HOW DO THE RICHEST 1% OWN 50% OF THE NATIONAL WEALTH IN AN INTEGRATED WALRASIAN EQUILIBRIUM AND NEOCLASSICAL GROWTH MODEL

WEI-BIN ZHANG
Ritsumeikan Asia Pacific University, Japan

Received: *Feb 23, 2015* Accepted: *March 24, 2015* Online Published: *August 10, 2015*

Abstract

This paper proposes a dynamic economic model of heterogeneous households to explain economic mechanisms of how the richest one per cent of the population own 50% of national wealth. We explain inequality in a purely competitive economic environment with endogenous wealth and human capital accumulation. The production technologies and economic structure follow the Uzawa two-sector model. In this study a household's disposable income is the traditional disposable income (which is the income that a household earns each period of time after taxes and transfers in the Solow model and many empirical studies) plus the value of the household's wealth. By applying Zhang's concept of disposable income and utility function, we describe consumers' wealth accumulation and consumption behavior. We show how wealth accumulation, human capital accumulation, and division of labor, and time distribution interact with each other under perfect competition. We simulate the model with three groups of the people, the rich, the middle, and the poor whose shares of the population are, respectively, the 1 %, 69%, and 20%. We demonstrate the existence of an equilibrium point at which the rich 1% own more than half of the national wealth and the poor 20% less than 10% of the national wealth. The rich household works only 4 hours a day and the poor household 11 hours a day. We show how the system moves to the equilibrium from an initial state and confirm that the equilibrium point is stable. We also demonstrate how changes in the total factor productivity of the capital goods sector, the rich's human capital utilization efficiency, the rich's efficiency of learning through consuming, and the rich's propensities to save, to consume, and to enjoy leisure, affect growth and inequality.

Keywords: Inequality and growth; learning by consuming; wealth and income distribution; heterogeneous households.

1. Introduction

It has been reported that the richest 1% of the world population is owning almost half of the world's wealth. Moreover, it does seem that inequality be enlarged in the near future in tandem with rapid economic globalization. There is a need to know determinants of inequality and dynamics of inequality. This need is emphasized by Forbes (2000) as follows: "careful reassessment of the relationship between these two variables (growth rate and income inequality) needs further theoretical and empirical work evaluating the channels through which inequality, growth, and any other variables are related." Surprisingly theoretical economics still has little to say about determinants and dynamics of economic growth and inequality. In a systematic review on the literature of economic growth and inequality of income and wealth, Zhang (2006) points out that although the importance of issues related to growth and inequality was well recognized long time ago by economists such as Marx and Kaldor, modern theoretical economics has failed in providing a proper analytical framework to analyze relations between growth and inequality.

Without a proper analytical framework, economics can hardly analyze extremely complicated dynamic issues with nonlinear interactions among many variables over time. Even mathematically it had been hopeless for any theoretical economists to deal with the issues in free market economies in an insightful and comprehensive manner even a few decades ago before computer was available for simulating high dimensional nonlinear dynamic models. The purpose of this study is to re-address issues related to growth and inequality with Zhang's concept of disposable income and utility function. We are especially interested in a phenomenon of contemporary free-market economies where the richest 1% of the population own more than almost half of wealth. By comparative dynamic analysis we also demonstrate some possibilities that inequalities will not be shrunk but will be enlarged in free market economies with rapid technological changes and connected markets. In order to analyze these issues, we introduce endogenous human capital and human capital externalities into the general equilibrium theory with heterogeneous households and endogenous wealth.

This study is based on Zhang's integrated Walrasian general equilibrium and neoclassical growth theory (Zhang, 2006, 2014). Both the Walrasian general equilibrium theory and neoclassical growth theory have played a key role in the development of formal economic theories in modern times. The Walrasian general equilibrium theory was initially developed by Walras (Walras, 1874). The theory was further developed and refined mainly in the 1950s by Arrow, Debreu and others (e.g., Arrow and Debreu, 1954; Gale, 1955; Nikaido, 1956, 1968; Debreu, 1959; McKenzie, 1959; Arrow and Hahn, 1971; Arrow, 1974; and Mas-Colell *et al.*, 1995). The theory solves equilibrium of pure economic exchanges with heterogeneous supplies and households. From the perspective of modern economies the theory has a serious shortcoming which is failures of properly including endogenous wealth (and other factors such as environment, resources, human capital and knowledge) irrespective many attempts done by many economists. Walras failed in developing a general equilibrium theory with endogenous saving and capital accumulation (e.g., Impicciatore *et al*., 2012).

Over years many economists attempted to further develop Walras' capital accumulation within Walras' framework (e.g., Morishima, 1964, 1977; Diewert, 1977; Eatwell, 1987; Dana et al. 1989; and Montesano, 2008). As it lacks proper economic mechanisms for determining wealth accumulation, the traditional Walrasian general equilibrium theory is not proper for addressing issues related to growth and inequality. On the other hand, the neoclassical growth theory deals with endogenous wealth accumulation with microeconomic foundation (e.g., Ramsey model). Nevertheless, the theory is not successful in dealing with growth with heterogeneous households. Almost all of the neoclassical growth models are built for a homogenous population. Some neoclassical growth models with endogenous wealth accumulation consider heterogeneous households. Nevertheless, the heterogeneity in these studies is by the differences in the initial endowments of wealth among different types of households rather than in preferences (see, for instance, Chatterjee, 1994; Caselli and Ventura, 2000; Maliar and Maliar, 2001; Penalosa and Turnovsky, 2006; and Turnovsky and Penalosa, 2006). Different households are essentially homogeneous in the sense that all the households have the same preference utility function in the traditional Ramsey approach. In our approach we consider different ethnic groups have different utility functions. It implies its limited power in analyzing growth and inequality. Zhang (2006, 2014) integrates the neoclassical growth theory with the Walrasian general equilibrium theory with his concept of disposable income and utility function.

This study is based on Zhang's approach to discuss growth and inequality. It should be noted that economists made efforts in integrating the neoclassical growth theory with the general equilibrium analysis (e.g., Jensen and Larsen, 2005). As far as the Walrasian general equilibrium theory and the traditional capital theory are concerned, the issues examined by Polterovich's approach with heterogeneous capital and heterogeneous households (Polterovich, 1977, 1983; Bewley, 1982; Amir and Evstigneev, 1999) are quite similar to the model in this study. The main difference between Polterovich's model and our approach is human capital dynamics the modeling of household behavior. Polterovich's approach to household is mainly based on the Ramsey model, while this study is based on Zhang's approach. Polterovich's approach does not take account of endogenous human capital.

Both physical wealth and human capital are the determinants of economic growth and inequality. Wealth differs between households partly because people have different propensities to save and human capital differs between people partly because they have different abilities and preferences in accumulating human capital. This study follows Uzawa's two sector growth model in describing economic structure (Uzawa, 1961). Uzawa's two-sector model has been generalized and extended in different ways over years (see, Diamond, 1965; Stiglitz, 1967; Mino, 1996; and Drugeon and Venditti, 2001). We will integrate the neoclassical growth theory with the Walrasian general equilibrium theory for studying dynamic interactions among growth, wealth and income distribution, and economic structures. A unique feature of our approach is to treat human capital accumulation as an endogenous process of economic growth. In economic theory there are only a few theoretical models which study inequality and growth with both endogenous wealth and human capital accumulation. Our approach to human capital accumulation is influenced by Arrow's learning by doing, Uzawa's learning formal education, and Zhang's learning through consuming (leisure creativity). This paper also extends a recent paper on economic growth with heterogeneous households by Zhang (2012, 2014). The main difference between this study and Zhang (2012) is that this study treats human capital as endogenous process while the previous study by Zhang considers human capital fixed. The main difference between this study and Zhang (2014) is that this study treats time distribution as endogenous and the previous study by Zhang neglects time distribution issues. The two studies have different human capital accumulation equations.

The rest of this paper is organized as follows. Section 2 defines the heterogeneous households neoclassical growth model with capital accumulation and human capital accumulation. Section 3 shows that the dynamics of the economy with J types of households can be described by $2J$-dimensional differential equations. As mathematical analysis of the system is too complicated, we demonstrate some of the dynamic properties by simulation when the economy consists of three types of households. Section 4 carries out comparative dynamic analysis with regard to the total factor productivity of the capital goods sector, the rich's human capital utilization efficiency, the rich's efficiency of learning through consuming, and the rich's propensities to save, to consume, and to enjoy leisure. Section 5 concludes the study.

2. The Basic Model

The economy consists of one capital goods and one consumer goods sectors. Most aspects of the production sectors are similar to the standard two-sector growth model by Uzawa (Uzawa, 1965; Burmeister and Dobell 1970; Azariadis, 1993; Barro and Sala-i-Martin, 1995). Households own assets of the economy and distribute their incomes to consume and to save. Firms use labor and physical capital inputs to supply goods and services.

Exchanges take place in perfectly competitive markets. Factor markets work well and the available factors are fully utilized at every moment. Saving is undertaken only by households. All earnings of firms are distributed in the form of payments to factors of production, labor, managerial skill and capital ownership. The population is classified into J groups. Each group has a fixed population, $\bar{N}_j$, ($j = 1, \ldots, J$). Let prices be measured in terms of the commodity and the price of the commodity be unit. Let $p(t)$ denote the price of consumer goods at time t. We denote wage and interest rates by $w_j(t)$ and $r(t)$, respectively. We use $H_j(t)$ to stand for group j's level of human capital. It should be noted that although we call it human capital, the variable $H_j(t)$ may consist of not only human capital such as skills and knowledge but also intangible assets such as social status, reputation, and social relations.

We use subscript index i and s to respectively stand for capital goods and consumer goods. We use $N_m(t)$ and $K_m(t)$ to stand for the labor force and capital stocks employed by sector m. Let $T_j(t)$ stand for the work time of a typical worker in group j. The variable $N(t)$ represents the total qualified labor force. A worker's labor force is $T_j(t)H_j^{m_j}(t)$, where m_j is a parameter measuring utilization efficiency of human capital by group j. The labor input is the work time by the effective human capital. A group's labor input is the group's population by each member the labor force, that is, $T_j(t)H_j^{m_j}(t)\bar{N}_j$. As the total qualified labor force is the sum of all the groups' labor forces, we have $N(t)$ as follows

$$N(t) = \sum_{j=1}^{J} T_j(t)H_j^{m_j}(t)\bar{N}_j, \qquad j = 1, \ldots, J. \tag{1}$$

Full employment of labor and capital

The total labor force is employed by the two sectors. The condition of full employment of labor force implies

$$N_i(t) + N_s(t) = N(t). \tag{2}$$

The total capital stock $K(t)$ is allocated between the two sectors. As full employment of capital is assumed, we have

$$K_i(t) + K_s(t) = K(t). \tag{3}$$

Let $\bar{k}_j(t)$ denote per capita wealth of group j at t. Group j's wealth is $\bar{k}_j(t)\bar{N}_j$. As wealth is held by the households, we have

$$K(t) = \sum_{j=1}^{J} \bar{k}_j(t)\bar{N}_j. \tag{4}$$

The capital goods sector

Let $F_m(t)$ stand for the production function of sector m, $m = i, s$. The production function of the capital goods sector is specified as follows

$$F_i(t) = A_i K_i^{\alpha_i}(t) N_i^{\beta_i}(t), \ \ \alpha_i, \beta_i > 0, \ \ \alpha_i + \beta_i = 1, \tag{5}$$

where A_i, α_i, and β_i are positive parameters. The capital goods sector employs two input factors, capital and labor force. We assume that all the markets are perfectly competitive. The marginal conditions for the capital goods sector are

$$r(t) + \delta_k = \frac{\alpha_i F_i(t)}{K_i(t)}, \ \ w(t) = \frac{\beta_i F_i(t)}{N_i(t)}. \tag{6}$$

The consumer goods sector

The production function of the consumer goods sector is specified as follows

$$F_s(t) = A_s K_s^{\alpha_s}(t) N_s^{\beta_s}(t), \ \ \alpha_s + \beta_s = 1, \ \ \alpha_s, \beta_s > 0, \tag{7}$$

where A_s, α_s, and β_s are technological parameters. The marginal conditions are

$$r(t) + \delta_k = \frac{\alpha_s\, p(t) F_s(t)}{K_s(t)}, \ \ w(t) = \frac{\beta_s\, p(t) F_s(t)}{N_s(t)}. \tag{8}$$

Consumer behaviors and wealth dynamics

Consumers make decisions on choice of leisure time, consumption levels of services and commodities as well as on how much to save. We note that the wage rate of group j is

$$w_j(t) = w(t) H_j^{m_j}(t), \ \ \ j = 1, \cdots, J. \tag{9}$$

Per capita current income from the interest payment $r(t)\bar{k}_j(t)$ and the wage payment $T_j(t) w_j(t)$ is

$$y_j(t) = r(t)\bar{k}_j(t) + T_j(t) w_j(t).$$

We call $y_j(t)$ the current income in the sense that it comes from consumers' payment for human capital and efforts and consumers' current earnings from ownership of wealth. The total value of wealth that consumers can use is $\bar{k}_j(t)$. Here, we assume that selling and buying wealth can be conducted instantaneously without any transaction cost. The per capita disposable income is given by

$$\hat{y}_j(t) = y_j(t) + \bar{k}_j(t) = (1 + r(t))\bar{k}_j(t) + W_j(t). \tag{10}$$

where $W_j(t) \equiv T_j(t)w_j(t)$ is the wage income. The disposable income is used for saving, consumption, and education. It should be noted that the value, $\bar{k}_j(t)$, (i.e., $p(t)\bar{k}_j(t)$ with $p(t) = 1$), in (10) is a flow variable. Under the assumption that selling wealth can be conducted instantaneously without any transaction cost, we may consider $\bar{k}_j(t)$ as the amount of the income that the consumer obtains at time t by selling all of his wealth. Hence, at time t the consumer has the total amount of income equaling $\hat{y}_j(t)$ to distribute between saving and consumption. It should be noted that in the traditional neoclassical growth theory and most empirical studies disposable income is defined as the income that a household earns each period of time after taxes and transfers. It is supposed to be the money available to the household for spending on goods and services. Indeed, when wealth plays minor role in analyzing behavior of households the traditional concept is not misleading. Nevertheless, when wealth is large and plays an important role in affecting household behavior, the omission in the money available for spending may be misleading. Obviously, a rich man with the net value of wealth US$70 billions will save a lot even if he had no current income (no pension and wealth earning a net zero rate of return) as common sense tells us.

According to the neoclassical growth theory (such as the most well-known Solow model in growth theory), the rich man makes neither consumption nor saving as his disposable income is zero. In our model, the man's disposable income is 0 + 70 = 70. If his consumption annually is 0.1 billions, his saving is 70 – 0.1 = 69.9 billions US dollars. His actual saving rate is saving/(disposable income) = 69.9/70, rather than 0 as in a national statistical record. In our approach rich people have a high propensity to save than poor people partly as the extremely rich have too much to spend. As our approach accumulated wealth will play another important role in protecting the social status as wealth helps the rich to accumulate more physical capital (due to interest returns of wealth) as well as human capital (due to easy access to best education, for instance), to build more useful social networks, and maintain reputation of being rich.

The typical consumer distributes the total available budget between saving $s_j(t)$, consumption of consumer goods $c_j(t)$. The budget constraint is

$$p(t)c_j(t) + s_j(t) = \hat{y}_j(t) = (1 + r(t))\bar{k}_j(t) + w_j(t)T_j(t), \tag{11}$$

The time constraint for everyone

$$T_j(t) + \overline{T}_j(t) = T_0\,, \tag{12}$$

where $\overline{T}_j(t)$ is the leisure time of the representative household and T_0 is the total available time. Substituting (12) into (11) yields

$$w_j(t)\overline{T}_j(t) + p(t)c_j(t) + s_j(t) = \bar{y}_j(t) \equiv (1 + r(t))\bar{k}_j(t) + T_0\, w_j(t). \tag{13}$$

The variable $\bar{y}_j(t)$ is the disposable income when the household spends all the available time on work. We assume that the consumer's utility function is dependent on $\overline{T}_j(t)$, $c_j(t)$, and $s_j(t)$ as follows

$$U(t) = \bar{T}_j^{\sigma_{j0}}(t) c^{\xi_{j0}}(t) s^{\lambda_{j0}}(t), \quad \sigma_{j0}, \xi_{j0}, \lambda_{j0} > 0, \tag{14}$$

where σ_{j0} is the propensity to use leisure time, ξ_{j0} is the propensity to consume, and λ_{j0} the propensity to own wealth. This utility function proposed by Zhang (1993) is applied to different economic problems. Maximizing $U_j(t)$ subject to (13) yields

$$\bar{T}_j(t) = \frac{\sigma_j \, \bar{y}_j(t)}{w_j(t)}, \quad c_j(t) = \frac{\xi_j \, \bar{y}_j(t)}{p_s(t)}, \quad s_j(t) = \lambda_j \, \bar{y}_j(t), \tag{15}$$

where

$$\sigma_{j0} \equiv \rho_j \sigma_{j0}, \quad \xi_{j0} \equiv \rho_j \xi_{j0}, \quad \lambda_{j0} \equiv \rho_j \lambda_{j0}, \quad \rho_j \equiv \frac{1}{\sigma_{j0} + \xi_{j0} + \lambda_{j0}}.$$

Change in the household wealth

According to the definitions of $s_j(t)$, the wealth accumulation of the representative household in group j is given by

$$\dot{\bar{k}}_j(t) = s_j(t) - \bar{k}_j(t). \tag{16}$$

This equation simply states that the change in wealth is equal to saving minus dissaving.

Dynamics of human capital

In economic theory there are three sources of improving human capital, through education, “learning by producing”, and “learning by leisure”. Arrow (1962) first introduced learning by doing into growth theory. The basic idea is that people accumulate more skills and have more ideas when they are engaged in economic production. Uzawa (1965) took account of trade-offs between investment in education and capital accumulation in his well-known two-sector model. The basic idea is that education uses social resources but enable people to have more skills and knowledge. Zhang (2007) introduced impact of consumption on human capital accumulation (via the so-called creative leisure) into growth theory. Arrow’s idea of learning by doing is that useful knowledge can be obtained mainly through working experiences. His idea has narrow implications as there are many other sources of accumulating skills and knowledge. Activities which are classified neither as formal education as in the Uzawa model nor as production as in the Arrow model, such as playing computer games, having social parties, being brought up by rich and educated parents, living in a decent society, touring different parts of the world, and being extremely rich, may have strong effects on human capital. Influencing by the three approaches just mentioned and being concerned with providing a case of richest 1% owing 50% wealth, we propose that the human capital accumulation is described as follows

$$\dot{H}_j(t) = \frac{\tilde{\upsilon}_j \, c_j^{a_j}(t) \bar{k}_j^{\upsilon_j}(t) T_j^{\theta_j}(t)}{H_j^{\pi_j}(t)} - \delta_{hj} H_j(t), \tag{17}$$

where δ_{hj} is the depreciation rates of human capital, $0 < \delta_{hj} < 1$. In (17), $\tilde{\upsilon}_j$, a_j, υ_j, and θ_j are non-negative parameters, and π_j is a parameter. In our approach different groups may have different depreciation rates of human capital. The human capital accumulation in this study is influenced by different approaches to human capital accumulation in the literature of endogenous human capital. We now interpret the items in $\tilde{\upsilon}_j c_j^{a_j} \bar{k}_j^{\upsilon_j} T_j^{\theta_j} / H_j^{\pi_j}$. The item $c_j^{a_j}$ which implies a positive relation between human capital accumulation and consumption is influenced by Uzawa's learning through education and Zhang's learning through consumption. As education is classified as the consumption of services, a higher level of consumption may imply a higher investment in education. On the other hand, a higher consumption also implies that the household may accumulate more through other consumption activities. The item $\bar{k}_j^{\upsilon_j}$ which implies a positive relation between wealth and human capital accumulation can be interpreted that more wealth means, for instance, a higher social status. More wealth may also help one to maintain professional reputation. The specification of $T_j^{\theta_j}$ is influenced by Arrow's learning by doing. More work accumulates more human capital. The term $H_j^{\pi_j}$ implies that more human capital makes it easier (more difficult) to accumulate knowledge in the case of $\pi_j < 0$ $(\pi_j > 0)$.

Demand of and supply for consumer goods

The output of the consumer goods sector is consumed only by the households. The demand for consumer goods from a group is $c_j(t)\bar{N}_j$. The condition that the total demand is equal to the total supply implies

$$\sum_{j=1}^{J} c_j(t)\bar{N}_j = F_s(t). \tag{18}$$

Demand of and supply for capital goods

As output of the capital goods sector is used only as capital goods, the output equals the depreciation of capital stock and the net savings. That is

$$\sum_{j=1}^{J} s_j(t)\bar{N}_j - K(t) + \delta_k K(t) = F_i(t). \tag{19}$$

We completed the model. The model is structurally general in the sense that some well-known models in theoretical economics can be considered as its special cases. For instance, if we fix wealth and human capital and allow the number of types of households equal the population, then the model is a Walrasian general equilibrium model. If the population is homogeneous, our model is structurally similar to the neoclassical growth model by Solow (1956) and Uzawa (1961). It is structurally similar to the multi-class models by Pasinetti and Samuelson (e.g., Samuelson, 1959; Pasinetti, 1960, 1974). We now examine dynamics of the model.

3. The Dynamics and Its Properties

As the system consists of any number of types of households, its dynamics is highly dimensional. The following lemma shows that the economic dynamics is represented by $2J$ dimensional differential equations. First we introduce a variable

$$z(t) \equiv \frac{r(t) + \delta_k}{w(t)}.$$

Lemma

The dynamics of the economy is governed by the following $2J$ dimensional differential equations system with $z(t)$, $\{\bar{k}_j(t)\}$, and $(H_j(t))$, where $\{\bar{k}_j(t)\} \equiv (\bar{k}_2(t), \cdots, \bar{k}_J(t))$ and $(H_j(t)) \equiv (H_1(t), \cdots, H_J(t))$, as the variables

$$\dot{z}(t) = \Lambda_1(z(t), (H_j(t)), \{\bar{k}_j(t)\}),$$
$$\dot{\bar{k}}_j(t) = \Lambda_j(z(t), (H_j(t)), \{\bar{k}_j(t)\}), \quad j = 2, ..., J,$$
$$\dot{H}_j(t) = \Omega_j(z(t), (H_j(t)), \{\bar{k}_j(t)\}), \quad j = 1, ..., J,$$

in which Λ_j and Ω_j are unique functions of $z(t)$, $\{\bar{k}_j(t)\}$, and $(H_j(t))$ at any point in time, defined in the appendix. For given $z(t)$, $\{\bar{k}_j(t)\}$, and $(H_j(t))$, the other variables are uniquely determined at any point in time by the following procedure: $r(t)$ and $w(t)$ by (A3) $\rightarrow$ $w_j(t)$ by (A4) $\rightarrow$ $p(t)$ by (A5) $\rightarrow$ $\bar{k}_1(t)$ by (A18) $\rightarrow$ $N_i(t)$ by (A12) $\rightarrow$ $N(t)$ by (A11) $\rightarrow$ $N_s(t)$ by (A8) $\rightarrow$ $\bar{y}_j(t)$ by (A6) $\rightarrow$ $K_i(t)$ and $K_s(t)$ by (A1) $\rightarrow$ $F_i(t)$ and $F_s(t)$ by the definitions $\rightarrow$ $\bar{T}_j(t)$, $c_j(t)$, and $s_j(t)$ by (15) $\rightarrow$ $K(t)$ by (4).

Following the lemma, we have a computational program to follow the motion of the dynamic economic system by simulating the dynamic equations with any number of types of households. As the system is nonlinear and is of high dimension, it is difficult to generally analyze behavior of the system. For illustration, we specify the parameters as follows:

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 69 \\ 20 \end{pmatrix}, \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.15 \\ 0.1 \end{pmatrix}, \begin{pmatrix} \xi_{10} \\ \xi_{20} \\ \xi_{30} \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.18 \\ 0.2 \end{pmatrix}, \begin{pmatrix} \lambda_{10} \\ \lambda_{20} \\ \lambda_{30} \end{pmatrix} = \begin{pmatrix} 0.94 \\ 0.65 \\ 0.6 \end{pmatrix}, \begin{pmatrix} \sigma_{10} \\ \sigma_{20} \\ \sigma_{30} \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.2 \\ 0.2 \end{pmatrix},$$
$$\begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \bar{v}_3 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.4 \\ 0.1 \end{pmatrix}, \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.2 \\ 0.1 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0.1 \\ 0.1 \end{pmatrix}, \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.35 \\ 0.4 \end{pmatrix}, \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.4 \end{pmatrix},$$
$$A_i = 1, \ A_s = 0.9, \ \alpha_i = 0.32, \ \alpha_s = 0.34, \ T_0 = 24, \ \delta_{h1} = 0.04, \ \delta_{h2} = 0.06,$$
$$\delta_{h3} = 0.08, \ \delta_k = 0.05. \tag{20}$$

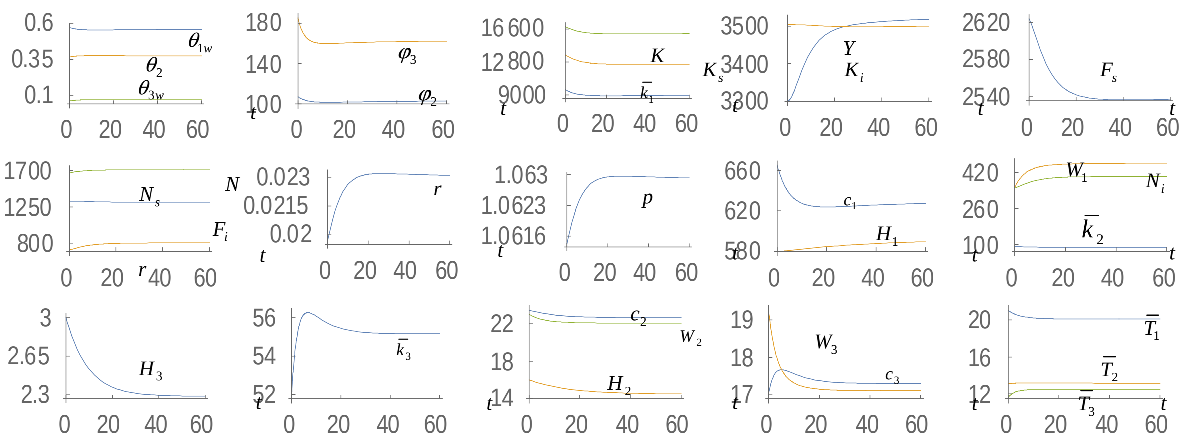
Group 1, 2 and 3's populations are respectively 1, 69 and 20. Group 2 has the largest population. The capital goods sector and consumer goods sector's total productivities are respectively 1 and 0.9. Group $1, 2$ and $3's$ utilization efficiency parameters, m_j, are respectively 0.7, 0.15 and 0.1. Group 1 utilizes human capital mostly effectively; group 2 next and group 3 lest effectively. We call the three groups respectively as the rich, the middle, and the poor. We We specify the values of the parameters, α_j, in the Cobb-Douglas productions approximately equal to 0.3. The rich's learning by doing parameter, $\bar{v}_1$, is the highest. The returns to scale parameters, π_j, are all positive, which implies that human capital accumulation exhibits decreasing returns to scale in human capital. The depreciation rates of human capital are specified in such a way that the rich has lowest rate. The rich's propensity to save is 0.94 and the rich's propensity to save is 0.6. It is assumed that the rich is most effective in learning through consuming and working. The value of the middle's propensity is between the rich and the poor. In Figure 1, we plot the motion of the system with the following initial conditions

$$z(0) = 0.135, \;\; \bar{k}_2(0) = 73, \;\; \bar{k}_3(0) = 45, \;\; H_1(0) = 460, \;\; H_2(0) = 17, \;\; H_3(0) = 3. \tag{21}$$

In Figure 1, the national output Y, the share of each group's wealth in the national wealth θ_{jw}, and the ratio between group 1's and another group's wealth φ_j, are respectively defined as

$$Y(t) = F_i(t) + p(t)F_s(t), \;\; \theta_{jw}(t) \equiv \frac{\bar{K}_j(t)}{K(t)}, \;\; \varphi_j(t) \equiv \frac{\bar{k}_1(t)}{\bar{k}_j(t)}, \;\; j = 1, 2, 3.$$

Figure 1 – The Motion of the Economic System.



With different initial conditions, the economy experiences different paths of development. Under (21), the national output and wealth experience negative growth over time. The rich's human capital is increased and the middle's and poor's human capital fall over time. A rich household works more hours and a household from the other two groups works less. The rich own more than half of the national wealth with 1 percent of the population and the poor own less 10 per cent of the national wealth with the 20 percent of the national population. The representative household from the rich owns more than 160 times wealth than the household from the poor.

We start with different initial states not far away from the equilibrium point and find that the system approaches to an equilibrium point. Under (21) we find that the system has a unique equilibrium. The equilibrium values are listed in (22). The rich has highest human capital and highest wage income. The rich household spends lest hours on working and the poor household spends longest time on working. The rich household's consumption level and wealth are also much higher than the household from the two other groups.

$$\theta_{1w} = 0.559,\ \theta_{2w} = 0.373,\ \theta_{3w} = 0.069,\ \varphi_2 = 103.4,\ \varphi_3 = 162.7,\ Y = 3503.5,\ K = 16092.3,$$
$$N = 1708.4,\ F_i = 804.6,\ F_s = 2539.2,\ K_i = 3526,\ K_s = 12566.3,\ N_i = 401.4,\ N_s = 1306.9,$$
$$r = 0.023,\ p = 1.06,\ w_1 = 118.9,\ w_2 = 2.04,\ w_3 = 1.48,\ W_1 = 462.4,\ W_2 = 22.1,$$
$$W_3 = 17.13,\ H_1 = 591.9,\ H_2 = 14.47,\ H_3 = 2.28,\ \bar{k}_1 = 8988.23,\ \bar{k}_2 = 86.96,\ \bar{k}_3 = 55.19,$$
$$c_1 = 629.7,\ c_2 = 22.66,\ c_3 = 17.31,\ \bar{T}_1 = 20.11,\ \bar{T}_2 = 13.15,\ \bar{T}_3 = 12.43. \quad (22)$$

It is straightforward to calculate the six eigenvalues as follows

$$-0.38,\ -0.32,\ -0.18,\ -0.11,\ -0.07,\ -0.03.$$

As all the eigenvalues are negative, we see that the equilibrium point is locally stable.

4. Comparative Dynamic Analysis

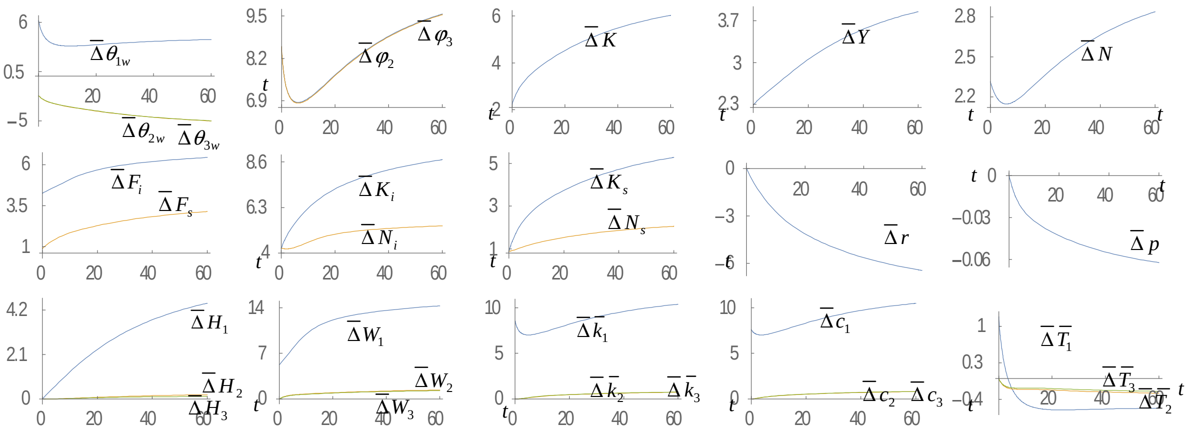
We simulated the motion of the dynamic system. It is important to ask questions such as how a change in one group's propensity to save or to obtain education affects the economic growth, inequality and each class's wealth and consumption. Before carrying out comparative dynamic analysis, we introduce a variable $\bar{\Delta} x_j(t)$ to stand for the change rate of the variable, $x_j(t)$, in percentage due to changes in a parameter.

The rich applying human capital more effectively

Different people in society have different opportunities to apply their human capital. It is expected that the rich has more opportunities to utilize and tends to be more capable of applying human capital. There are many possible determinants of inequality of income and wealth in modern societies. It is expected that the rich get richer in the near future. For instance, if the society is developed toward such a direction that enables the rich (and successful ones) to apply their human capital more effectively, one may expect changes in inequality between the rich and the poor. We now increase the rich's human capital utilization efficiency as follows: $m_1 : 0.7 \Rightarrow 0.71$. As the rich increases their efficiency in applying human capital, the inequality between the rich and the poor is greatly enlarged.

The rich get higher share of the national wealth and the ratios of per household wealth between the poor and rich and between the middle and rich are increased. The improved efficiency by the rich benefits the growth of the national wealth, GDP and total labor supply. The output levels and two input factors of the two sectors are augmented. The rate of interest falls in tandem with rising national wealth. The price of consumer goods falls. The rich's human capital as well as the poor and the middle are all enhanced, with the rich's human capital being increased much higher than the poor and middle's. It should be noted that an improvement in the rich's human capital utilization efficiency increases not only the rich's per household wealth, consumption level of services and wage income, but also the middle's and the poor's. In summary, an improvement in the rich's human capital utilization efficiency enlarges the gaps between the rich and the poor and the rich and middle, and benefits everyone in society (except everyone working longer hours). By the way it should be remarked that the impact of human capital is currently a main topic in economic theory and empirical research (e.g., Easterlin, 1981, Hanushek and Kimko, 2000; Barro, 2001; Krueger and Lindahl, 2001; Bandyopadhyay and Tang, 2011; Castelló-Climent and Hidalgo-Cabrillana, 2012). There are different empirical conclusions about inequalities and human capital (e.g., Tilak, 1989; Could *et al*. 2001; Tselios, 2008; Fleisher *et al.* 2011). Our study addresses issues related to dynamic interactions among growth, inequality and distribution by assuming heterogeneity in preferences and human capital utilization efficiencies among different types of people. Our conclusion with regard to the rich's human capital change is that it benefits everyone and worsen equality.

Figure 2 – The Rich Applying Human Capital More Effectively

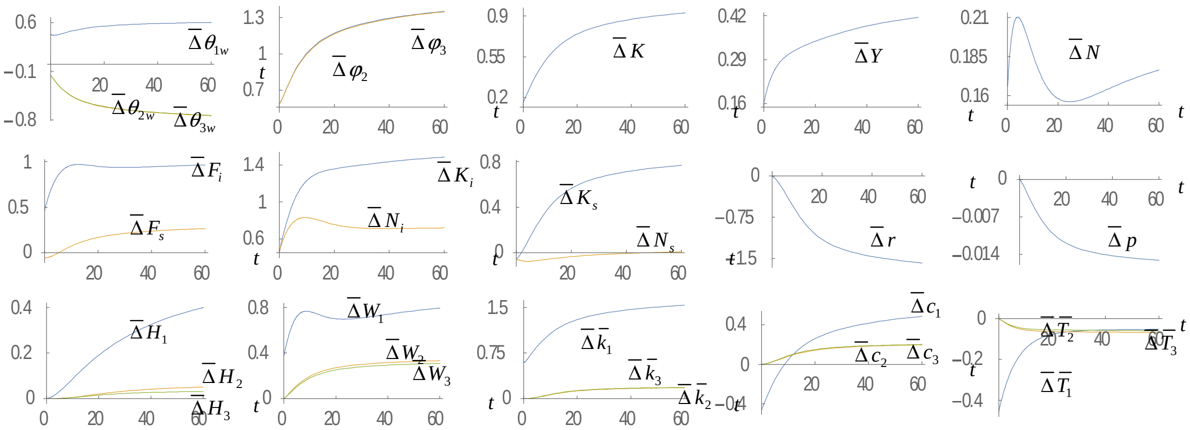


The rich's propensity to save being augmented

If the rich has become too rich to spend their wealth and become less interested in showing off with wealth, their propensity to save tends to be enhanced. We now increase the rich's propensity to save in the following way: $\lambda_{10} : 0.94 \Rightarrow 0.95$. The simulation results are plotted in Figure 3. The effects caused by the rise in the rich's propensity to save are quite similar to the effects of an improvement in the rich's human capital utilization efficient. As the rich increases their propensity to save, the inequality between the rich and the poor is enlarged. The rich get higher share of the national wealth and the ratios of per household wealth between the poor and rich and between the middle and rich are increased.

The national wealth, GDP and total labor supply are augmented. The output levels and two input factors of the two sectors are increased. The rate of interest and the price of consumer goods are lowered. The rich's human capital as well as the poor and the middle are all enhanced. It should be noted that although the rich's consumption of consumer goods is reduced in the short term, it is enhanced in the long term. This occurs as in the short term the increased propensity to save makes the rich consume less from the disposable income. But their consumption is increased in the long term as more wealth enables the rich to accumulate more human capital and the rich work long hours due to the change in the preference. We see that in the long term any household's per household wealth, consumption level of services and wage income are augmented. In summary, a rise in the rich's propensity to save benefits the national economic variables, enlarges the gaps between the rich and the poor and the rich and middle, and benefits everyone in society (except everyone working longer hours).

Figure 3 – The Rich's Propensity to Save Being Augmented

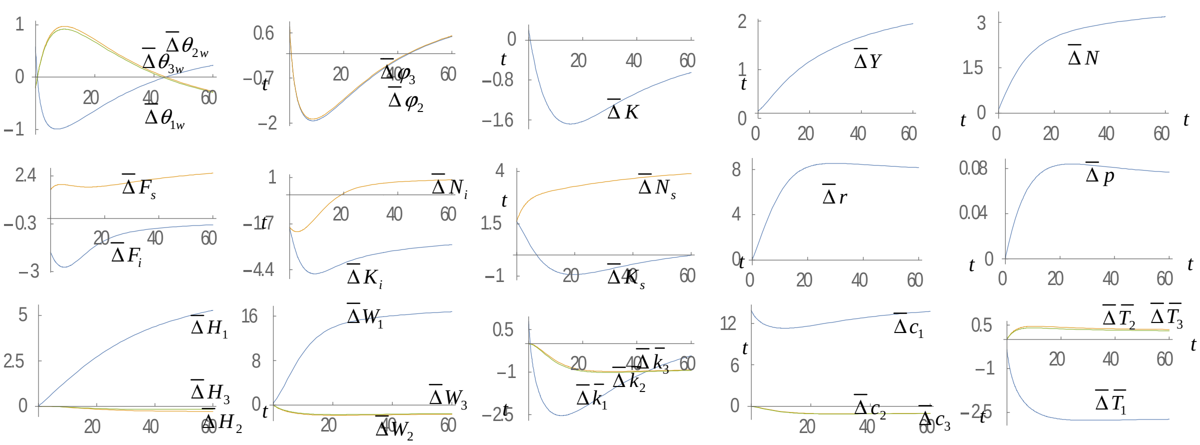


The rich's propensity to consume being increased

We now increase the rich's propensity to consume in the following way: $\xi_{10} : 0.07 \Rightarrow 0.08$. The simulation results are plotted in Figure 4. Per household of all the households is reduced by the rich's preference change. In the short term the inequalities are "improved" as the share of the rich in the national wealth falls and the ratios of the per household's wealth levels are shrunk. In the long term the inequalities are "deteriorated" as the share of the rich in the national wealth rises and the ratios of the per household's wealth levels are enhanced. This occurs partly because in our approach working experiences, owning wealth and consuming experiences all can affect human capital accumulation. As the rich consume more, they may grasp more business opportunities, learn more about business and build more productive/useful human relations.

The rich's human capital is thus enhanced. As they earn more income, their share of national wealth will be increased more rapidly than the other two groups. The rich work more hours and the middle and the poor work less hours. The rich get more wage income and the other two groups less. The national wealth is reduced and the national GDP and labor supply are augmented. The rate of interest and price of consumer goods rise. The output of the consumer goods sector is enhanced and the output of the capital goods sector is reduced.

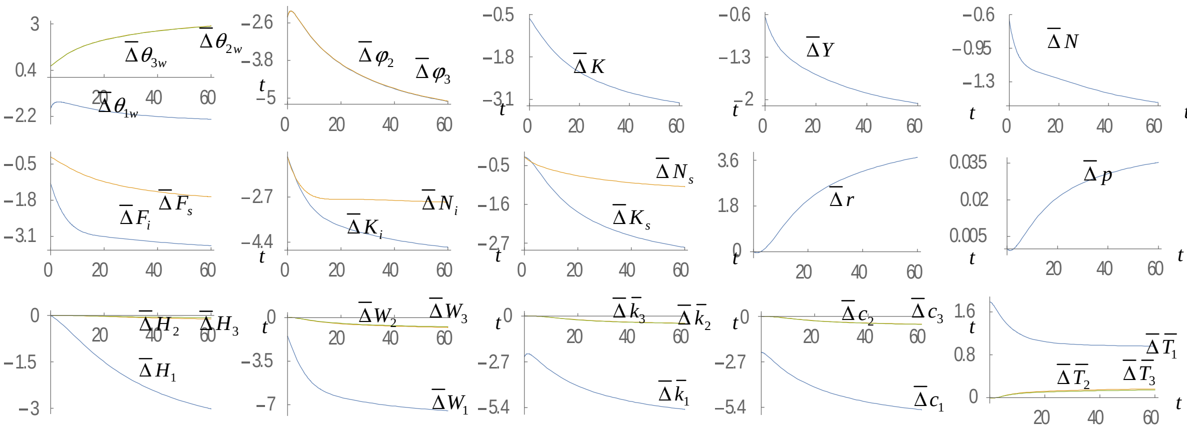
Figure 4 – The Rich's Propensity to Consume Being Increased



The rich's propensity to enjoy leisure being increased

Another change in the rich's preference is that they prefer more consuming leisure. To examine the impact of their preference change, we now increase the rich's propensity to enjoy leisure as follows: $\sigma_{10} : 0.25 \Rightarrow 0.26$. The simulation results are plotted in Figure 5. The inequalities are improved as the share of the rich in the national wealth falls and the ratios of the per household's wealth levels are shrunk. As the rich spend more time leisure, their human capital fall. The national wealth, GDP and labor supply all fall. The rate of interest and price of consumer goods rise. The output levels of the two sectors and each sector's two inputs are enhanced. The household from any group has lower human capital, works less hours, consumes less goods, and owns less wealth. In summary as the rich tend to enjoy more leisure, the inequalities are reduced but everyone worsens off (except having more leisure hours).

Figure 5 – The Rich's Propensity to Enjoy Leisure Being Increased

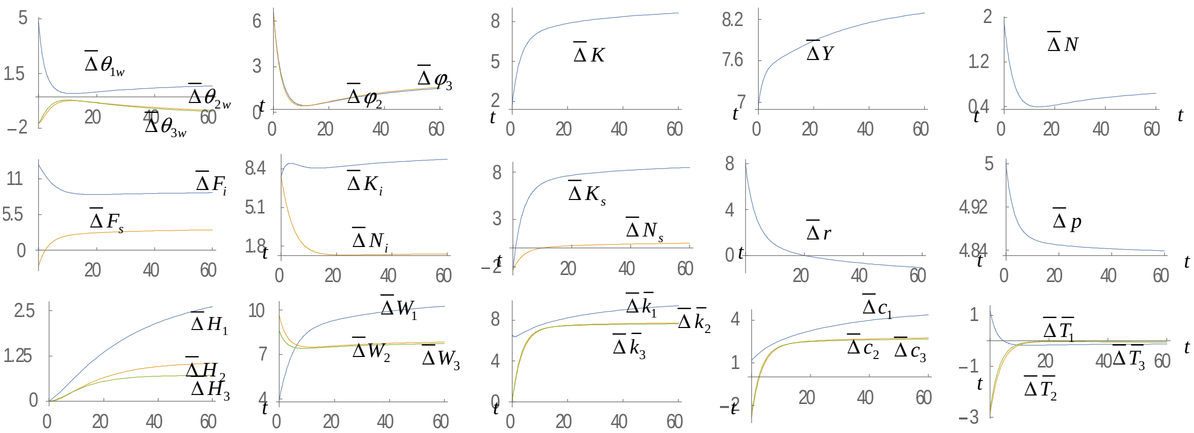


A rise in the total factor productivity of the capital goods sector

Another important question is what will happen to different people and the national economy if the total factor productivity of the capital goods sector is increased. We increase the total productivity in the following way: $A_i : 1 \Rightarrow 1.05$. The simulation results are plotted in Figure 6. The rise in the productivity increases human capital and wage incomes of all the groups. The rate of interest rises initially and falls in the long term. The price of consumer goods rises. The distribution of the total labor force is slightly affected. The two sectors increase the output levels in the long term. The wealth and consumption levels of all the groups are increased in the long term. The national wealth, GDP and total labor supply are all increased. The inequality between the rich and the poor is enlarged. The rich get higher share of the national wealth and the ratios of per household wealth between the poor and rich and between the middle and rich are increased. It should be further noted that economists have been concerned with relations between wealth and income distribution and growth have long time ago. Kaldor (1956) argues that as income inequality is enlarged, growth should be encouraged as savings are promoted. This positive relation between income inequality and growth is also observed in studies, for instance, by Bourguignon (1981), Forbes (2000), and Frank (2009).

There are other studies which find negative relations between income inequality and economic growth. Some mathematical models which predicate negative relations are referred to, for instance, Galor and Zeira (1993) and Galor and Moav (2004), and Benabou (2002). Some empirical studies by, for instance, Persson and Tabellini (1994), also confirm negative relations. From our simulation, we see that relations between inequality and economic growth are complicated in the sense that these relations are determined by many factors. The relation are expectably ambiguous or development-dependent in the sense that one may observe positive or negative relations according the parameter values combinations and state of economic development. We already demonstrated that if the rich increase their human capital utilization efficiency or the total factor productivity of the capital goods sector is increased, the economy experiences positive growth and inequalities are enlarged. It is also straightforward to show that if either the poor or the middle increase their human capital utilization efficiency, the economy will experience positive growth and inequalities between are shrunk.

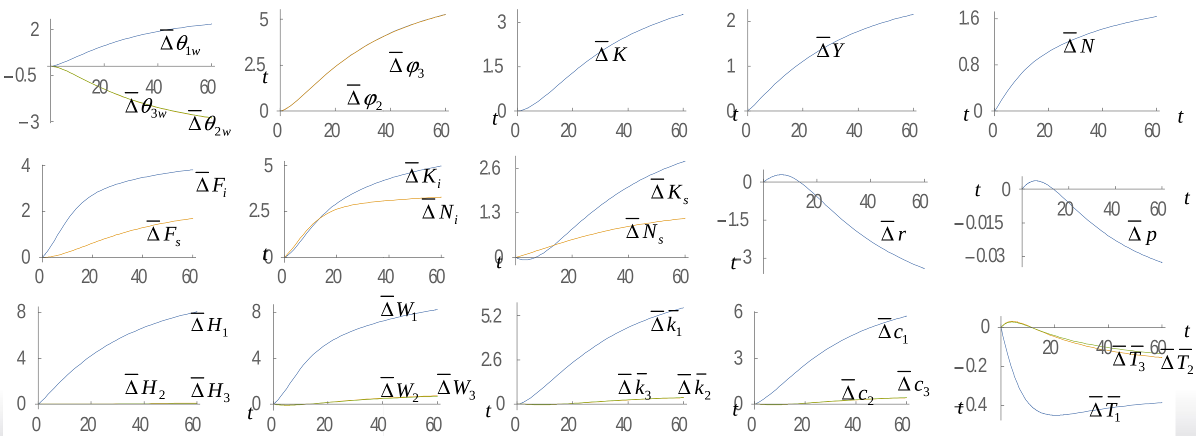
Figure 6 – A Rise in the Total Factor Productivity of the Capital Goods Sector



The rich's learning through consuming being strengthened

We now strengthen the impact of the rich's learning through consuming upon human capital accumulation as follows: $a_1 : 0.3 \Rightarrow 0.31$. As the rich's consumption affects human capital accumulation more strongly, the wealth gap between the rich and the poor is enlarged. The rich get higher share of the national wealth and the ratios of per household wealth between the poor and rich and between the middle and rich are enhanced. The rich's human capital is augmented, which also increases the growth of the national wealth, GDP and total labor supply. The output levels and two input factors of the two sectors are augmented. The rate of interest and the price of consumer goods fall. We see that the parameter change enlarges the gaps between the rich and the poor and the rich and middle, and benefits everyone in society (except everyone working longer hours).

Figure 7 – The Rich's Learning through Consuming Being Strengthened



5. Concluding Remarks

This paper built a dynamic economic model of heterogeneous households to explain some economic mechanisms of how the richest one per cent of the population own 50% of national wealth. The main determinants of growth and inequality are endogenous wealth and human capital accumulation under perfectly competitive conditions. The production technologies and economic structure follow the Uzawa two-sector model. In this study a household's disposable income is the traditional disposable income (which is the income that a household earns each period of time after taxes and transfers in the Solow model and many empirical studies) plus the value of the household's wealth.

By applying Zhang's concept of disposable income and utility approach, we describe consumers' wealth accumulation and consumption behavior. We showed how wealth accumulation, human capital accumulation, and division of labor, and time distribution interact with each other under perfect competition. We simulated the model with three groups of the population, the rich 1 %, the middle 69%, and the poor 20%. We demonstrated the existence of an equilibrium point at which the rich 1% do own more than half of the national wealth and the poor 20% less than 10% of the national wealth. We showed how the system moves to the equilibrium from an initial state and confirm that the equilibrium point is stable. We also demonstrated how changes in the total factor productivity of the capital goods sector, the rich's human capital utilization efficiency, the rich's efficiency of learning through consuming, and the rich's propensities to save, to consume, and to enjoy leisure, affect growth and inequality.

Although our comparative dynamic analysis does not find a situation of 'the rich get richer and the poor get poorer', we show that inequalities can be enlarged in tandem with economic growth, for instance, when the total factor productivity of the capital goods sector is increased.

The study has many obvious limitations when we look at real economic systems. For instance, we assume that there is no social mobility in the economic system. Although for a "mature" social and economic system, it is rate for the poor to become rich. In a case like China, about three decades ago there was almost no rich in the entire country. The educated and rich could rarely survive during the Mao period. Today there are so many really rich people who had never dreamt of becoming rich even two decades ago. Our model does not explain this kind of phenomena.

This study does not consider the role of the government in redistributing wealth and income. It is important to see how the government can affect distribution with various policies. We carried out comparative dynamic analysis only with respect change in a single parameter. It is more insightful to allow multiple parameters to be changed simultaneously. Another important issue is how to introduce endogenous change in preferences of different people. We may extend the model in some other directions. We may introduce education and allow some kind of government intervention in education. In this study, we don't consider public provision or subsidy of education.

Acknowledgements: The author thanks the anonymous referees and Editor Fabio Musso for the constructive comments. The author would also like to extend his appreciation to the financial support from the Grants-in-Aid for Scientific Research (C), Project No. 25380246, Japan Society for the Promotion of Science.

References

1. Amir, R., & Evstigneev, I.V. (1999). Stochastic Version of Polterovich's Model: Exponential Turnpike Theorems for Equilibrium Paths. *Macroeconomic Dynamics*, 3(2), 149-66.
2. Arrow, K.J. (1962). The Economic Implications of Learning by Doing. *Review of Economic Studies*, 29(3), 155-73.
3. Arrow, K. J. (1974). General Economic Equilibrium: Purpose, Analytic Techniques, Collective Choice. *American Economic Review*, 64(3), 253-72.
4. Arrow, K. J. (1974). General Economic Equilibrium: Purpose, Analytic Techniques, Collective Choice. *American Economic Review*, 64(3), 253-72.
5. Arrow K.J., & Debreu, G. (1954). Existence of an Equilibrium for a Competitive Economy. *Econometrica*, 22(3), 265–90.
6. Arrow, K.J., & Hahn, F.H. (1971). *General Competitive Analysis*. San Francisco: Holden-Day, Inc.
7. Azariadis, C. (1993). *Intertemporal Macroeconomics*. Oxford: Blackwell.
8. Barro, R.J. (2001). Human Capital and Growth. *American Economic Review*, 91(2), 12-17.
9. Barro, R.J., & X. Sala-i-Martin (1995). *Economic Growth*. New York: McGraw-Hill, Inc.
10. Benabou, R. (2002). Tax and Education Policy in a Heterogeneous-Agent Economy: What Levels of Redistrbution Maximize Growth and Efficiency? *Econometrica* 70(2), 481-517.
11. Bewley, T. (1982). An Integration of Equilibrium Theory and Turnpike Theory. *Journal of Mathematical Economics*, 10(1982), 233-67.
12. Bourguignon, F. (1981). Parento-superiority of Unegalitarian Equilibria in Stiglitz's Model of Wealth Distribution with Convex Saving Function. *Econometrica*, 49(6), 1469-75.
13. Burmeister, E., & Dobell, A.R. (1970). *Mathematical Theories of Economic Growth*. London: Collier Macmillan Publishers.
14. Caselli, F., & Ventura, J. (2000). A Representative Consumer Theory of Distribution. *American Economic Review* 90(4), 909–26.
15. Castelló-Climent, A., & Hidalgo-Cabrillana, A. (2012). The Role of Education Quality and Quantity in the Process of Economic Development. *Economics of Education Review*, 31(4), 391-409.
16. Chatterjee, S. (1994). Transitional Dynamics and the Distribution of Wealth in a Neoclassical Growth Model. *Journal of Public Economics*, 54(1), 97–119.
17. Dana, R.A, Florenzano, M., & Levy, D. (1989). Production Prices and General Equilibrium Prices: A Long-run Property of a Leontief Economy. *Journal of Mathematical Economics*, 18(3), 263–80.
18. Debreu, G. (1959). *Theory of Value: An Axiomatic Analysis of Equilibrium*. Yale University Press, London.
19. Diamond, P.A. (1965). Disembodied Technical Change in a Two-Sector Model. *Review of Economic Studies*, 32(2), 161-68.
20. Diewert, W. (1977). Walras' Theory of Capital Formation and the Existence of a Temporary Equilibrium. In G. Schwödiauer (Ed.). *Equilibrium and Disequilibrium in Economic Theory*. Dordrecht: Reidel.
21. Drugeon, J.P., & Venditti, A. (2001). Intersectoral External Effects, Multiplicities & Indeterminacies. *Journal of Economic Dynamics & Control*, 25(5), 765-87.
22. Easterlin, R. (1981). Why Isn't the Whole World Developed? *Journal of Economic History*, 41(1), 1-19.
23. Eatwell, J. (1987). Walras' Theory of Capital. In J., Eatwell, M., Milgate, & P. Newmann (Eds.) *The New Palgrave Dictionary of Political Economy*. London: MacMillan.

24. Fleisher, B., Hu, Y.F., Li, H.Z., & Kim, S.H. (2011). Economic Transition, Higher Education and Worker Productivity in China. *Journal of Development Economics*, 94(1), 86-94.
25. Forbes, K. (2000). A Reassessment of the Relationship between Inequality and Growth. *American Economic Review*, 90(4), 869-70.
26. Frank, M.W. (2009). Inequality and Growth in the United States: Evidence from a New State-level Panel of Income Inequality Measures. *Economic Inquiry*, 47(1), 55-68.
27. Gale, D. (1955). The Law of Supply and Demand. *Mathematica Scandinavica* 3, 33–44
28. Galor, O., & Moav, O. (2004). From Physical to Human Capital Accumulation: Inequality in the Process of Development. *Review of Economic Studies*, 71(4), 1001-26.
29. Galor, O., & Zeira, J. (1993). Income Distribution and Macroeconomics. *Review of Economic Studies* 60(1), 35-52.
30. Hanushek, E., & Kimko, D. (2000). Schooling, Labor-Force Quality and the Growth of Nations. *American Economic Review*, 90(5), 1194-204.
31. Impicciatore, G., Panaccione, L., & Ruscitti, F. (2012). Walras's Theory of Capital Formation: An Intertemporal Equilibrium Reformation. *Journal of Economic Theory*, 106(2), 99-118.
32. Jensen, B.S., & Larsen, M.E. (2005). General Equilibrium Dynamics of Multi-Sector Growth Models. *Journal of Economics*, 10(suppl.), 17-56.
33. Kaldor, N. (1956). Alternative Theories of Distribution. *Review of Economic Studies* 23, 83–100.
34. Krueger, A.B., & Lindahl, M. (2001). Education for Growth: Why and for Whom. *Journal of Economic Literature*, 39(December), 1101-36.
35. Maliar, L., & Maliar, S. (2001). Heterogeneity in Capital and Skills in a Neoclassical Stochastic Growth Model. *Journal of Economic Dynamics and Control*, 38(9), 635–54.
36. Mas-Colell, A., Whinston, M.D., & Green, J.R. (1995). *Microeconomic Theory*. New York: Oxford University Press.
37. McKenzie, L.W. (1959). On the Existence of General Equilibrium for a Competitive Market. *Econometrica*, 27(1), 54–71.
38. Mino, K. (1996). Analysis of a Two-Sector Model of Endogenous Growth with Capital Income Taxation. *International Economic Review*, 37(1), 227-51.
39. Montesano, A. (2008). A Restatement of Walras' Theory of Capitalization and Money. *History of Economics Review*, 47(winter), 86–109.
40. Morishima, M. (1964). *Equilibrium, Stability and Growth*. Oxford: Oxford University Press.
41. Morishima, M. (1977). *Walras' Economics*. Cambridge: Cambridge University Press.
42. Nikaido, H. (1956). On the Classical Multilateral Exchange Problem. *Metroeconomica*, 8(3), 135–45.
43. Nikaido, H. (1968). *Convex Structures and Economic Theory*. New York: Academic Press.
44. Pasinetti, L.L. (1960). A Mathematical Formulation of the Ricardian System. *Review of Economic Studies*, 27(2), 78-98.
45. Pasinetti, L.L. (1974). *Growth and Income Distribution - Essays in Economic Theory*. Cambridge: Cambridge University Press.
46. Penalosa, C.G., & Turnovsky, S.J. (2006). Growth and Income Inequality: A Canonical Model. *Economic Theory*, 28(1), 25-49.
47. Persson, T., & Tabellini, G.E. (1994). Is Inequality Harmful for Growth? Theory and Evidence. *American Economic Review*, 84(3), 600–21.
48. Polterovich, V. (1977). Models of Equilibrium Economic Growth. *Matecon,* 13(1), 3-24.
49. Polterovich, V. (1983). Equilibrium Trajectories of Economic Growth. *Econometrica*,

51(3), 693-730.
50. Samuelson, P.A. (1959). A Modern Treatment of the Ricardian Economy: I. The Pricing of Goods and Labor and Land Services. *The Quarterly Journal of Economics*, 73(1), 1-35.
51. Solow, R. (1956). A Contribution to the Theory of Growth. *Quarterly Journal of Economics*, 70(1) 65-94.
52. Stiglitz, J.E. (1967). A Two Sector Two Class Model of Economic Growth. *Review of Economic Studies*, 34(2), 227-38.
53. Tilak, J.B.C. (1989). *Education and Its Relation to Economic Growth, Poverty and Income Distribution: Past Evidence and Future Analysis*. Washington: World Bank.
54. Tselios, V. (2008). Income and Educational Inequalities in the Regions of the European Union: Geographically Spillovers under Welfare State Restrictions. *Papers in Regional Science*, 87(3), 403-30.
55. Turnovsky, S.J., & Penalosa, C.G. (2006). Distributional Dynamics in a Neoclassical Growth Model: The Role of Elastic Labor Supply. *Journal of Economic Dynamics & Control*, 32(5), 1399-431.
56. Uzawa, H. (1961). On a Two-Sector Model of Economic Growth. *Review of Economic Studies*, 29(1), 47-70.
57. Uzawa, H. (1965). Optimal Technical Change in an Aggregative Model of Economic Growth. *International Economic Review*, 6(1), 18-31.
58. Walras, L. (1874). *Elements of Pure Economics*, translated from the French by W. Jaffé, 1954. London: Allen and Unwin.
59. Zhang, W.B. (1993). A Woman's Labor Participation and Economic Growth: Creativity, Knowledge Utilization and Family Preference. *Economics Letters*, 42(4), 105-10.
60. Zhang, W.B. (2006). *Growth with Income and Wealth Distribution*. London: Macmillan.
61. Zhang, W.B. (2007). Economic Growth with Learning by Producing, Learning by Education, and Learning by Consuming. *Interdisciplinary Description of Complex Systems* 5(1), 21-38.
62. Zhang, W.B. (2012). Economic Growth with Heterogeneous Households, Gender Division of Labor, and Elastic Labor Supply. *Journal of Income Distribution* 21(2), 15-37.
63. Zhang, W.B. (2014). Ethnic Human Capital Externalities and Inequality in a General Equilibrium Growth Model. *Journal of the New Economic Association*, 21(1), 33-54.

Appendix: Identifying the Differential Equations in the Lemma

By (6) and (8) we obtain

$$z \equiv \frac{r + \delta_k}{w} = \frac{N_q}{\bar{\beta}_q K_q}, \ q = i, s, \tag{A1}$$

where $\bar{\beta}_q \equiv \beta_q / \alpha_q$. From (A1) and (3), we obtain

$$\frac{N_i}{\bar{\beta}_i} + \frac{N_s}{\bar{\beta}_s} = z \sum_{j=1}^{J} \bar{k}_j \bar{N}_j , \tag{A2}$$

where we also use (4). Insert (A1) in (6)

$$r(z) = \alpha_r z^{\beta_i} - \delta_k , \ \ w(z) = \alpha z^{-\alpha_i} , \tag{A3}$$

where

$$\alpha_r = \alpha_i A_i \bar{\beta}_i^{\beta_i} , \ \ \alpha = \beta_i A_i \bar{\beta}_i^{-\alpha_i} .$$

We have

$$w_j(z, H_j) = H_j^{m_j} w. \tag{A4}$$

Hence, we determine the rate of interest and the wage rates as functions of z and (H_j). From (7) and (8), we have

$$p(z) = \frac{\bar{\beta}_s^{\alpha_s} z^{\alpha_s} w}{\beta_s A_s}. \tag{A5}$$

From (A4) and the definitions of $\bar{y}_j$, we have

$$\bar{y}_j = (1 + r)\bar{k}_j + T_0 w_j . \tag{A6}$$

Insert $p c_j = \xi_j \bar{y}_j$ in (18)

$$\sum_{j=1}^{J} \xi_j \bar{N}_j \bar{y}_j = p F_s . \tag{A7}$$

Substituting (A6) in (A7) yields

$$N_s = \sum_{j=1}^{J} \tilde{g}_j \bar{k}_j + \tilde{g}, \tag{A8}$$

where we use $pF_s = wN_s / \beta_s$ and

$$\tilde{g}_j(z) \equiv \bar{r}\beta_s \xi_j \bar{N}_j, \ \ \bar{r}(z) \equiv \frac{1+r}{w}, \ \ \tilde{g}(z, (H_j)) \equiv \beta_s T_0 \sum_{j=1}^{J} H_j^{m_j} \xi_j \bar{N}_j.$$

Inserting $w_j \bar{T}_j = \sigma_j \bar{y}_j$ in (12), we have

$$T_j = T_0 - \frac{\sigma_j \bar{y}_j}{w_j}. \tag{A9}$$

Insert (A6) in (A9)

$$T_j = (1-\sigma_j)T_0 - \frac{(1+r)\sigma_j \bar{k}_j}{w_j}. \tag{A10}$$

Insert (A10) in (1)

$$N = n_0 - \sum_{j=1}^{J} n_j \bar{k}_j, \tag{A11}$$

where

$$n_0(z, (H_j)) \equiv T_0 \sum_{j=1}^{J} (1-\sigma_j) \bar{N}_j H_j^{m_j}, \ \ n_j(z, (H_j)) \equiv \bar{r}\sigma_j \bar{N}_j.$$

Substituting (A8) and (A11) into yields

$$N_i(z, (H_j), (\bar{k}_j)) = n_0 - \tilde{g} - \sum_{j=1}^{J} (n_j + \tilde{g}_j)\bar{k}_j. \tag{A12}$$

Insert (A12) and (A8) in (A2)

$$\bar{k}_1 = \varphi(z, \{\bar{k}_j\}, (H_j)) \equiv \left(\frac{n_0 - \tilde{g}}{\beta_i} + \frac{\tilde{g}}{\beta_s} - \sum_{j=2}^{J} \phi_j \bar{k}_j \right) \frac{1}{\phi_1}, \tag{A13}$$

in which

$$\phi_j(z, (H_j)) \equiv z\bar{N}_j + \frac{n_j + \tilde{g}_j}{\beta_i} - \frac{\tilde{g}_j}{\beta_s}, \ \ \{\bar{k}_j\} \equiv (\bar{k}_2, ..., \bar{k}_J).$$

It is straightforward to confirm that all the variables can be expressed as functions of z, $\{\bar{k}_j\}$ and (H_j) by the following procedure: r and w by (A3) $\to$ w_j by (A4) $\to$ p by (A5) $\to$ $\bar{k}_1$ by (A18) $\to$ N_i by (A12) $\to$ N by (A11) $\to$ N_s by (A8) $\to$ $\bar{y}_j$ by (A6) $\to$ K_i and K_s by (A1) $\to$ F_i and F_s, by the definitions $\to$ $\bar{T}_j$, c_j, and s_j by (15) $\to K$ by (4). From this procedure, (A13), (16), and (17), we have

$$\dot{\bar{k}}_1 = \bar{\Omega}_1\left(z, \{\bar{k}_j\}, (H_j)\right) \equiv \lambda_1 \bar{y}_1 - \varphi, \tag{A14}$$

$$\dot{\bar{k}}_j = \Lambda_j\left(z, \{\bar{k}_j\}, (H_j)\right) \equiv \lambda_j \bar{y}_j - \bar{k}_j, \quad j = 2, ..., J,$$
$$\dot{H}_j = \Omega_j\left(z, \{\bar{k}_j\}, (H_j)\right), \quad j = 1, ..., J, \tag{A15}$$

Taking derivatives of equation (A13) with respect to t and combining with (A15), we get

$$\dot{\bar{k}}_1 = \frac{\partial \varphi}{\partial z}\dot{z} + \sum_{j=2}^{J}\Lambda_j \frac{\partial \varphi}{\partial \bar{k}_j} + \sum_{j=1}^{J}\Omega_j \frac{\partial \varphi}{\partial H_j}. \tag{A16}$$

Equaling the right-hand sizes of equations (A14) and (A16), we get

$$\dot{z} = \Lambda_1\left(z, \{\bar{k}_j\}, (H_j)\right) \equiv \left[\bar{\Omega}_1 - \sum_{j=2}^{J}\Lambda_j \frac{\partial \varphi}{\partial \bar{k}_j} - \sum_{j=1}^{J}\Omega_j \frac{\partial \varphi}{\partial H_j}\right]\left(\frac{\partial \varphi}{\partial z}\right)^{-1}. \tag{A17}$$

In summary, we proved the lemma.