Computation Results of Finding All Efficient Points in Multiobjective Combinatorial Optimization

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Abstract: The number of efficient points in criteria space of multiple objective combinatorial optimization problems is considered in this paper. It is concluded that under certain assumptions, that number grows polynomially although the number of Pareto optimal solutions grows exponentially with the problem size. In order to perform experiments, an original algorithm for obtaining all efficient points was formulated and implemented for three classical multiobjective combinatorial optimization problems. Experimental results with the shortest path problem, the Steiner tree problem on graphs and the traveling salesman problem show that the number of efficient points is much lower than a polynomial upper bound.
Keywords: multiple objective optimization, combinatorial optimization, complexity of computation.

1 Introduction

For all combinatorial problems cardinality of the feasible solution set grows exponentially with the problem size. For one group of combinatorial problems (e.g. the shortest path problem, the shortest spanning tree problem, assignment problem etc.) algorithms that can find a solution of a single-criterion problem in polynomial time are known. This problem class is denoted by $\mathcal{P}$. For other combinatorial problems (e.g. traveling salesman problem, Steiner tree problem, knapsack problem etc.) so called *non deterministic polynomial* algorithms exist. These problems belong to the class denoted by $\mathcal{NP}$. For this problem's class it is not proved whether polynomial algorithms exist or not. There is a third class of problems (e.g. finding all spanning trees for a given graph) for which it is known that they can be solved only by exponential algorithms. In such problems result usually consists of exponential amount of data, so the exponential time is needed just to represent them. Let's denote this class by $\mathcal{E}$.

In all known literature which concerns multiobjective combinatorial optimization (MOCO), mostly the set of Pareto optimal solutions is being observed, and it has been stated that its size can grow exponentially with the problem size. Moreover, with sufficient number of uncorrelated criteria it is possible to achieve that every feasible solution is Pareto optimal [1]. This implies that there is no efficient method of determining all Pareto optimal solutions for problems of bigger dimensions, because such a procedure would not be even $\mathcal{NP}$ hard, but strictly exponential, i.e. it would belong to class $\mathcal{E}$. Such results are confirmed for many known problems, such as: the shortest spanning tree, the shortest path problem, traveling salesman problem, assignment problem and knapsack problem [2, 3, 4, 5, 7].

In Section 2 some multiobjective optimization models are introduced. Section 3 presents the implemented algorithm which is applied in order to find all efficient points of multiobjective combinatorial optimization problems. Experimental results described in Section 4 show that the number of efficient points is even much lower than a polynomial upper bound. In Section 5 some concluding remarks are formulated.

2 Multiobjective optimization models

The general model of multiobjective optimization problem can be briefly formulated as follows.

$$\min_{x \in X} f(x) \tag{1}$$

where x is the decision variable, X is the feasible solution set, and $f = (f_1,\ldots,f_p)$ is the p-dimensional vector of objective functions.

The solution $x^* \in X$ of Problem (1) is Pareto optimal if there is no other solution $x \in X$ such that $f_k(x) \leq f_k(x^*) \ \forall k = 1,\ldots,p$, where at least one of these inequalities is a strict inequality. If x^* is Pareto optimal solution of Problem (1), the point $y^* = f(x^*)$ is called the efficient point in the criteria space.

The set of all Pareto optimal solutions is called the Pareto set and it is denoted by X_{Par}. The set of all efficient points in the criteria space is called the efficient points set and it is denoted by Y_{eff}.

A solution obtained by minimizing the objective function $f_k(x)$ over X, is called the marginal solution for the k-th criterion of Problem (1).

One marginal solution which is Pareto optimal is called Pareto optimal marginal solution. If $\bar{x}$ is a Pareto optimal marginal solution then $f(\bar{x})$ is called efficient marginal point.

Remark 1. For each criterion $k = 1,\ldots,p$ at least one Pareto optimal marginal solution exists. It can be obtained by lexicographic optimization putting k-th criterion to have the first priority.

All notions regarding Problem (1) can be applied to the next models which are defined as its particular cases.

The model of multiobjective combinatorial optimization problem can be formulated as follows.

$$\min_{x \in \chi} f(x) \tag{2}$$

where x is the decision variable, χ is the feasible solution set of the model which is a subset of the power set of a finite set E (i.e. $\chi \subset \wp(E)$, $E = \{e_1,\ldots,e_m\}$), and $f = (f_1,\ldots,f_p)$ is the $p-$dimensional vector of objective functions.

Moreover, $x \in \chi$ can be represented by a binary m-dimensional variable $(x_1,x_2,\ldots,x_m) \in \{0,1\}^m$ where $x_k = 1$ if and only if the corresponding element $e_k \in E$ belongs to x and $x_k = 0$ if and only if the corresponding element $e_k \in E$ does not belong to x.

The experiments related to the number of efficient points of multiobjective optimization problems were done on three types of multiple objective network problems: multiobjective shortest path problem (SPP), multiobjective Steiner tree problem on graphs (STP) and multiobjective traveling salesman problem (TSP).

For all the three problems, an undirected graph $G = (V,E)$ with $|V| = v$ vertices, $|E| = m$ edges and $w_k : E \rightarrow \mathbb{Z}^+$, $k = 1,2,\ldots,p$ weight functions on the edges is given. In addition, for SPP starting vertex $s \in V$ and target vertex $t \in V$ and for STP set of terminal vertices $T \subset V$ are given.

For each of the problems, each feasible solution represents a specific graph structure. For SPP it is a path from s to t and the feasible set χ is a set of all such paths. For STP, χ represents set of all Steiner trees of graph G and terminal nodes T. And for TSP χ is a set of all Hamiltonian cycles.

For any of the problems, a feasible solution $x \in \chi$ is a set of edges that belong to a feasible graph structure. Alternatively, a feasible solution is a vector $(x_1,x_2,\ldots,x_m) \in \{0,1\}^m$ that satisfies a set of constraints specific to each problem.

The form of goal functions in each of above mentioned problems can be twofold, depending on the type of k-th criterion:

- When a criterion represents length or weight, the corresponding goal function is linear as follows

$$f_k(x) = \sum_{j=1}^{m} c_j^k x_j \tag{3}$$

and it is minimized.

- When a criterion type represents capacity, the corresponding goal function is

$$f_k(x) = \min_{1 \leq j \leq m} c_j^k x_j \tag{4}$$

and it is maximized.

Problems of minimizing a function of type (3) are called *minisum*, while problems of maximizing a function of type (4) are called *maximin* or *bottleneck* problems. Coefficients c_j^k, $j = 1, \ldots, m$, $k = 1, \ldots, p$ in general case are real numbers that represent length, height, weight or capacity of elements of set E. In our experiments it is presumed that coefficients c_j^k, $j = 1, \ldots, m$, $k = 1, \ldots, p$ have integer values. Practically that does not mean a loss of generality because in real life problems coefficients are rational numbers which can be transformed to integers.

In one multiobjective combinatorial optimization problem both kinds of goal function can exist. If functions of type (4) exist, they can be transformed to

$$f_k(x) = \max_{1 \leq j \leq m} \left(-c_j^k x_j \right), \tag{5}$$

that have to be minimized, in order to match the general formulation (2).

3   Applied algorithm

In order to find all efficient points for mentioned problems an original algorithm was formulated and implemented. The algorithm is based on ε-constraints method which is usualy used in apriori approach (when the decision maker's preferences about criteria are known before optimization). Our method is developed for aposteriori approach (when optimization is done with purpose to present information about nondominated solutions to decision maker).
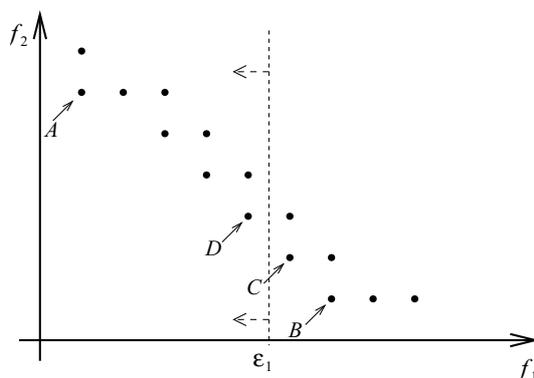

Figure 1: Sketch of applying algorithm

For finding all efficient points of a combinatorial optimization problem with two objective functions

$$\min \left\{ (f_1(x), f_2(x)) \mid x \in X \right\}$$

the following algorithm was applied.

Algorithm 2.
   ***Input:** Parameters of the problem.*

1. *For both criteria find efficient marginal points $f^k = \left(f_1^k, f_2^k \right), k = 1, 2$.*

2. *Identify index s such as $\left| f_s^1 - f_s^2 \right| = \min \left\{ \left| f_1^1 - f_1^2 \right|, \left| f_2^1 - f_2^2 \right| \right\}$. Use f_s as a search criterion and another one, denoted by f_o, as an optimization criterion.*

3. *Initialize* $Y_{eff} = \left\{ \left(f_1^1, f_2^1\right), \left(f_1^2, f_2^1\right) \right\}$ *and* $\varepsilon_s = f_s^o - 1$.

4. *Solve problem* $\min \left\{ f_o(x) \mid x \in X, f_s(x) \leq \varepsilon_s \right\}$ *and denote by* x^* *its solution.*

5. $Y_{eff} = Y_{eff} \cup \left\{ \left(f_1(x^*), f_2(x^*)\right) \right\}$.

6. *Put* $\varepsilon_s = f_s(x^*) - 1$. *If* $\varepsilon_s > f_s^s$ *then go to Step 4, otherwise STOP.*

Output: *Set* Y_{eff}.

An example of a part of criteria space and efficient points obtained by Algorithm 2 is graphically represented in Figure 1.

Points *A* and *B* represent eficient marginal points $\left(f_1^1, f_2^1\right)$, $\left(f_1^2, f_2^2\right)$ of the problem. They were obtained by independent optimization of criteria f_1 and f_2 respectively and they are efficient points of the multiobjective problem. In this example f_1 is the search criterion and f_2 is the optimization criterion. Points *C* and *D* are also efficient points of the problem and they were obtained in two succesive iterations. Efficient point *C* is obtained in the first iteration. In the next iteration point *D* will be obtained by optimizing second objective function over the initial feasible set reduced by the additional constraint $f_1(x) \leq \varepsilon_1$.

Remark 3. Algorithm 2 can be extended for problems with more than two criteria by putting one criterion to be optimization criterion and all others to be search criteria. Efficiency of such procedure can be questionable because all search criteria have to be recursively checked. This implies time and resource consumption problems. Also, efficient marginal points can be difficult to be determined in this case.

4   Experiments

Problem SPP, in its single objective version, belongs to the class $\mathcal{P}$ and its multiobjective version represents one of the most studied problems of MOCO. Problem STP in its single objective version with a *minisum* criterion belongs to the class $\mathcal{NP}$, but if objective function is of type *maximin*, it belongs to the class $\mathcal{P}$ [9]. The single objective TSP, with any type of objective function (*minisum* or *maximin*) is an $\mathcal{NP}$ hard problem.

4.1   Description of the developed computer programs

For each of the problems, a specific computer program was developed by authors. Each program determines all efficient points (and one solution for each of them) using Algorithm 2 adapted to the corresponding problem.

The programs for SPP and STP support instances with both kinds of objective functions *minisum* and *maximin*. Program for TSP supports only instances with *minisum* criteria type.

For all the experiments random instances with certain characteristics were generated. All instances had two non correlated criteria. Each result of experiments is obtained as an average of 10 randomly generated instances with the same characteristics.

For SPP problem instances had specific structure in order to provide paths to have at least $\sqrt{m}$ edges. Graph density varied between 36% and 60% depending on the number of vertices. Instances with smaller number of vertices had higher density.

Instances for examining STP problem were generated so each vertex has a certain probability to be connected to any of other vertices. Both, too dense or too spare graphs would not be proper for this kind of problem. Average number of a vertex degree was 7 for all dimensions, e.g. average density was 35% for graphs with 20 vertices and 14% for graphs with 50 vertices. The number of terminals was 5 for all instances.

TSP instances were generated in a similar way as STP, but density was higher, 50% for all graph dimensions.

4.2   Example

In order to present the procedure of performing experiments a short example of finding all efficient points of a small instance of STP problem is given below.

The instance has the following characteristic: unoriented graph with 10 vertices and 20 edges (consequently, graph density is 44% and average number of vertex degree is 4), 5 terminal nodes, two criteria of type *minisum*. Edges weights were generated randomly from interval $[10, 99]$. Criteria were not correlated.

The upper triangular matrix of edges weights for both criteria is given below.

$$
\begin{bmatrix}
0 & 98/95 & - & - & 45/17 & - & 92/12 & - & - & 41/45 \\
 & 0 & - & 67/58 & - & 12/89 & - & - & 32/47 & - \\
 & & 0 & 65/98 & 22/61 & - & 71/67 & 19/32 & - & - \\
 & & & 0 & - & 33/49 & - & 83/32 & - & - \\
 & & & & 0 & 18/70 & 11/93 & 51/47 & 10/53 & - \\
 & & & & & 0 & - & 44/51 & - & - \\
 & & & & & & 0 & - & - & 12/29 \\
 & & & & & & & 0 & - & - \\
 & & & & & & & & 0 & 15/25 \\
 & & & & & & & & & 0
\end{bmatrix}
$$

Results obtained by developed program using Algoritm 2 are presented in Table 1.

Table 1: Output results - coordinates of efficient points

No.	First criterion values	Second criterion values	Remarks
1	130	286	supported point
2	174	276	-
3	176	236	supported point
4	199	234	-
5	232	231	-
6	236	200	supported point
7	275	186	supported point

Seven efficient points were obtained and their coordinates in criteria space are given in Columns 2 and 3 of Table 1. Also, for each efficient point it is determined wether they are supported or non supported points (information given in Column 4 of Table 1).

4.3   Types of the performed experiments

Tree groups of experiments were performed.

The first group was inspired by a supposition that the upper bound for the number of efficient points depends on the length of the intervals from which edges lengths can get integer values. Three such intervals were defined: I1=[10, 99], I2=[100, 999] and I3=[1000, 9999] (as sets of two, three and four digits numbers) which contain sets of integer values of cardinality 90, 900 and 9000, respectively. Instances were generated so the lengths of edges would get random values from a certain interval, independently for each criterion. All combinations of the intervals for the first (C1) and second (C2) criterion were

checked. All the experiments from this group were performed on graphs with 20 vertices. Both criteria were of *minisum* type.

For all problems, SPP, STP and TSP, two values were observed: upper bound (UB) and actual numbers of efficient points (EFF).

The upper bound was calculated by the formula

$$UB = \min\left\{f_1^2 - f_1^1, f_2^1 - f_2^2\right\} + 1 \tag{6}$$

where $\left(f_1^k, f_2^k\right), k = 1, 2$ are efficient marginal points of the k-th criterion. Both UB and EFF are calculated as the average of results obtained from 10 randomly generated instances.

In paper [6] a different upper bound was also considered. Although that upper bound has a polynomial growth with the problems size, it is more rough than UB. On the other side for its calculation were used only parameters of instances. Upper bound UB presented in this paper is far more precise. But, in order to obtain it, p^2 optimizations per instance are necessary to be performed in order to obtain efficient marginal points.

The analysis and comparisons of upper bound and the actual number of efficient points were performed in order to get an idea about order of magnitude and relations between the values of the UB and EFF. Results of this group of experiments are given in Table 2.

Table 2: Dependence of the upper bound and efficient points number on the range of edge lengths for SPP, STP and TSP

prob:		SPP		STP		TSP	
C1	C2	UB	EFF	UB	EFF	UB	EFF
I1	I1	136	9.6	133	7.5	483	40.5
I1	I2	147	6.6	172	8.9	546	44.1
I1	I3	166	7.6	159	7.9	498	39.3
I2	I2	1027	6.5	905	8.7	4951	44.0
I2	I3	1870	9.8	1804	9.3	5489	45.1
I3	I3	13967	8.5	13461	8.0	47606	39.1
average			8.1		8.4		42.0

The second group of experiments was performed in order to check the dependency of the number of efficient points on the graph size. Because of the exponential complexity of the algorithms for finding all efficient points for all tree problem types when objective functions are of *minisum* type, the experiments were performed for instances with up to 50 vertices. Performing experiments it was concluded that the number of efficient points for STP with both criteria types *minisum* and *maximin*, significantly more depends on the number of terminal vertices than on total number of vertices in graph. Because of that, instances of problems SPP and TSP with *minisum* criteria types were compared separately.

Experiments results for this group are represented in Table 3.

In order to demonstrate "independence" of efficient points number on graphs size for STP, some experimental results are given for *maximin* criteria types. For this problem, algorithm is polynomial, so instances with much bigger number of vertices were able to be solved. These results are given in Table 4.

The third group of experiments considered the types of criteria. Three combinations were observed: when both criteria were of type *minisum* (S/S), when the first criterion was of type *maximin* and second was of type *minisum* (M/S) and when both criteria were of type *maximin* (M/M).

This time the TSP problem was excluded from the experiments because the available software did not support solving TSP with *maximin* criterion. Instances with both, 20 and 50 vertices were observed.

Table 3: Dependence of the efficient points number on the graph size for SPP and TSP

problem	SPP		TSP	
v	UB	EFF	UB	EFF
10	622	3.5	771	4.7
20	1096	8.3	4809	43.9
30	2158	15.0	8365	105.5
40	2333	19.1	14116	223.5
50	2371	16.8	18521	373.9

Table 4: Depedence of the efficient points number on the graph size for *maximin* STP

intervals:	*I1*		*I2*		*I3*		average for size
v	UB	EFF	UB	EFF	UB	EFF	EFF
50	35	5,1	224	3,9	3928	7,1	5,4
100	33	5,7	235	3,6	4269	7,2	5,5
200	24	4,0	267	5,5	3233	5,0	4,8
500	26	4,3	181	3,1	2357	3,7	3,7
1000	13	2,3	153	4,1	1858	3,9	3,4
2000	14	3,6	161	4,9	1830	4,1	4,2
average for interval		4,2		4,2		5,2	4,5

The results of the third group of experiments are represented in Table 5.

Table 5: Dependence of the efficient points number on the type of criteria for SPP and STP

criteria type:		C=S/S		C=M/S		C=M/M	
problem	v	UB	EFF	UB	EFF	UB	EFF
SPP	20	2886	8.1	938	5.4	309	2.6
	50	5689	20.0	1141	12.8	739	5.4
STP	20	2772	8.4	908	7.8	667	5.5
	50	2815	8.9	818	9.6	674	5.8

5   Conclusion

We can make the following conclusions which are based on results presented in Tables 2, 3, 5.

Although the problems SPP, STP and TSP have very different nature, their number of efficient points show very similar characteristics.

The most interesting are the values in columns EFF. First of all, they are surprisingly small, and second, the influence of the observed parameters to it is very low or even insignificant.

Observing Table 2, it is obvious and expected that upper bound UB grows with the size of the intervals from which edges take their values. Very unexpected is that the actual number of efficient points does

not show any dependence on the size of the intervals for all three problems.

It is also obvious that TSP has about five times bigger number of efficient points than SPP and STP which have similar number. Explanation is that for instances we used, TSP solutions contain more edges than solutions of SPP and STP. Consequences are that the distance between the two efficient marginal points is bigger, because of that UB is also bigger and it is expected that bigger number of efficient points can be between them.

Observing the results from Table 3 a little deviation can be noticed for SPP between graphs with 40 and 50 vertices. Namely, the number of efficient points on this stage starts to decrease. However, we concluded that the deviation is accidental, especially because in the next group of experiments, on different instances with the same characteristics (SPP, 50 vertices, S/S) is obtained value 20.0 (shown in Table 5) which matches the value expected in Table 3. Here the number of efficient points is bigger for TSP and moreover, it grows faster than for STP. This is in accordance to the previous explanation because the number of edges in solution for TSP grows linearly with number of vertices and for SPP is approximately $\sqrt{m}$.

Finally, the results from Table 5 show that the number of efficient points is smaller for *maximin* than for *minisum* type of criteria, i.e. if more criteria are of *maximin* type, smaller will be the number of efficient points. A slightly deviation of that rule is in the last row where for M/S combination of criteria types is a bigger number of efficient points than for S/S. Still we consider this as an accidental deviation.

It was mentioned before and presented in Table 4 that the number of efficient points for STP does not depend much on the number of vertices and it is also obvious from the last two rows of Table 5.

In all the considerations in this paper it was assumed that criteria are not correlated. On the other hand, the number of efficient points decreases with the increase of correlation between criteria. Since it is known that between many criteria used in practice correlation exists (the length, time and price of path, price and reliability etc.), we can expect even less number of efficient points when it comes to practical problems.

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