 & \text{otherwise}. \end{cases} \tag{7}$$

Proof. Suppose C is a Hermitian LCD $[n, 2, d]$ code. Let G be a generator matrix of the code C. Let r_1, r_2 be the first and second rows of G respectively. The number of columns of G equals to the length n. Thus, it holds that

$$2 + a_1 + a_2 + a_3 + a_4 + a_5 = n. \tag{8}$$

Since the minimum weight of C is d, the following holds: $\mathrm{wt}(r_2) \geq d$, $\mathrm{wt}(r_1 + r_2) \geq d$, $\mathrm{wt}(r_1 + \omega r_2) \geq d$, $\mathrm{wt}(r_1 + \omega^2 r_2) \geq d$. By (3), we have $\mathrm{wt}(r_1) = d$. Substituting (8) in each equation, we obtain (4) through (6).

The code C is a Hermitian LCD code if and only if

$$\det G\overline{G}^T = (r_1, r_1)_h (r_2, r_2)_h - (r_2, r_1)_h (r_1, r_2)_h \neq 0,$$

where

$$\begin{aligned} (r_1, r_1)_h &= 1 + a_2 + a_3 + a_4 + a_5 = d, \\ (r_1, r_2)_h &= a_3 + \omega a_5 + \omega^2 a_4 = a_3 + a_4 + \omega(a_4 + a_5), \\ (r_2, r_1)_h &= a_3 + \omega a_4 + \omega^2 a_5 = a_3 + a_5 + \omega(a_4 + a_5), \\ (r_2, r_2)_h &= 1 + a_1 + a_3 + a_4 + a_5. \end{aligned}$$

Here we regard n, d, a_3, a_4, a_5 as elements of $\mathbb{F}_4$. Therefore, (4) through (7) hold if C is a Hermitian LCD $[n, 2, d]$ code and vice versa. $\square$

Given an $[n, 2, d]$ code $C(\boldsymbol{a})$, we define the following:

$$b_i \coloneqq (n - d) - a_i \text{ for } 1 \leq i \leq 5. \tag{9}$$

Lemma 2.4. *Let C be an $[n, 2, d]$ code $C(\boldsymbol{a})$. Then C is a Hermitian LCD code if and only if C satisfies the following conditions:*

$$b_1 = 1, \tag{10}$$
$$b_2 \geq 1, \tag{11}$$
$$b_3, b_4, b_5 \geq 0,$$
$$\begin{cases} b_3 + b_4 + b_5 + b_3b_4 + b_4b_5 + b_5b_3 \not\equiv 0 \pmod 2 & \text{if } d \text{ is even}, \\ b_3b_4 + b_4b_5 + b_5b_3 \not\equiv 0 \pmod 2 & \text{otherwise}. \end{cases}$$

Proof. The result follows from Lemma 2.3. □

Given an $[n, 2, d]$ code $C(\boldsymbol{a})$, we define the following:

$$\Delta := 4n - 5d. \tag{12}$$

Lemma 2.5. *Let C be a Hermitian LCD $[n, 2, d]$ code. Then C is optimal if and only if the value of Δ with respect to n is given as follows:*

$$\Delta = \begin{cases} 5 & \text{if } n \equiv 0 \pmod 5, \\ 4 & \text{if } n \equiv 1 \pmod 5, \\ 3 & \text{if } n \equiv 2 \pmod 5, \\ 2 & \text{if } n \equiv 3 \pmod 5, \\ 6 & \text{if } n \equiv 4 \pmod 5. \end{cases}$$

Proof. The result follows from (12). □

Lemma 2.6. *Let C be a code $C(\boldsymbol{a})$. If C is a Hermitian LCD code, then it holds that*

$$0 \le b_3, b_4, b_5 \le \Delta.$$

Proof. Substituting $b_2, b_3, b_4, b_5, \Delta$ in (3), we obtain

$$b_2 = \Delta + 1 - (b_3 + b_4 + b_5). \tag{13}$$

Combining with (11), we obtain $b_3+b_4+b_5 \le \Delta$. Since $0 \le b_3, b_4, b_5$, it follows that $0 \le b_3, b_4, b_5 \le \Delta$. □

Lemma 2.7.

$$C(\boldsymbol{a}) \simeq C(a_1, a_2, a_5, a_3, a_4). \tag{14}$$

Proof. Multiply the second row of $G(\boldsymbol{a})$ by ω. Permuting the columns, the result follows. Note that we may assume that the nonzero entry of a column is 1, provided that the entry of the other column is 0. □

By Lemma 2.7, we may assume $a_3 \ge a_4, a_5$. Notice that $a_3 \ge a_4, a_5$ if and only if $b_3 \le b_4, b_5$ by (9).

3. Optimal Hermitian LCD codes of dimension 2

By Lemmas 2.4 through 2.7, it suffices to calculate all b_3, b_4, b_5 satisfying

$$0 \le b_3 \le b_4, b_5 \le \Delta, \tag{15}$$

$$\begin{cases} b_3 + b_4 + b_5 + b_3 b_4 + b_4 b_5 + b_5 b_3 \ne 0 & \text{if } d \text{ is even}, \\ b_3 b_4 + b_4 b_5 + b_5 b_3 \ne 0 & \text{otherwise}, \end{cases} \tag{16}$$

in order to obtain optimal Hermitian LCD codes of dimension 2, including all inequivalent codes. Notice that b_1, b_2 are obtained by (10), (13) respectively. Our computer search found all integers b_3, b_4, b_5 satisfying (15) and (16). This calculation was done by Magma [2]. Recall that a_1, a_2, a_3, a_4, a_5 are obtained from b_1, b_2, b_3, b_4, b_5 by (9). For optimal Hermitian LCD codes of dimension 2, the integers a_1, a_2, a_3, a_4, a_5 are listed in Table 1, where the rows are in lexicographical order with respect to a_1, a_2, a_3, a_4, a_5, and m is a nonnegative integer.

Table 1: Optimal Hermitian LCD codes of dimension 2

n	Code	$(a_1, a_2, a_3, a_4, a_5)$	m
$n = 5m$	$C_{5m,1}$	$(m, m, m, m, m-2)$	$m \geq 2$
	$C_{5m,2}$	$(m, m, m, m-2, m)$	$m \geq 2$
	$C_{5m,3}$	$(m, m-1, m+1, m, m-2)$	$m \geq 2$
	$C_{5m,4}$	$(m, m-1, m+1, m-2, m)$	$m \geq 2$
	$C_{5m,5}$	$(m, m-1, m, m, m-1)$	$m \geq 1$
	$C_{5m,6}$	$(m, m-1, m, m-1, m)$	$m \geq 1$
	$C_{5m,7}$	$(m, m-2, m, m, m)$	$m \geq 2$
	$C_{5m,8}$	$(m, m-3, m+1, m, m)$	$m \geq 3$
$n = 5m+1$	$C_{5m+1,1}$	$(m, m, m+1, m, m-2)$	$m \geq 2$
	$C_{5m+1,2}$	$(m, m, m+1, m-2, m)$	$m \geq 2$
	$C_{5m+1,3}$	$(m, m, m, m, m-1)$	$m \geq 1$
	$C_{5m+1,4}$	$(m, m, m, m-1, m)$	$m \geq 1$
	$C_{5m+1,5}$	$(m, m-1, m+1, m+1, m-2)$	$m \geq 2$
	$C_{5m+1,6}$	$(m, m-1, m+1, m, m-1)$	$m \geq 1$
	$C_{5m+1,7}$	$(m, m-1, m+1, m-1, m)$	$m \geq 1$
	$C_{5m+1,8}$	$(m, m-1, m+1, m-2, m+1)$	$m \geq 2$
	$C_{5m+1,9}$	$(m, m-2, m+1, m, m)$	$m \geq 2$
	$C_{5m+1,10}$	$(m, m-3, m+1, m+1, m)$	$m \geq 3$
	$C_{5m+1,11}$	$(m, m-3, m+1, m, m+1)$	$m \geq 3$
$n = 5m+2$	$C_{5m+2,1}$	(m, m, m, m, m)	$m \geq 0$
	$C_{5m+2,2}$	$(m, m-1, m+1, m, m)$	$m \geq 1$
$n = 5m+3$	$C_{5m+3,1}$	$(m, m-1, m+1, m+1, m)$	$m \geq 1$
	$C_{5m+3,2}$	$(m, m, m+1, m, m)$	$m \geq 0$
	$C_{5m+3,3}$	$(m, m-1, m+1, m, m+1)$	$m \geq 1$
$n = 5m+4$	$C_{5m+4,1}$	$(m+1, m+1, m+1, m+1, m-2)$	$m \geq 2$
	$C_{5m+4,2}$	$(m+1, m+1, m+1, m, m-1)$	$m \geq 1$
	$C_{5m+4,3}$	$(m+1, m+1, m+1, m-1, m)$	$m \geq 1$
	$C_{5m+4,4}$	$(m+1, m+1, m+1, m-2, m+1)$	$m \geq 2$
	$C_{5m+4,5}$	$(m+1, m+1, m+2, m+1, m-3)$	$m \geq 3$
	$C_{5m+4,6}$	$(m+1, m+1, m+2, m-1, m-1)$	$m \geq 1$
	$C_{5m+4,7}$	$(m+1, m+1, m+2, m-3, m+1)$	$m \geq 3$
	$C_{5m+4,8}$	$(m+1, m, m+1, m, m)$	$m \geq 0$
	$C_{5m+4,9}$	$(m+1, m, m+2, m+1, m-2)$	$m \geq 2$
	$C_{5m+4,10}$	$(m+1, m, m+2, m+2, m-3)$	$m \geq 3$
	$C_{5m+4,11}$	$(m+1, m, m+2, m, m-1)$	$m \geq 1$
	$C_{5m+4,12}$	$(m+1, m, m+2, m-1, m)$	$m \geq 1$
	$C_{5m+4,13}$	$(m+1, m, m+2, m-2, m+1)$	$m \geq 2$
	$C_{5m+4,14}$	$(m+1, m, m+2, m-3, m+2)$	$m \geq 3$
	$C_{5m+4,15}$	$(m+1, m-1, m+1, m+1, m)$	$m \geq 1$
	$C_{5m+4,16}$	$(m+1, m-1, m+1, m, m+1)$	$m \geq 1$
	$C_{5m+4,17}$	$(m+1, m-1, m+2, m+1, m-1)$	$m \geq 1$
	$C_{5m+4,18}$	$(m+1, m-1, m+2, m-1, m+1)$	$m \geq 1$
	$C_{5m+4,19}$	$(m+1, m-2, m+2, m+1, m)$	$m \geq 2$
	$C_{5m+4,20}$	$(m+1, m-2, m+2, m+2, m-1)$	$m \geq 2$
	$C_{5m+4,21}$	$(m+1, m-2, m+2, m, m+1)$	$m \geq 2$
	$C_{5m+4,22}$	$(m+1, m-2, m+2, m-1, m+2)$	$m \geq 2$
	$C_{5m+4,23}$	$(m+1, m-3, m+2, m+1, m+1)$	$m \geq 3$
	$C_{5m+4,24}$	$(m+1, m-4, m+2, m+1, m+2)$	$m \geq 4$
	$C_{5m+4,25}$	$(m+1, m-4, m+2, m+2, m+1)$	$m \geq 4$

Lemma 3.1. *Suppose a_3 is a positive integer. Then*

$$C(\boldsymbol{a}) \simeq C(a_1, a_3 - 1, a_2 + 1, a_5, a_4). \tag{17}$$

Proof. Add the second row of $G(\boldsymbol{a})$ to the first row. Permuting the columns, the result follows. Recall that $C(\boldsymbol{a})$ is defined in (2). $\square$

Table 2. Equivalent optimal Hermitian LCD codes of dimension 2

n	Code
$n = 5m$	$C_{5m,7} \simeq_{17,14} C_{5m,6} \simeq_{14} C_{5m,5}$
	$C_{5m,8} \simeq_{17,14,17} C_{5m,3} \simeq_{17} C_{5m,2} \simeq_{14} C_{5m,1} \simeq_{17} C_{5m,4}$
$n = 5m+1$	$C_{5m+1,9} \simeq_{17,14,17} C_{5m+1,6} \simeq_{17} C_{5m+1,4} \simeq_{14} C_{5m+1,3} \simeq_{17} C_{5m+1,7}$
	$C_{5m+1,11} \simeq_{17,14,17} C_{5m+1,5} \simeq_{17,14} C_{5m+1,1} \simeq_{18} C_{5m+1,2}$
	$\simeq_{14,17} C_{5m+1,8} \simeq_{17,14,17} C_{5m+1,10}$
$n = 5m+2$	$C_{5m+2,1} \simeq_{17} C_{5m+2,2}$
$n = 5m+3$	$C_{5m+3,1} \simeq_{17,14} C_{5m+3,2} \simeq_{14,17} C_{5m+3,3}$
$n = 5m+4$	$C_{5m+4,8} \simeq_{14,17} C_{5m+4,16} \simeq_{14} C_{5m+4,15}$
	$C_{5m+4,6} \simeq_{14,17} C_{5m+4,22} \simeq_{14} C_{5m+4,20}$
	$C_{5m+4,21} \simeq_{17,14,17} C_{5m+4,17} \simeq_{17,14,17} C_{5m+4,12}$
	$\simeq_{17} C_{5m+4,2} \simeq_{18} C_{5m+4,3} \simeq_{17} C_{5m+4,11}$
	$\simeq_{17,14,14,17} C_{5m+4,19} \simeq_{17,14,14,17} C_{5m+4,18}$
	$C_{5m+4,23} \simeq_{17,14,17} C_{5m+4,9}$
	$\simeq_{17} C_{5m+4,4} \simeq_{18} C_{5m+4,1} \simeq_{17} C_{5m+4,13}$
	$C_{5m+4,24} \simeq_{17,14,17} C_{5m+4,10} \simeq_{17,14} C_{5m+4,5} \simeq_{18} C_{5m+4,7}$
	$\simeq_{14,17} C_{5m+4,14} \simeq_{17,14,17} C_{5m+4,25}$

Lemma 3.2. *Suppose* $1 + a_1 + a_3 + a_4 + a_5 = d$. *Then*

$$C(\boldsymbol{a}) \simeq C(a_2, a_1, a_3, a_5, a_4). \tag{18}$$

Proof. Interchange the first row and the second row of $G(\boldsymbol{a})$. Permuting the columns, the result follows. □

By Lemmas 2.7 through 3.2, we have the equivalences among some codes listed in Table 1, which are displayed in Table 2. Note that $C \simeq_i C'$ denotes the two codes C, C' are equivalent by (i). Also, $C \simeq_{i,j} C'$ denotes that, given two codes C, C', there exists a code C'' such that $C \simeq_i C'' \simeq_j C'$.

Table 3 gives the weight enumerators of representatives in Table 2. The weight enumerator is given by $1 + 3y^{\mathrm{wt}(r_1)} + 3y^{\mathrm{wt}(r_2)} + 3y^{\mathrm{wt}(r_1+r_2)} + 3y^{\mathrm{wt}(r_1+\omega r_2)} + 3y^{\mathrm{wt}(r_1+\omega^2 r_2)}$, where r_1, r_2 be the first and second rows of $G(\boldsymbol{a})$ respectively. Since the weight enumerators are distinct, the codes in Table 3 are inequivalent. Table 4 gives the classification of optimal Hermitian LCD codes of dimension 2, with the case where n is so small that some codes in Table 1 do not exist.

Recall that we have assumed $a_0 = 0$ by Lemma 2.1. It follows from (1) that there exists an optimal Hermitian LCD $[n, 2]$ code C such that the minimum weight of $C^{\perp h}$ equals to 1 if and only if $n \equiv 4 \pmod 5$. Consequently, we obtain the following theorem.

Theorem 3.3. (i) *Up to equivalence, there exist two optimal Hermitian LCD* $[5m, 2, 4m-1]$ *codes for every integer* m *with* $m \geq 2$.

(ii) *Up to equivalence, there exist two optimal Hermitian LCD* $[5m+1, 2, 4m]$ *codes for every integer* m *with* $m \geq 2$.

(iii) *Up to equivalence, there exists a unique optimal Hermitian LCD* $[5m+2, 2, 4m+1]$ *code for every integer* m *with* $m \geq 0$.

(iv) *Up to equivalence, there exists a unique optimal Hermitian LCD* $[5m+3, 2, 4m+2]$ *code for every integer* m *with* $m \geq 0$.

(v) *Up to equivalence, there exist six optimal Hermitian LCD* $[5m+4, 2, 4m+2]$ *codes for every integer* m *with* $m \geq 3$. *One of them is the code such that the minimum weight of the Hermitian dual code is* 1.

Table 3. Weight enumerators of representatives

n	Code	Weight Enumerator
$n = 5m$	$C_{5m,7}$	$1 + 3y^{4m-1} + 9y^{4m} + 3y^{4m+1}$
	$C_{5m,8}$	$1 + 6y^{4m-1} + 6y^{4m} + 3y^{4m+2}$
$n = 5m + 1$	$C_{5m+1,9}$	$1 + 6y^{4m} + 6y^{4m+1} + 3y^{4m+2}$
	$C_{5m+1,1}$	$1 + 9y^{4m} + 3y^{4m+1} + 3y^{4m+3}$
$n = 5m + 2$	$C_{5m+2,1}$	$1 + 6y^{4m+1} + 6y^{4m+2} + 3y^{4m+3}$
$n = 5m + 3$	$C_{5m+3,1}$	$1 + 9y^{4m+2} + 6y^{4m+3}$
$n = 5m + 4$	$C_{5m+4,8}$	$1 + 3y^{4m+2} + 6y^{4m+3} + 6y^{4m+4}$
	$C_{5m+4,6}$	$1 + 9y^{4m+2} + 6y^{4m+5}$
	$C_{5m+4,21}$	$1 + 6y^{4m+2} + 3y^{4m+3} + 3y^{4m+4} + 3y^{4m+5}$
	$C_{5m+4,23}$	$1 + 6y^{4m+2} + 6y^{4m+3} + 3y^{4m+6}$
	$C_{5m+4,24}$	$1 + 9y^{4m+2} + 3y^{4m+3} + 3y^{4m+7}$

Table 4. Classification of optimal Hermitian LCD codes of dimension 2

n	m	Code
$n = 5m$	$m = 1$	$C_{5m,7}$
	$m \geq 2$	$C_{5m,7}, C_{5m,8}$
$n = 5m + 1$	$m = 1$	$C_{5m+1,9}$
	$m \geq 2$	$C_{5m+1,9}, C_{5m,1}$
$n = 5m + 2$	$m \geq 0$	$C_{5m+2,1}$
$n = 5m + 3$	$m \geq 0$	$C_{5m+3,1}$
$n = 5m + 4$	$m = 0$	$C_{5m+4,8}$
	$m = 1$	$C_{5m+4,8}, C_{5m+4,6}, C_{5m+4,21}$
	$m = 2$	$C_{5m+4,8}, C_{5m+4,6}, C_{5m+4,21}, C_{5m+4,23}$
	$m \geq 3$	$C_{5m+4,8}, C_{5m+4,6}, C_{5m+4,21}, C_{5m+4,23}, C_{5m+4,24}$

4. Concluding remarks

A natural extension of this work is to classify the quaternary Hermitian LCD codes of larger dimensions. In the case where the dimension is 3, there are 64 different codewords. By a method similar to that in Proposition 2.2, the number of column vectors we need consider is reduced to 22. However, this is still a large number. Therefore, it is difficult to extend our method to classify Hermitian LCD codes of larger dimensions.

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