En español

Implementación MEF-DtN para un material incompresible en un dominio no acotado

Liliana Camargo [1] y Jairo Duque [2]

RESUMEN

En este trabajo presentamos la implementación de un método de elementos finitos combinado con la aplicación Dirichlet-to-Neumann (DtN), obtenida en términos de series de Fourier, para estudiar la existencia de soluciones de un problema exterior que proviene de la teoría de elasticidad lineal incompresible bidimensional. Finalmente, se presenta un método numérico que demuestra la precisión de nuestra aplicación DtN, puesto que sólo se necesitan unos cuantos términos de la serie de Fourier para obtener una buena aproximación de la solución. Para la discretización del problema se emplea el elemento estable Taylor-Hood.

Palabras clave: método de elementos finitos mixto, elemento Taylor-Hood, técnica DtN, elasticidad lineal, condición inf-sup.

Recibido: septiembre 4 de 2009
Aceptado: noviembre 15 de 2010

Introducción

En este trabajo se explica un procedimiento para estudiar la aproximación de Galerkin de un material incompresible en un dominio exterior no acotado Ω. El procedimiento emplea la aplicación Dirichlet-to-Neumann (DtN) (Han y Bao, 1997; Gatica, Gatica y Stephan, 2003; Kako y Touda, 2004), que consiste en introducir una frontera artificial dibujando en Ω un círculo Γ_R en $\mathbb{R}^2$ de radio R. Entonces Ω se divide en la parte acotada Ω_i y la no acotada Ω_R. Para resolver el problema en el dominio acotado Ω_i se dan condiciones de frontera exactas y aproximadas sobre la frontera artificial Γ_R (Figura 1).

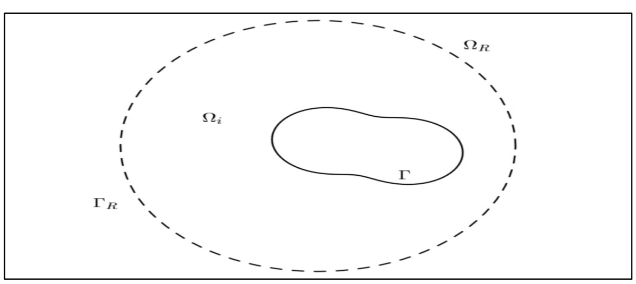

Figura 1. Dominio $\Omega = \Omega_i \cup \Omega_R$ y frontera artificial Γ_R.

Sea $\Omega \subset \mathbb{R}^2$ un dominio no acotado y simplemente conexo con frontera Lipschitz continua Γ. Teniendo en cuenta las condiciones de frontera tipo Dirichlet dadas sobre Γ se define el espacio $H_\Gamma^1 = \{v : \Omega \to \mathbb{R} : v \in H^1(\Omega),\ v|_\Gamma = 0\}$. Así, el problema de elasticidad lineal por resolver consiste en encontrar $(u,p) \in [H_\Gamma^1(\Omega)]^2 \times L^2(\Omega)$ tal que:

[1] Matemática, Universidad del Valle, Cali, Colombia. Universidad del Valle. Lilicaz85@gmail.com.
[2] Dr. rer. nat., Universidad del Valle, Cali, Colombia. jjduque@univalle.edu.co.

In English

Implementing FEM-DtN for an incompressible material on an unbounded domain

Liliana Camargo [3] and Jairo Duque [4]

ABSTRACT

This paper presents the implementation of the finite element method combined with Dirichlet-to-Neumann (DtN) mapping (derived in terms of an infinite Fourier series) for studying the solvability of an exterior problem arising in linear incomepressible 2D-elasticity. A reliable numerical experiment is also presented showing the accuracy of DtN mapping; only a few Fourier series terms were needed to get a good approximation to the solution. The stable Taylor-Hood element was used for finite element discretisation.

Keywords: mixed finite element method, Taylor-Hood element, DtN technique, linear elasticity, inf-sup condition.

Received: september 4th 2009
Accepted: november 15th 2010

Introduction

This article presents a procedure for studying the Galerkin approximation to an incompressible material on unbounded exterior domain Ω. This procedure used Dirichlet-to-Neumann mapping (DtN), (Han and Bao, 1997; Gatica, Gatica and Stephan, 2003; Kako and Touda, 2004), consisting of introducing an artificial boundary by drawing circumference Γ_R in $\mathbb{R}^2$ with radius R in domain Ω; Ω was then divided in bounded part Ω_i and unbounded part Ω_R. Exact and approximate boundary conditions were given on artificial boundary Γ_R to solve the problem on bounded domain (Figure 1).

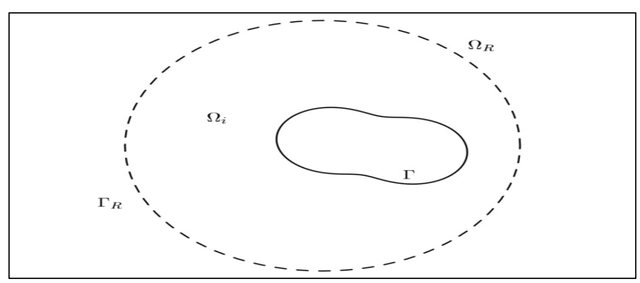

Figure 1. Domain $\Omega = \Omega_i \cup \Omega_R$ and artificial boundary Γ_R

$\Omega \subset \mathbb{R}^2$ was an unbounded and simply connected domain with Lipschitz continuous boundary Γ. Because of Dirichlet boundary conditions on Γ, $H_\Gamma^1 = \{v : \Omega \to \mathbb{R} : v \in H^1(\Omega),\ v|_\Gamma = 0\}$ space was defined. Hence, the problem of linear elasticity consisted in finding $(u,p) \in [H_\Gamma^1(\Omega)]^2 \times L^2(\Omega)$ so that:

[3] Mathematics, Universidad del Valle, Cali, Colombia. Universidad del Valle. Lilicaz85@gmail.com.
[4] Dr. rer. nat., Universidad del Valle, Cali, Colombia. jjduque@univalle.edu.co.

En español

$$-2\mu \operatorname{div} \varepsilon(u) + \operatorname{grad} p = f, \quad \text{in } \Omega,$$

$$\operatorname{div}(u) = 0, \quad \text{in } \Omega, \qquad (1)$$

Aquí $u: \Omega \to \mathbb{R}$ representa el desplazamiento y p es la presión del material en cada punto $x := (x_1, x_2)^T \in \Omega$ por el efecto de la fuerza externa f, $\mu > 0$ es la constante de Lamé y $\varepsilon(u)$ representa el tensor de esfuerzos de componentes $\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, para $i, j = 1, 2$. De este material suponemos válida la relación tensión-esfuerzo para pequeñas deformaciones, tal como se discute en Necas (1986); véanse también Necas (1981) y Zeidler (1988). En este trabajo suponemos que la fuerza f tiene soporte compacto y que está contenida en la región Ω_i.

Sean $X = [H^1_\Gamma(\Omega)]^2$ y $M = L^2(\Omega)$ espacios dotados con las normas de $L^2(\Omega)$ y $H^1(\Omega)$, respectivamente. Entonces, siguiendo el procedimiento estándar de integración por partes, se llega a la siguiente formulación variacional del problema (1):

$$2\mu \int_\Omega \varepsilon(u) : \varepsilon(v)\, dx - \int_\Omega p \operatorname{div} v \; dx = \int_\Omega f \cdot v\, dx$$

$$\int_\Omega q \operatorname{div} v \; dx = 0.$$

Nótese que este problema está definido en el dominio no acotado Ω y Ω_i es el dominio computacional para nuestro método de elementos finitos, que se obtiene al introducir la frontera artificial Γ_R. Entonces es necesario deducir la formulación variacional del problema (1) en Ω_i; nuevamente mediante integración por partes, y teniendo en cuenta que para el tensor distorsión $\sigma(u,p) := 2\mu\varepsilon(u) - pI$, donde I es la matriz identidad de orden 2, se tiene que $-\operatorname{div}\left(\sigma(u,p)\right) = -2\mu \operatorname{div} \varepsilon(u) + \operatorname{grad} p$, y a partir de esto se obtiene la formulación débil:

$$2\mu \int_{\Omega_i} \varepsilon(u) : \varepsilon(v)\, dx - \int_{\Gamma_R} \sigma(u,p) \cdot v \cdot \vec{\eta}\, ds - \int_{\Omega_i} p \operatorname{div} v \; dx = \int_{\Omega_i} f \cdot v\, dx \qquad (2)$$

$$\int_{\Omega_i} q \operatorname{div} v \; dx = 0.$$

El análisis se concentra ahora en la integral de frontera, $\int_{\Gamma_R} \sigma(u,p) \cdot v \cdot \vec{\eta}\, ds$. Nuestro objetivo es obtener una expresión para el tensor $\sigma(u,p)$ a lo largo de la frontera artificial Γ_R. Consideremos entonces la siguiente descomposición del desplazamiento u, en el dominio no acotado $\Omega_R = \left\{x = (x_1, x_2)^T : r = \sqrt{x_1^2 + x_2^2} > R\right\}$,

$$u_j = G_j + (r^2 - R^2)\frac{\partial W}{\partial x_j}, \qquad j = 1, 2, \qquad (3)$$

siendo G_1, G_2 y W funciones armónicas que determinaremos más adelante. En particular, las funciones G_j satisfacen el siguiente problema de valores de frontera:

$$\Delta G_j = 0, \quad \text{en } \Omega_R,$$

$$G_j|_{\Gamma_R} = u_j(R, \theta),$$

$$G_j \text{ es acotado, cuando } r \to +\infty.$$

Nótese que los valores de G_j y u_j coinciden a lo largo de Γ_R. Usando el método de separación de variables obtenemos las siguientes representaciones de G_j en términos de los coeficientes de Fourier:

$$G_1 = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\theta) + b_n \operatorname{sen}(n\theta)\right) r^{-n},$$

$$G_2 = \frac{c_0}{2} + \sum_{n=1}^{\infty} \left(c_n \cos(n\theta) + d_n \operatorname{sen}(n\theta)\right) r^{-n},$$

donde los coeficientes a_n, b_n, c_n y d_n están dados por:

In English

$$-2\mu \operatorname{div} \varepsilon(u) + \operatorname{grad} p = f, \quad \text{in } \Omega,$$

$$\operatorname{div}(u) = 0, \quad \text{in } \Omega, \qquad (1)$$

Here $u: \Omega \to \mathbb{R}$ was the displacement and p was the pressure of a material subjected to some external force f, $\mu > 0$ was the Lamé constant and $\varepsilon(u)$ the strain tensor given by $\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, for $i, j = 1, 2$. It was assumed for this material that the stress-strain ratio was valid for small deformations, as discussed in Necas (Necas 1986; Necas, 1981; Zeidler, 1988). It was supposed that force f had compact support in this work and was defined inside region Ω_i.

Next, notation $X = [H^1_\Gamma(\Omega)]^2$ and $M = L^2(\Omega)$ was adopted with norms $L^2(\Omega)$ and $H^1(\Omega)$, respectively. Standard integration by parts led to the following variational formulation of the problem (1)

$$2\mu \int_\Omega \varepsilon(u) : \varepsilon(v)\, dx - \int_\Omega p \operatorname{div} v \; dx = \int_\Omega f \cdot v\, dx$$

$$\int_\Omega q \operatorname{div} v \; dx = 0.$$

Notice that problem (1) defined on unbounded domain Ω and Ω_i was the computational domain for the finite element method used here which was obtained by introducing artificial boundary Γ_R. The variational formulation of problem (1) in Ω_i had to be deduced next. Integration by parts was used, by virtue of stress tensor $\sigma(u,p) := 2\mu\varepsilon(u) - pI$, where I was the identity matrix of $\mathbb{R}^{2\times 2}$, this gave $-\operatorname{div}\left(\sigma(u,p)\right) = -2\mu \operatorname{div} \varepsilon(u) + \operatorname{grad} p$, leading to weak formulation:

$$2\mu \int_{\Omega_i} \varepsilon(u) : \varepsilon(v)\, dx - \int_{\Gamma_R} \sigma(u,p) \cdot v \cdot \vec{\eta}\, ds - \int_{\Omega_i} p \operatorname{div} v \; dx = \int_{\Omega_i} f \cdot v\, dx \qquad (2)$$

$$\int_{\Omega_i} q \operatorname{div} v \; dx = 0.$$

Boundary integral $\int_{\Gamma_R} \sigma(u,p) \cdot v \cdot \vec{\eta}\, ds$ was then analyzed. The next goal was to obtain one expression representing tensor $\sigma(u,p)$ on artificial boundary Γ_R. The following decomposition of displacement u on unbounded domain $\Omega_R = \left\{x = (x_1, x_2)^T : r = \sqrt{x_1^2 + x_2^2} > R\right\}$,

$$u_j = G_j + (r^2 - R^2)\frac{\partial W}{\partial x_j}, \qquad j = 1, 2, \qquad (3)$$

was then considered, where G_1, G_2 and W were the harmonic functions to be determined. In particular, function G_j satisfied the following boundary-value problem

$$\Delta G_j = 0, \quad \text{in } \Omega_R,$$

$$G_j|_{\Gamma_R} = u_j(R, \theta),$$

$$G_j \text{ is bounded, when } r \to +\infty.$$

Notice that values for G_j and u_j coincided along Γ_R. Using the separation of variables method the following representations of G_j were obtained using Fourier series development

$$G_1 = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) r^{-n},$$

$$G_2 = \frac{c_0}{2} + \sum_{n=1}^{\infty} \left(c_n \cos(n\theta) + d_n \sin(n\theta)\right) r^{-n},$$

where coefficients a_n, b_n, c_n and d_n were given by:

En español

$$a_n = \frac{R^n}{\pi}\int_{\Gamma_R} u_1(R,\theta)\cos(n\theta)\,d\theta,$$
$$b_n = \frac{R^n}{\pi}\int_{\Gamma_R} u_1(R,\theta)\,\mathrm{sen}(n\theta)\,d\theta,$$
$$c_n = \frac{R^n}{\pi}\int_{\Gamma_R} u_2(R,\theta)\cos(n\theta)\,d\theta,$$
$$d_n = \frac{R^n}{\pi}\int_{\Gamma_R} u_2(R,\theta)\,\mathrm{sen}(n\theta)\,d\theta.$$

Usando la condición div $(u) = 0$ y la representación (3) se obtiene que $-\frac{1}{2}\mathrm{div}\, G_j = r\frac{\partial W}{\partial r}$, y de aquí

$$W = -\sum_{n=1}^{\infty}\frac{n}{2(n+1)r^{n+1}}\left[(a_n - d_n)\cos\big((n+1)\theta\big) + (b_n - c_n)\,\mathrm{sen}\big((n+1)\theta\big)\right].$$

Análogamente, usando la ecuación de balance en Ω_R, $\mathrm{grad}\, p = 2\mu\,\mathrm{div}\,\varepsilon(u)$, obtenemos $\mathrm{grad}\, p = \mathrm{grad}(4\mu\vec{x}\cdot\mathrm{grad}\, W)$ y de aquí se deduce que $p = 4\mu r\frac{\partial W}{\partial r}$. Ahora estamos en condiciones de calcular $\sigma(u,p)\cdot\vec{\eta}$ a lo largo de Γ_R. Recordando que los valores de u_j y G_j coinciden sobre Γ_R, y que la presión p se conoce en términos de W se obtiene:

$$\int_{\Gamma_R}\sigma(u,p)\cdot v\cdot\vec{\eta}\,ds = \int_{\Gamma_R}\mathrm{T}(u)\cdot v\,ds,$$

donde

$$\mathrm{T}(u)_j = \left(\frac{2\mu}{\pi R}\sum_{n=1}^{\infty}\int_0^{2\pi}\frac{\cos\big(n(\varphi-\theta)\big)}{n}\cdot\frac{\partial u_j(R,\varphi)}{\partial\varphi}d\varphi\right),\quad j = 1, 2,$$

que sólo depende de la variación de u a lo largo de Γ_R. El operador $T(u)$ se conoce como *aplicación Dirichlet to Neumann*. Obsérvese que el problema variacional (2) es la formulación débil del problema:

$$-2\mu\,\mathrm{div}\,\varepsilon(u) + \mathrm{grad}\, p = f, \quad \text{en } \Omega_i,$$
$$\mathrm{div}\,(u) = 0, \quad \text{en } \Omega_i,$$
$$u|_\Gamma = 0,$$
$$(\sigma\cdot\vec{\eta})_1 = T(u)_1, \qquad (\sigma\cdot\vec{\eta})_2 = T(u)_2, \text{ sobre } \Gamma_R.$$

Existencia y unicidad de soluciones

Consideremos las formas bilineales $A_0: X\times X\to\mathbb{R}$, $A: X\times X\to\mathbb{R}$, $B: X\times M\to\mathbb{R}$ y el funcional lineal $F: X\to\mathbb{R}$ definidos mediante:

$$A_0(u,v) = -\frac{2\mu}{\pi}\sum_{j=1}^{2}\sum_{n=1}^{\infty}\int_0^{2\pi}\int_0^{2\pi}\frac{\cos\big(n(\varphi-\theta)\big)}{n}\cdot\frac{\partial u_j(R,\varphi)}{\partial\varphi}\cdot\frac{\partial v_j(R,\theta)}{\partial\theta}d\varphi\, d\theta,$$
$$A(u,v) = 2\mu\int_{\Omega_i}\varepsilon(u):\varepsilon(v)\,dx,$$
$$\mathrm{B}(q,v) = -\int_{\Omega_i} q\cdot\mathrm{div}\, v\, dx,$$

y $F(v) = \int_{\Omega_i} f\cdot v\,dx$, respectivamente. Entonces el problema variacional (2) consiste en encontrar $(u,p)\in X\times M$ tal que

$$A(u,v) + A_0(u,v) + B(p,v) = F(v), \quad \forall\, v\in X, \tag{4}$$
$$B(u,q) = 0, \qquad \forall\, q\in M.$$

La demostración de existencia y unicidad de este problema se basa en el teorema de Brezzi, puesto que la forma $A(u,v) + A_0(u,v)$ es

In English

$$a_n = \frac{R^n}{\pi}\int_{\Gamma_R} u_1(R,\theta)\cos(n\theta)\,d\theta,$$
$$b_n = \frac{R^n}{\pi}\int_{\Gamma_R} u_1(R,\theta)\sin(n\theta)\,d\theta,$$
$$c_n = \frac{R^n}{\pi}\int_{\Gamma_R} u_2(R,\theta)\cos(n\theta)\,d\theta,$$
$$d_n = \frac{R^n}{\pi}\int_{\Gamma_R} u_2(R,\theta)\sin(n\theta)\,d\theta.$$

Using condition div $(u) = 0$ and representation (3) the following equation was obtained $-\frac{1}{2}\mathrm{div}\, G_j = r\frac{\partial W}{\partial r}$, and so

$$W = -\sum_{n=1}^{\infty}\frac{n}{2(n+1)r^{n+1}}\left[(a_n - d_n)\cos\big((n+1)\theta\big) + (b_n - c_n)\sin\big((n+1)\theta\big)\right].$$

Similarly, using the balanced equation in Ω_R, $\mathrm{grad}\, p = 2\mu\,\mathrm{div}\,\varepsilon(u)$, $\mathrm{grad}\, p = \mathrm{grad}(4\mu\vec{x}\cdot\mathrm{grad}\, W)$ was obtained and hence $p = 4\mu r\frac{\partial W}{\partial r}$. Next $\sigma(u,p)\cdot\vec{\eta}$ could be estimated along Γ_R. To this end, it was first recalled that the values of u_j and G_j coincided on Γ_R, and pressure p was known in terms of W, leading to

$$\int_{\Gamma_R}\sigma(u,p)\cdot v\cdot\vec{\eta}\,ds = \int_{\Gamma_R}\mathrm{T}(u)\cdot v\,ds,$$

where

$$\mathrm{T}(u)_j = \left(\frac{2\mu}{\pi R}\sum_{n=1}^{\infty}\int_0^{2\pi}\frac{\cos\big(n(\varphi-\theta)\big)}{n}\cdot\frac{\partial u_j(R,\varphi)}{\partial\varphi}d\varphi\right),\quad j = 1, 2,$$

just depending on the variation of u along Γ_R. Operator $T(u)$ represented *Dirichlet to Neumann mapping*. It was stated that variational formulation (2) was the weak formulation of the problem

$$-2\mu\,\mathrm{div}\,\varepsilon(u) + \mathrm{grad}\, p = f, \quad \text{in } \Omega_i,$$
$$\mathrm{div}\,(u) = 0, \quad \text{in } \Omega_i,$$
$$u|_\Gamma = 0,$$
$$(\sigma\cdot\vec{\eta})_1 = T(u)_1, \qquad (\sigma\cdot\vec{\eta})_2 = T(u)_2, \text{ on } \Gamma_R.$$

Existence and uniqueness of solutions

Bilinear forms $A_0: X\times X\to\mathbb{R}$, $A: X\times X\to\mathbb{R}$, $B: X\times M\to\mathbb{R}$ and linear functional $F: X\to\mathbb{R}$ defined by

$$A_0(u,v) = -\frac{2\mu}{\pi}\sum_{j=1}^{2}\sum_{n=1}^{\infty}\int_0^{2\pi}\int_0^{2\pi}\frac{\cos\big(n(\varphi-\theta)\big)}{n}\cdot\frac{\partial u_j(R,\varphi)}{\partial\varphi}\cdot\frac{\partial v_j(R,\theta)}{\partial\theta}d\varphi\, d\theta,$$
$$A(u,v) = 2\mu\int_{\Omega_i}\varepsilon(u):\varepsilon(v)\,dx,$$
$$\mathrm{B}(q,v) = -\int_{\Omega_i} q\cdot\mathrm{div}\, v\, dx,$$

and $F(v) = \int_{\Omega_i} f\cdot v\,dx$, respectively, were considered. Variational formulation of (2) would then read: *Find* $(u,p)\in X\times M$ *so that*

$$A(u,v) + A_0(u,v) + B(p,v) = F(v), \quad \forall\, v\in X, \tag{4}$$
$$B(u,q) = 0, \qquad \forall\, q\in M.$$

The proof of existence and uniqueness was based on Brezzi's theorem, since the form $A(u,v) + A_0(u,v)$ was continuous and

En español

continua y coerciva, y el operador B satisface la condición inf-sup (Han y Bao, 1997; Camargo, 2008).

Método de elementos finitos

El método de elementos finitos que presentamos en este trabajo no aproxima la solución del problema (4), sino la solución del problema que se obtiene truncando la serie de Fourier, en términos de la cual está definido el operador $T(u)$. Al reemplazar $\sum_{n=1}^{\infty}$ por $\sum_{n=1}^{N}$, para un valor entero de N, obtenemos el problema aproximado: encontrar $(u_N, p_N) \in X \times M$ tal que

$$A(u_N, v) + A_0^N(u_N, v) + B(p_N, v) = F(v), \quad \forall v \in X, \quad (5)$$

$$B(u_N, q) = 0, \quad \forall q \in M,$$

donde $A_0^N(u,v) = -\int_{\Gamma_R} \mathrm{T}^{\mathrm{N}}(u) \cdot v \, ds$, y

$$\mathrm{T}_j^{\mathrm{N}}(u) = \frac{2\mu}{\pi R} \sum_{n=1}^{N} \frac{\partial}{\partial \theta} \int_0^{2\pi} \frac{\cos(n(\varphi - \theta))}{n} \cdot \frac{\partial u_j(R,\varphi)}{\partial \varphi} d\varphi,$$

con $j = 1, 2$. El estudio de existencia y unicidad del problema (5) es análogo al del problema exacto (Han y Bao, 1997; Camargo, 2008). Además, se tiene el siguiente estimativo para el error de la aproximación (u_N, p_N).

Teorema 1. Sean $(u,p) \in X \times M$ y $(u_N, p_N) \in X \times M$ las soluciones de los problemas variacionales (4) y (5), respectivamente. Además, suponga $u|_{\Gamma_R} \in H^{m+\frac{1}{2}}(\Gamma_R)$ con $m \in \mathbb{Z}$. Entonces el siguiente estimativo del error se cumple:

$$\|u - u_N\|_X - \|p - p_N\|_M \le C \frac{1}{(N+1)^m} |u|_{m+\frac{1}{2},\ \Gamma_R},$$

donde C es una constante independiente de N y m.

Ahora, sean $X_h \subset X$ y $M_h \subset M$ los espacios de elementos finitos correspondientes al elemento Taylor-Hood, (Brezzi, 1991), y denotemos por $\mathcal{T}_h$ la partición regular del dominio Ω_i, en elementos triangulares curvilíneos T de diámetro h_T, donde $h = \sup_{T \in \mathcal{T}_h} h_T$. Entonces, el método de elementos finitos para aproximar el problema (5) con-siste en encontrar $(u_h, p_h) \in X_h \times M_h$ tal que

$$A(u_h, v_h) + A_0^N(u_h, v_h) + B(p_h, v_h) = F(v_h), \quad \forall v_h \in X_h, \quad (6)$$

$$B(u_h, q_h) = 0, \quad \forall q_h \in M_h.$$

Nótese que el elemento Taylor-Hood satisface la condición inf-sup (Brezzi, 1991) y nuevamente por el teorema de Brezzi es inmediata la existencia de una única solución $(u_h, q_h) \in X_h \times M_h$ para (6).

Teorema 2. Suponga que $(u,p) \in X \times M$ es la solución del problema (4) y $(u_h, q_h) \in X_h \times M_h$ es la solución del problema (6), así que el siguiente estimativo de error se cumple:

$$\|u - u_h\|_X + \|p - p_h\|_M \le C \left\{ \inf_{v_h \in X_h} \|u - v_h\|_X + \inf_{q_h \in M_h} \|p - q_h\|_M \right\} + C \frac{1}{(N+1)^m} |u|_{m+\frac{1}{2},\ \Gamma_R},$$

donde C es una constante independiente de h y N.

Demostración. Sea $(u_N, p_N) \in X \times M$ la solución del problema (5) para cualquier entero $N \ge 0$ y consideremos los errores $\|u_N - u_h\|_X$ y $\|p_N - p_h\|_M$; empleando la técnica estándar del método de elementos finitos mixto se sigue que existe $C_0 \in \mathbb{Z}^+$ independiente de N y el diámetro h de la triangulación Ω_i tal que:

In English

coercive, and operator B satisfied the inf-sup condition (Han and Bao, 1997; Camargo, 2008).

Finite element method

The finite element method presented in this article did not approximate the solution of the problem (4), but provided the solution to the problem obtained by truncating operator $T(u)$ in the corresponding Fourier series. If $\sum_{n=1}^{\infty}$ were replaced by $\sum_{n=1}^{N}$ for integer value N, the variational formulation of the approximate problem would read: *Find* $(u_N, p_N) \in X \times M$ *so that*

$$A(u_N, v) + A_0^N(u_N, v) + B(p_N, v) = F(v), \quad \forall v \in X, \quad (5)$$

$$B(u_N, q) = 0, \quad \forall q \in M,$$

where $A_0^N(u,v) = -\int_{\Gamma_R} \mathrm{T}^{\mathrm{N}}(u) \cdot v \, ds$, and

$$\mathrm{T}_j^{\mathrm{N}}(u) = \frac{2\mu}{\pi R} \sum_{n=1}^{N} \frac{\partial}{\partial \theta} \int_0^{2\pi} \frac{\cos(n(\varphi - \theta))}{n} \cdot \frac{\partial u_j(R,\varphi)}{\partial \varphi} d\varphi,$$

with $j = 1, 2$. The study of existence and uniqueness of the problem (5) was similar to the exact problem (Han and Bao, 1997; Camargo, 2008). Furthermore, the following error estimate of the approximation (u_N, p_N) held.

Theorem 1. $(u,p) \in X \times M$ and $(u_N, p_N) \in X \times M$ were the solutions to variational problems (4) and (5), respectively. In addition, it was supposed that $u|_{\Gamma_R} \in H^{m+\frac{1}{2}}(\Gamma_R)$ with $m \in \mathbb{Z}$. Then the following abstract error estimate would hold

$$\|u - u_N\|_X - \|p - p_N\|_M \le C \frac{1}{(N+1)^m} |u|_{m+\frac{1}{2},\ \Gamma_R},$$

where C was a constant independent of N and m.

Next, $X_h \subset X$ and $M_h \subset M$ were the spaces of finite element corresponding to Taylor-Hood element (Brezzi, 1991). The regular partition of domain Ω_i was denoted by $\mathcal{T}_h$ using curved triangular elements T having diameter h_T, where $h = \sup_{T \in \mathcal{T}_h} h_T$. Then, the finite element method of (5) would read: *Find* $(u_h, p_h) \in X_h \times M_h$ *so that*

$$A(u_h, v_h) + A_0^N(u_h, v_h) + B(p_h, v_h) = F(v_h), \quad \forall v_h \in X_h, \quad (6)$$

$$B(u_h, q_h) = 0, \quad \forall q_h \in M_h.$$

Notice that the Taylor-Hood element satisfied the inf-sup condition (Brezzi, 1991) and again, by Brezzi's theorem, the unique solvability of the variational formulation (6) was obtained.

Theorem 2. $(u,p) \in X \times M$ was the solution to problem (4) and $(u_h, q_h) \in X_h \times M_h$ was the solution to problem (6), then the following error estimate would hold

$$\|u - u_h\|_X + \|p - p_h\|_M \le C \left\{ \inf_{v_h \in X_h} \|u - v_h\|_X + \inf_{q_h \in M_h} \|p - q_h\|_M \right\} + C \frac{1}{(N+1)^m} |u|_{m+\frac{1}{2},\ \Gamma_R},$$

where C was a constant independent of h and N.

Proof. Let $(u_N, p_N) \in X \times M$ be the solution of problem (5) for any integer $N \ge 0$ and consider errors $\|u_N - u_h\|_X$ and $\|p_N - p_h\|_M$; then according to the standard technique of mixed finite element method, $C_0 \in \mathbb{Z}^+$ would exist independently of N and diameter h of triangulation Ω_i so that

En español

$$\|u_N - u_h\|_X + \|p_N - p_h\|_M \leq C_0 \left\{ \inf_{v_h \in X_h} \|u_N - u_h\|_X + \inf_{q_h \in M_h} \|p_N - p_h\|_M \right\}. \qquad (7)$$

Además,

$$\|u - u_N + u_N - u_h\|_X + \|p - p_N + p_N - p_h\|_M \leq \|u - u_N\|_X + \|u_N - u_h\|_X + \|p - p_N\|_M + \|p_N - p_h\|_M,$$

y empleando (7) se sigue que

$$\begin{aligned} \|u - u_h\|_X + \|p - q_h\|_M &\leq C_0 \left\{ \inf_{v_h \in X_h} \|u_N - v_h\|_X + \inf_{q_h \in M_h} \|p_N - q_h\|_M \right\} + \|u - u_N\|_X + \|p - p_N\|_M \\ &\leq C_0 \left\{ \inf_{v_h \in X_h} \|u - v_h\|_X + \inf_{q_h \in M_h} \|p - q_h\|_M \right\} + (C_0 + 1)(\|u - u_N\|_X + \|p - p_N\|_M), \end{aligned}$$

lo que concluye la demostración del teorema.

Implementación de la técnica DtN

Sea Ω el dominio no acotado $\Omega = \{x \in \mathbb{R}^2 : 1 < |x_1| \text{ or } 1 < |x_2|\}$, Γ_R la circunferencia de radio $R = 2$ y

$$u_1^0 := \frac{1}{4\mu}\left(\frac{(x_1 - x_1^0)^2}{|x - x^0|^2} - \frac{(x_1 - x_1^1)^2}{|x - x^1|^2} - \frac{|x - x^0|}{|x - x^1|}\right),$$

$$u_2^0 := \frac{1}{4\mu}\left(\frac{(x_1 - x_1^0)(x_2 - x_2^0)}{|x - x^0|^2} - \frac{(x_1 - x_1^1)(x_2 - x_2^1)}{|x - x^1|^2}\right),$$

$$p_0 := \frac{1}{2}\left(\frac{x_1 - x_1^0}{|x - x^0|^2} - \frac{x_1 - x_1^1}{|x - x^1|^2}\right),$$

donde $x^0 := \left(0, \frac{1}{2}\right)$ y $x^1 := \left(0, -\frac{1}{2}\right)$. Entonces, $u^0(x) = \left(u_1^0(x), u_2^0(x), p_0(x)\right)^T$ es la única solución del siguiente problema de valores de frontera:

$$\begin{gathered} -2\mu \operatorname{div} \varepsilon(u) + \operatorname{grad} p = 0, \quad \text{en } \Omega, \\ \operatorname{div}(u) = 0, \quad \text{en } \Omega, \\ u|_\Gamma = u^0, \\ u \text{ es acotado}, \qquad p \to 0 \text{ cuando } |x| \to +\infty. \end{gathered}$$

Dado $n \in \mathbb{N}$, sea $0 = t_0 \leq t_1 < \cdots < t_n = 2\pi$ una partición uniforme de $[0,2\pi[\to \Gamma_R$ y sea $\gamma : [0,2\pi[\to \Gamma_R$ la parametrización del círculo definida por $\gamma(t) = R(\cos(t), \operatorname{sen}(t))^T$. Entonces denotamos con Ω_h la región anular acotada por $(\cos(t), \operatorname{sen}(t))\Gamma = \{(x_1, x_2) : |x_i| = 1,\ i = 1, 2\}$ y la línea poligonal cerrada Γ_h con vértices $\{\gamma(t_1), \cdots, \gamma(t_n)\}$, y sea $\mathcal{T}_h$ una triangulación regular de Ω_h con triángulos T de diámetro h_T tal que $h = \sup_{T \in \mathcal{T}_h} h_T$.

Ahora consideramos el triángulo canónico con vértices $\widehat{P_1} = (0,0)^T$, $\widehat{P_2} = (1,0)^T$ y $\widehat{P_3} = (0,1)^T$ como un triángulo de referencia $\hat{T}$, e introducimos la familia de transformaciones afines biyectivas $\{F_T\}_{T \in \mathcal{T}_h}$ tal que $F_T(\hat{T}) = T$ para todo $T \in \mathcal{T}_h$. Es bien conocido que $F_T(\hat{x}) = B_T \hat{x} + d_T$, $\forall\, \hat{x} := (\widehat{x_1} + \widehat{x_2})^T$, donde $B_T \in \mathbb{R}^{2\times 2}$ y $d_T \in \mathbb{R}^2$ depende de los vértices del triángulo T. Además, dado un entero $l > 0$ y un subconjunto S of $\mathbb{R}^2$, denotamos con $\mathcal{P}_l(S)$ el espacio de polinomios en dos variables definidos en S de grado total a lo más l.

Con el fin de especificar X_h y M_h tomamos

$$\mathcal{P}_2(T) := \left\{ v : v = \hat{v} \circ F_T^{-1}, \hat{v} \in \mathcal{P}_2(\hat{T}) \right\}$$

In English

$$\|u_N - u_h\|_X + \|p_N - p_h\|_M \leq C_0 \left\{ \inf_{v_h \in X_h} \|u_N - u_h\|_X + \inf_{q_h \in M_h} \|p_N - p_h\|_M \right\}. \qquad (7)$$

Therefore,

$$\|u - u_N + u_N - u_h\|_X + \|p - p_N + p_N - p_h\|_M \leq \|u - u_N\|_X + \|u_N - u_h\|_X + \|p - p_N\|_M + \|p_N - p_h\|_M,$$

and (7) would yield

$$\begin{aligned} \|u - u_h\|_X + \|p - q_h\|_M &\leq C_0 \left\{ \inf_{v_h \in X_h} \|u_N - v_h\|_X + \inf_{q_h \in M_h} \|p_N - q_h\|_M \right\} + \|u - u_N\|_X + \|p - p_N\|_M \\ &\leq C_0 \left\{ \inf_{v_h \in X_h} \|u - v_h\|_X + \inf_{q_h \in M_h} \|p - q_h\|_M \right\} + (C_0 + 1)(\|u - u_N\|_X + \|p - p_N\|_M), \end{aligned}$$

concluding the theorem proof.

Implementing the DtN technique

Let Ω be unbounded domain $\Omega = \{x \in \mathbb{R}^2 : 1 < |x_1| \text{ or } 1 < |x_2|\}$, Γ_R the circumference with radius $R = 2$ and

$$u_1^0 := \frac{1}{4\mu}\left(\frac{(x_1 - x_1^0)^2}{|x - x^0|^2} - \frac{(x_1 - x_1^1)^2}{|x - x^1|^2} - \frac{|x - x^0|}{|x - x^1|}\right),$$

$$u_2^0 := \frac{1}{4\mu}\left(\frac{(x_1 - x_1^0)(x_2 - x_2^0)}{|x - x^0|^2} - \frac{(x_1 - x_1^1)(x_2 - x_2^1)}{|x - x^1|^2}\right),$$

$$p_0 := \frac{1}{2}\left(\frac{x_1 - x_1^0}{|x - x^0|^2} - \frac{x_1 - x_1^1}{|x - x^1|^2}\right),$$

where $x^0 := \left(0, \frac{1}{2}\right)$ and $x^1 := \left(0, -\frac{1}{2}\right)$. Then, $u^0(x) = \left(u_1^0(x), u_2^0(x), p_0(x)\right)^T$ would be the unique solution for the following boundary value problem

$$\begin{gathered} -2\mu \operatorname{div} \varepsilon(u) + \operatorname{grad} p = 0, \quad \text{in } \Omega, \\ \operatorname{div}(u) = 0, \quad \text{in } \Omega, \\ u|_\Gamma = u^0, \\ u \text{ is bounded}, \qquad p \to 0 \text{ when } |x| \to +\infty. \end{gathered}$$

Given $n \in \mathbb{N}$, let $0 = t_0 \leq t_1 < \cdots < t_n = 2\pi$ a partition uniform of $[0,2\pi[\to \Gamma_R$ and let $\gamma : [0,2\pi[\to \Gamma_R$ be the parametrisation of a circle defined by $\gamma(t) = R(\cos(t), \sin(t))^T$. The annular region bounded by $\Gamma = \{(x_1, x_2) : |x_i| = 1,\ i = 1, 2\}$ was then denoted by Ω_h and closed polygonal line Γ_h with vertices $\{\gamma(t_1), \cdots, \gamma(t_n)\}$, and $\mathcal{T}_h$ was regular triangulation of Ω_h with triangles T of diameter h_T so that $h = \sup_{T \in \mathcal{T}_h} h_T$.

Reference triangle $\hat{T}$ with vertices $\widehat{P_1} = (0,0)^T$, $\widehat{P_2} = (1,0)^T$ and $\widehat{P_3} = (0,1)^T$ was then considered and the family of similar bijective transformations $\{F_T\}_{T \in \mathcal{T}_h}$ introduced so that $F_T(\hat{T}) = T$ for all $T \in \mathcal{T}_h$. It is well-known that $F_T(\hat{x}) = B_T \hat{x} + d_T$, $\forall\, \hat{x} := (\widehat{x_1} + \widehat{x_2})^T$, whe-re $B_T \in \mathbb{R}^{2\times 2}$ and $d_T \in \mathbb{R}^2$ depend on the vertices of triangle T. Therefore, given integer $l > 0$ and subset S of $\mathbb{R}^2$, $\mathcal{P}_l(S)$ denoted the space of polynomials defined in S having two variables and great degrees of freedom at most l.

Next,

$$\mathcal{P}_2(T) := \left\{ v : v = \hat{v} \circ F_T^{-1}, \hat{v} \in \mathcal{P}_2(\hat{T}) \right\}$$

En español

$$\mathcal{P}_1(T) := \{q: q = \hat{q} \circ F_T^{-1}, \hat{q} \in \mathcal{P}_1(\hat{T})\}.$$

Entonces discretizamos X usando

$$X_h = \mathcal{P}_{2,h}^2 := \{v_h \in [H_\Gamma^1(\Omega)]^2 : v_h|_T \in [\mathcal{P}_2(T)]^2, \forall T \in \mathcal{T}_h\},$$

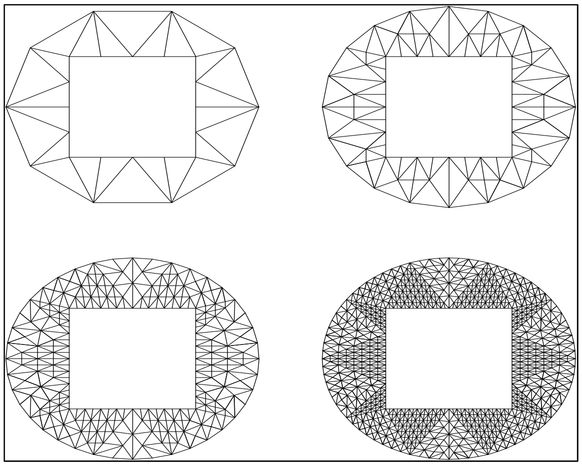

Figura 2. Secuencia $\mathcal{T}_{0,h}, \dots, \mathcal{T}_{3,h}$ de mallas usadas en la implementación.

y M con

$$M_h = \mathcal{P}_{1,h} := \{q_h \in L^2 : q_h|_T \in \mathcal{P}_1(T), \forall T \in \mathcal{T}_h\}.$$

Los espacios X_h y M_h constituyen los subespacios de elementos finitos Hood y Taylor más sencillos que satisfacen la condición inf-sup. En nuestra implementación usamos varias triangulaciones. Nosotros arrancamos la experimentación numérica con la malla inicial $\mathcal{T}_{0,h}$ con 26 elementos. Cada malla refinada uniformemente $\mathcal{T}_{1,h}, \dots, \mathcal{T}_{3,h}$ se generó dividiendo cada triángulo en cuatro más pequeños. El resultado se puede observar en la figura 2.

Tabla 1. Grados de libertad (Gdl) y norma L^2 de los errores cuando $N = 7$.

Malla	$\mathcal{T}_{0,h}$	$\mathcal{T}_{1,h}$	$\mathcal{T}_{2,h}$	$\mathcal{T}_{3,h}$
Gdl	102	438	1812	7368
$\|u - u_{h,N}\|_{0,2,\Omega_i}$	0.0289022	0.0096389	0.0040739	0.0018375
$\|p - p_{h,N}\|_{0,2,\Omega_i}$	0.1138212	0.0463215	0.0170638	0.0102138

La tabla 1 muestra la norma L^2 de los errores $u - u_{h,N}$, $p - p_{h,N}$ cuando $N = 7$ para las mallas $\mathcal{T}_{0,h}, \dots, \mathcal{T}_{3,h}$.

Tabla 2. Norma L^2 de los errores para $\mathcal{T}_{3,h}$ cuando $N = 0, 1, 3, 5, 7$.

N	0	1	3	5	7
$\|u - u_{h,N}\|_{0,2,\Omega_i}$	0.3954247	0.0417149	0.0070330	0.0019081	0.0018375
$\|p - p_{h,N}\|_{0,2,\Omega_i}$	0.5005008	0.4686165	0.0434668	0.0106238	0.0102138

La tabla 2 muestra la norma L^2 de los errores $u - u_{h,N}$, $p - p_{h,N}$ para la malla $\mathcal{T}_{3,h}$ cuando $N = 0, 1, 3, 5, 7$.

La figura 3 muestra los valores de $|u_h|$ y p_h para todas las mallas y $N = 7$. Como muestra la figura, $N = 7$ produce buena aproximación y pocos términos se necesitan en la forma bilineal A_0 para obtener resultados precisos.

Conclusiones

En este artículo se usa una aplicación DtN para aproximar la solución de un problema exterior no acotado. Con este fin se introduce una frontera artificial en el dominio computacional acotado, en el que se aproxima la solución usando el método de elementos finitos.

In English

$$\mathcal{P}_1(T) := \{q: q = \hat{q} \circ F_T^{-1}, \hat{q} \in \mathcal{P}_1(\hat{T})\}.$$

was defined to specify X_h and M_h and discretised X and M using

$$X_h = \mathcal{P}_{2,h}^2 := \{v_h \in [H_\Gamma^1(\Omega)]^2 : v_h|_T \in [\mathcal{P}_2(T)]^2, \forall T \in \mathcal{T}_h\},$$

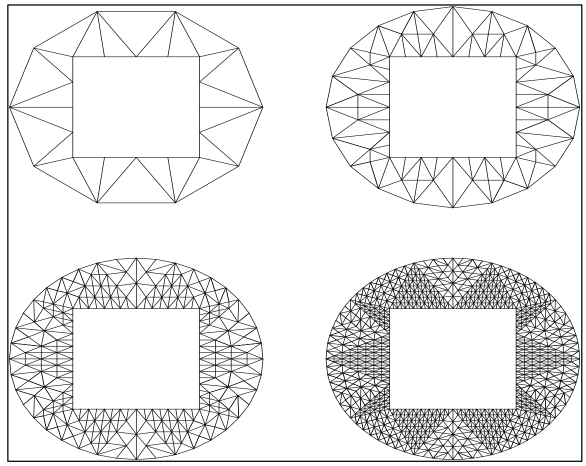

Figure 2. Sequence $\mathcal{T}_{0,h}, \dots, \mathcal{T}_{3,h}$ of meshes used in the implementation

and

$$M_h = \mathcal{P}_{1,h} := \{q_h \in L^2 : q_h|_T \in \mathcal{P}_1(T), \forall T \in \mathcal{T}_h\}.$$

Spaces X_h and M_h were Hood and Taylor finite element subspaces satisfying inf-sup condition. Several triangulations were used in our numerical experiment. Numerical experimentation was started with initial mesh $\mathcal{T}_{0,h}$ which had 26 elements. Each uniformly refined mesh $\mathcal{T}_{1,h}, \dots, \mathcal{T}_{3,h}$ was generated by dividing each triangle into four smaller triangles. The result can be seen in Figure 2.

Table 1. Degrees of freedom and norm L^2 of errors when $N = 7$

Mesh	$\mathcal{T}_{0,h}$	$\mathcal{T}_{1,h}$	$\mathcal{T}_{2,h}$	$\mathcal{T}_{3,h}$
Dof	102	438	1812	7368
$\|u - u_{h,N}\|_{0,2,\Omega_i}$	0.0289022	0.0096389	0.0040739	0.0018375
$\|p - p_{h,N}\|_{0,2,\Omega_i}$	0.1138212	0.0463215	0.0170638	0.0102138

Table 1 shows norm L^2 of errors $u - u_{h,N}$, $p - p_{h,N}$ when $N = 7$ for meshes $\mathcal{T}_{0,h}, \dots, \mathcal{T}_{3,h}$.

Table 2. Norm L^2 of errors for $\mathcal{T}_{3,h}$ when $N = 0, 1, 3, 5, 7$

N	0	1	3	5	7
$\|u - u_{h,N}\|_{0,2,\Omega_i}$	0.3954247	0.0417149	0.0070330	0.0019081	0.0018375
$\|p - p_{h,N}\|_{0,2,\Omega_i}$	0.5005008	0.4686165	0.0434668	0.0106238	0.0102138

Table 2 shows norm L^2 of errors $u - u_{h,N}$, $p - p_{h,N}$ for mesh $\mathcal{T}_{3,h}$ when $N = 0, 1, 3, 5, 7$.

Figure 3 shows values of $|u_h|$ and p_h for meshes $\mathcal{T}_{0,h}$, $\mathcal{T}_{1,h}$, $\mathcal{T}_{3,h}$ and $N = 7$. As Figure 3 shows, $N = 7$ produced a good approximation and only a few terms were needed in bilinear form A_0 to obtain good results.

Conclusions

DtN mapping was used this paper to approximate the solution of an unbounded exterior problem. An artificial boundary was introduced into the domain to approximate the solution on a bounded computational domain by using a finite element method.

En español

Específicamente, se implementa el elemento Taylor-Hood, el cual se define en términos de una serie de Fourier sobre la frontera artificial. Los cálculos numéricos muestran que truncando la serie en siete términos se logra una buena convergencia. Sin embargo, en nuestra aproximación el operador de frontera tiene asociada una matriz de rigidez densa y se usaron precondicionadores especiales para obtener buena convergencia.

In English

The Taylor-Hood element defined in terms of a Fourier series on the artificial domain was implemented in this work as finite element method. The numerical calculations showed that good convergence had been obtained just by truncating the Fourier series to seven terms. However, our discretisation produced a dense stiffness matrix, corresponding to the boundary operator; special preconditioners have to be used to obtain good convergence by very refined meshes.

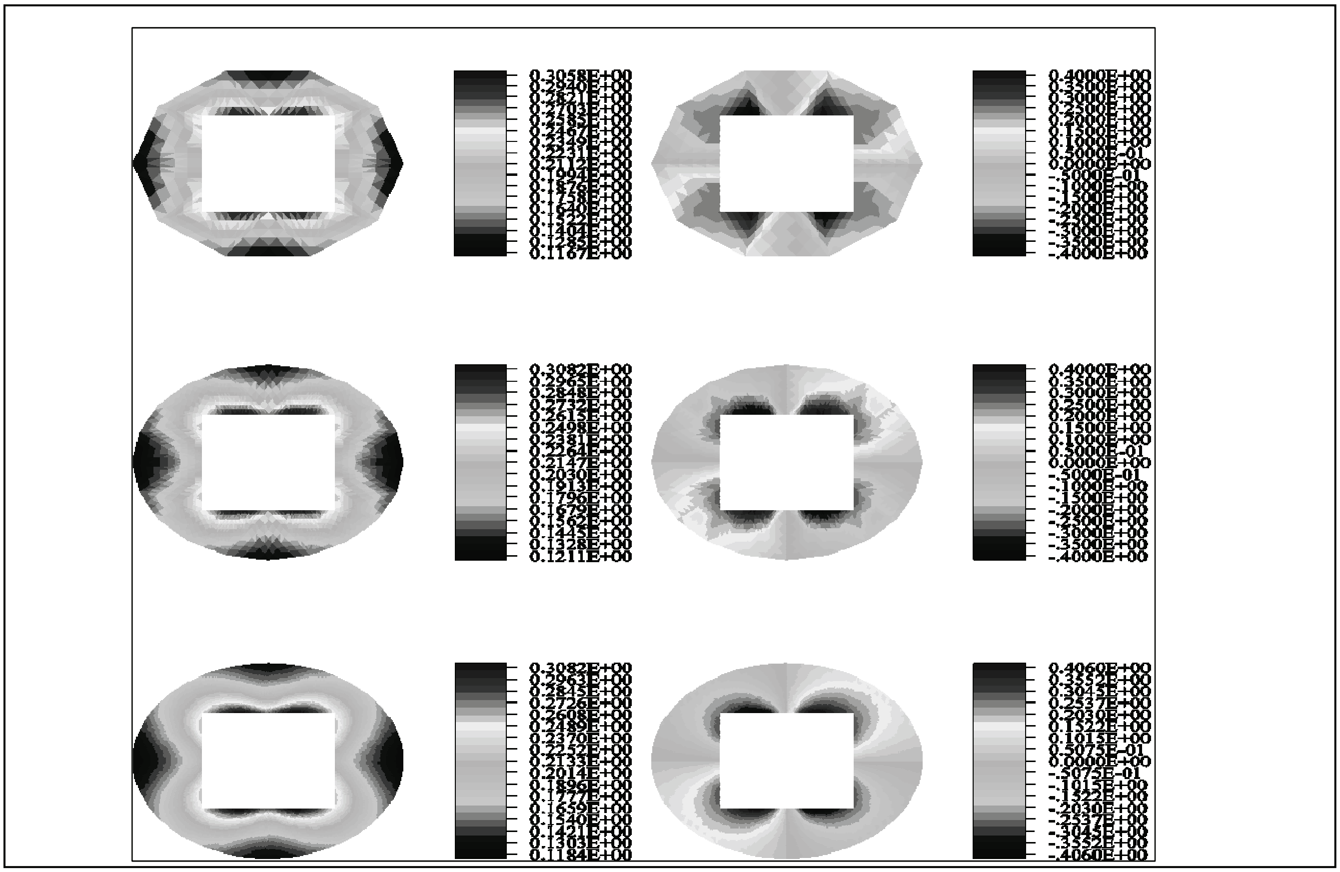

Figura 3. $|u_h|$ y p para las mallas $\mathcal{T}_{0,h}$, $\mathcal{T}_{1,h}$ y $\mathcal{T}_{3,h}$ cuando $N = 7$.

Figure 3. $|u_h|$ and p_h for the meshes $\mathcal{T}_{0,h}$, $\mathcal{T}_{1,h}$ and $\mathcal{T}_{3,h}$ when $N = 7$

Bibliografía / References

Brezzi F., Fortin M., Mixed and hybrid finite element methods., Springer-Verlag, 1991.

Camargo, L., Aplicación DtN-FEM a un problema elástico lineal., BSc Mathematics thesis, Universidad del Valle, 2008.

Gatica G. N., Gatica L. F., Stephan E. P., A FEM–DtN formulation for a non-linear exterior problem in incompressible elasticity., Mathematical Methods in the Applied sciences, 26, 2003, pp. 151–170.

Han H., Bao W., The artificial boundary conditions for incompressible materials on an unbounded domain., Numerische Mathematik, 1997.

Kako T., Touda K., Numerical simulation of voice generation based on finite element method in unbounded region., Department of computer science, Japan, 2004.

Necas, J. Introduction to the theory of nonlinear elliptic equations.., John Wiley & Sons, 1986.

Necas, J., Hlavacek, I., Mathematical theory of elastic and elasto-plastic bodies: an introduction., Elsevier scientific publishing company, 1981.

Zeidler, E., Nonlinear functional analysis and its applications IV., Springer-Verlag, New York, 1988.