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The Fuzzification of Classical Structures: A General View

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Abstract: The aim of this survey article, dedicated to the 50th anniversary of Zadeh's pioneering paper "Fuzzy Sets" (1965), is to offer a unitary view to some important spaces in fuzzy mathematics: fuzzy real line, fuzzy topological spaces, fuzzy metric spaces, fuzzy topological vector spaces, fuzzy normed linear spaces. We believe that this paper will be a support for future research in this field.
Keywords: Fuzzy real line, fuzzy topological spaces, fuzzy metric spaces, fuzzy topological vector spaces, fuzzy normed linear spaces, fuzzy F-space.

1 Introduction

An introduction in the classical set theory begins, in general, in the following way: by a set we understand a collection of objects, well individualized, such that we can decide without any ambiguity whether a given element belongs to that set or not. What should we do when we cannot answer this question? Can we talk about sets described in natural language such as "the set of beautiful women" or "the set of tall men"? Although these questions are natural they were formulated only in 1965 by Lotfi A. Zadeh. In order to give answers to these questions, L.A. Zadeh [61] introduced the concept of fuzzy set.

We present bellow some thoughts of Lotfi A. Zadeh, remembering the beginnings and the current impact of fuzzy sets theory.

In [63] Lotfi A. Zadeh said: "In July of 1964, I was attending a conference in New York and was staying at the home of my parents. They were away. I had a dinner engagement but it had to be canceled. I was alone in the apartment. My thoughts turned to the unsharpness of class boundaries. It was at that point that the simple concept of a fuzzy set occurred to me. It did not take me long to put my thoughts together and write a paper on the subject. This was the genesis of fuzzy set theory. I knew that the word "fuzzy" would make the theory controversial. Knowing how the real world functions, I submitted my paper to Information and Control because I was a member of the Editorial Board. There was just one review-which was very lukewarm. I believe that my paper would have been rejected if I were not on the Editorial Board. Today (20 Dec. 2010), with over 26,000 Google Scholar citations, "Fuzzy Sets"is by far the highest cited paper in Information and Control.

My paper was a turning point in my research. Since 1965, almost all of my papers relate to fuzzy set theory and fuzzy logic. As I expected, my 1965 paper drew a mixed reaction, partly because the word "fuzzy" is generally used in a pejorative sense, but, more substantively, because unsharpness of class boundaries was not considered in science and engineering. In large measure, comments of my paper were skeptical or hostile. An exception was Japan. In 1968, I began to receive letters from Japan expressing interest in application of fuzzy set theory to pattern recognition. In the years which followed, in Japan fuzzy set theory and fuzzy logic became objects of extensive research and wide-ranging application, especially in the realm of consumer

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products. A very visible application was the subway system in the city of Sendai - a fuzzy logic-based system designed by Hitachi and Kawasaki Heavy Industry. The system began to operate in 1987 and is considered to be a great success."

On October 2, 2015 the paper "Fuzzy Sets" has already over 58,540 citations in Google Scholar and all Zadeh's papers have over 151,300 citations.

"Computation with information described in natural language (NL) is closely related to Computing with Words. NL-Computation is of intrinsic importance because much of human knowledge is described in natural language. This is particularly true in such fields as economics, data mining, systems engineering, risk assessment and emergency management. It is safe to predict that as we move further into the age of machine intelligence and mechanized decision-making, NL-Computation will grow in visibility and importance." (L.A. Zadeh, [65]).

"What is thought-provoking is that neither traditional mathematics nor standard probability theory has the capability to deal with computational problems which are stated in a natural language. Not having this capability, it is traditional to dismiss such problems as ill-posed. In this perspective, perhaps the most remarkable contribution of Computing with Words (CW) is that it opens the door to empowering of mathematics with a fascinating capability - the capability to construct mathematical solutions of computational problems which are stated in a natural language. The basic importance of this capability derives from the fact that much of human knowledge, and especially world knowledge, is described in natural language. In conclusion, only recently did I begin to realize that the formalism of CW suggests a new and challenging direction in mathematics - mathematical solution of computational problems which are stated in a natural language. For mathematics, this is an unexplored territory." (L.A. Zadeh, [64]).

Since then many authors have developed the theory of fuzzy set and its applications. Especially, many mathematicians tried to extend in fuzzy context classical mathematics results. The success of the research undertaken has been demonstrated in a variety of areas such as: artificial intelligence, computer science, quantum particle physics, control engineering, robotics and many more. Perhaps the main reason for this rapid development is that the world that surrounds us is full of uncertainty, the data we collect from the environment are, in general, vague and incorrect. So the notion of fuzzy set allows us to study the degree of uncertainty mentioned above in a purely mathematical way.

2 Fuzzy Sets

The concept of fuzzy set was introduced by L.A. Zadeh [61] in 1965.

Definition 1. [61] A fuzzy set in X is a function $\mu : X \to [0, 1]$. We denote by $\mathcal{F}(X)$ the family of all fuzzy sets in X.

Remark 2. In fact μ is the membership function of a fuzzy set A of X and the value $\mu(x)$ represents "the grade of membership" of x to fuzzy set A. But, in this paper, we adopt the convention to identify fuzzy sets with their membership functions. This identification was first used by J.A. Goguen [19].

Remark 3. As any subset of X can be identified with its characteristic function we remark that fuzzy sets generalize subsets.

Definition 4. [61] Let μ, ν be fuzzy sets in X. The union of fuzzy sets μ şi ν, denoted $\mu \vee \nu$, the intersection of fuzzy sets μ şi ν, denoted $\mu \wedge \nu$, the complement of fuzzy set μ, denoted $1 - \mu$, are fuzzy sets in X, defined by

$$(\mu \vee \nu)(x) = \max\{\mu(x), \nu(x)\} \quad (1)$$
$$(\mu \wedge \nu)(x) = \min\{\mu(x), \nu(x)\} \quad (2)$$
$$\mathcal{C}(\mu)(x) = 1 - \mu(x) \quad (3)$$

Definition 5. The union of the fuzzy sets $\{\mu_i\}_{i \in I}$ is defined by

$$\left(\bigvee_{i \in I} \mu_i\right)(x) = \sup\{\mu_i(x) \ : \ i \in I\} .$$

The intersection of the fuzzy sets $\{\mu_i\}_{i \in I}$ is defined by

$$\left(\bigwedge_{i \in I} \mu_i\right)(x) = \inf\{\mu_i(x) \ : \ i \in I\} .$$

Definition 6. Let $\alpha \in (0, 1]$, and let μ be a fuzzy set in X. The α-level set $[\mu]_\alpha$ is defined by

$$[\mu]_\alpha := \{x \in X \ : \ \mu(x) \geq \alpha\} .$$

The support of μ is

$$\text{supp}\ \mu := \{x \in X \ : \ \mu(x) > 0\} .$$

Definition 7. [61] Let X be a vector space over a field $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$). A fuzzy set μ is called convex if

$$\mu(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu(x_1), \mu(x_2)\} , \ (\forall)x_1, x_2 \in X, (\forall)\lambda \in [0, 1] .$$

The extension principle is undoubtedly one of the most important of Zadeh's contribution in fuzzy set theory, allowing to extend in a fuzzy context almost any mathematical concept. The extension principle was introduced by Zadeh [61] in 1965, and then it suffered many changes: Zadeh [62]; Jain [24]; Dubois & Prade [14]. For more details of this principle and its extensions we refer the reader to [66], [30].

Let $X = X_1 \times X_2 \times \cdots \times X_r$ and $\mu_1, \mu_2, \cdots, \mu_r$ be fuzzy sets in $X_1, X_2, \cdots, X_r$, respectively. Let $f : X \to Y$. The extension principle allows us to define a fuzzy set in Y by

$$\mu(y) = \begin{cases} \sup\limits_{(x_1, \cdots, x_r) \in f^{-1}(y)} \min\{\mu_1(x_1), \cdots, \mu_r(x_r)\} & \text{if} \quad f^{-1}(y) \neq \emptyset \\ 0 & \text{if} \quad f^{-1}(y) = \emptyset \end{cases} .$$

3 Fuzzy relations

It is well known that the fuzzy relations play an important role in fuzzy modeling and fuzzy control and they also have important applications in relational databases, approximate reasoning, preference modeling, medical diagnosis.

The concept of fuzzy relation was introduced by L.A. Zadeh in his classical paper [61]. According to L.A. Zadeh a fuzzy relation T between two nonempty sets X and Y is a fuzzy set in $X \times Y$, i.e. it is a mapping $T : X \times Y \to [0, 1]$. We denote by $FR(X, Y)$ the family of all fuzzy relations between X and Y. For $x \in X$ we denote by T_x the fuzzy set in Y defined by $T_x(y) = T(x, y)$. Thus, a fuzzy relation can be seen as a mapping $X \ni x \mapsto T_x \in \mathcal{F}(Y)$, where $\mathcal{F}(Y)$ represents the family of all fuzzy sets in Y.

Such mappings were investigated by various mathematicians under different aspects. Thus N. Papageorgiou [46] called these mappings fuzzy multifunctions and studied the continuity of these mappings. E. Tsiporkova, B. De Baets, E. Kerre [56, 57] called these maps fuzzy multivalued mappings and they defined lower and upper semi-continuous fuzzy multivalued mapping. The relationships between these two types were studied completely. The continuity of fuzzy multifunctions was also studied by I. Beg [6]. In papers [7, 8], I.Beg studied the linear fuzzy multivalued operators and vector-valued fuzzy multifunctions. An application $T : \mathbb{R}^m \to \mathcal{F}(\mathbb{R}^n)$ is called a fuzzy process (see Y. Chalco-Cano, M.A. Rojas-Medar, R. Osuna-Gómez [9]).

A special attention was given to convex fuzzy processes. They were introduced by M. Matloka [36] in 2000. Another concept of convex fuzzy process was proposed by Y. Syau, C. Low and T. Wu [55] in 2002. A comparative study of these fuzzy convex processes was made in 2010 by D. Qiu, F. Yang, L. Shu [47]. To avoid any confusion D. Qiu, F. Yang and L. Shu called the former M-convex fuzzy process and the latter SLW-convex process.

In paper [41] special types of fuzzy relations on vector spaces were considered : affine fuzzy relations, linear fuzzy relations, convex fuzzy relations, M-convex fuzzy relations. Some fundamental properties of fuzzy linear relations between vector spaces are considered in [43].

The domain $D(T)$ of T is a fuzzy set in X defined by $D(T)(x) := \sup_{y\in Y} T(x, y)$ (see [56]). We note that

$$\text{supp } D(T) = \{x \in X \ : \ T_x \neq \emptyset\} = \{x \in X \ : \ (\exists)y \in Y \text{ such that } T(x, y) > 0\} .$$

If for all $x \in \text{supp } D(T)$ there exists unique $y \in Y$ such that $T(x, y) > 0$, then T is called fuzzy function (or single-valued fuzzy function). In this case, we denote this unique y by $T(x)$.

If $\mu \in \mathcal{F}(X)$, then $T(\mu) \in \mathcal{F}(Y)$ is defined by $T(\mu)(y) := \sup_{x\in X}[T(x, y) \wedge \mu(x)]$ (see [6]). In particular, the range $R(T)$ of T is a fuzzy set in Y defined by $R(T)(y) := \sup_{x\in X} T(x, y)$ [56].

Let $T \in FR(X, Y), S \in FR(Y, Z)$. The composition $S \circ T \in FR(X, Z)$ (or simply ST) is defined by $(S \circ T)(x, z) := \sup_{y\in Y}[T(x, y) \wedge S(y, z)]$ [61].

Proposition 8. Let $T \in FR(X, Y), S \in FR(Y, Z)$. Then $(S \circ T)_x = S(T_x), (\forall)x \in X$.

Proposition 9. The operation " $\circ$ " is associative.

The inverse (or reverse relation) T^{-1} of a fuzzy relation $T \in FR(X, Y)$ is a fuzzy set in $Y \times X$ defined by $T^{-1}(y, x) = T(x, y)$. It is obvious that $R(T) = D(T^{-1})$ and $R(T^{-1}) = D(T)$. We remark that, for $\mu \in \mathcal{F}(Y)$, we have $T^{-1}(\mu)(x) = \sup_{y\in Y}[T^{-1}(y, x) \wedge \mu(x)] = \sup_{y\in Y}[T(x, y) \wedge \mu(x)]$. This type of inverse is usually called lower inverse [6].

4 Fuzzy real numbers

For the concept of fuzzy real number, arithmetic operation and ordering on the set of all fuzzy real numbers we refer the reader to the papers [13, 14, 17, 25, 26, 38, 59].

Definition 10. A fuzzy set in $\mathbb{R}$, namely a mapping $x : \mathbb{R} \to [0, 1]$, with the following properties:

1. x is convex, i.e. $x(t) \geq \min\{x(s), x(r)\}$, for $s \leq t \leq r$;

2. x is normal, i.e. $(\exists)t_0 \in \mathbb{R} \ : \ x(t_0) = 1$;

3. x is upper semicontinuous, i.e.

$$(\forall)t \in \mathbb{R}, (\forall)\alpha \in (0, 1] \ : \ x(t) < \alpha,$$
$$(\exists)\delta > 0 \text{ such that } |s - t| < \delta \Rightarrow x(s) < \alpha$$

is called a fuzzy real number. We will denote by $\mathbb{R}(I)$ the set of all fuzzy real numbers.

Remark 11. Let $x \in \mathbb{R}(I)$. For all $\alpha \in (0,1]$, the α-level sets $[x]_\alpha = \{t : x(t) \geq \alpha\}$ are closed intervals $[a_\alpha, b_\alpha]$, where the values $a_\alpha = -\infty$ and $b_\alpha = \infty$ are admissible. When $a_\alpha = -\infty$, the interval $[a_\alpha, b_\alpha]$ will be denoted by $(-\infty, b_\alpha]$.

Definition 12. A fuzzy real number x is called non-negative if $x(t) = 0, (\forall)t < 0$. The set of all non-negative real numbers will be denoted by $\mathbb{R}^*(I)$.

Remark 13. For each $r \in \mathbb{R}$ we can consider the fuzzy real number $\overline{r}$ defined by

$$\overline{r}(t) = \begin{cases} 1 & \text{if} \quad t = r \\ 0 & \text{if} \quad t \neq r \end{cases} .$$

These fuzzy numbers are called crisp. Thus $\mathbb{R}$ can be embedded in $\mathbb{R}(I)$.

Definition 14. [38] The arithmetic operations $+, -, \cdot, /$ on $\mathbb{R}(I)$, are defined by:

$$(x+y)(t) = \bigvee_{s\in\mathbb{R}} \min\{x(s), y(t-s)\}, \ (\forall)t \in \mathbb{R} \tag{4}$$

$$(x-y)(t) = \bigvee_{s\in\mathbb{R}} \min\{x(s), y(s-t)\}, \ (\forall)t \in \mathbb{R} \tag{5}$$

$$(xy)(t) = \bigvee_{s\in\mathbb{R}^*} \min\{x(s), y(t/s)\}, \ (\forall)t \in \mathbb{R} \tag{6}$$

$$(x/y)(t) = \bigvee_{s\in\mathbb{R}} \min\{x(ts), y(s)\}, \ (\forall)t \in \mathbb{R} \tag{7}$$

Remark 15. Previous definitions are special cases of Zadeh's extension principle.

Remark 16. The additive and multiplicative operations are associative and commutative with the identities $\overline{0}$ and $\overline{1}$, where

$$\overline{0}(t) = \begin{cases} 1 & \text{if} \quad t = 0 \\ 0 & \text{if} \quad t \neq 0 \end{cases} , \ \overline{1}(t) = \begin{cases} 1 & \text{if} \quad t = 1 \\ 0 & \text{if} \quad t \neq 1 \end{cases} .$$

Remark 17. It is obvious that

1. $-x = \overline{0} - x$;
2. $(-x)(t) = x(-t)$;
3. $x - y = x + (-y)$;
4. $-(x+y) = (-x) + (-y)$.

Definition 18. The absolute value $|x|$ of $x \in \mathbb{R}(I)$ is defined by

$$|x|(t) = \begin{cases} \max\{x(t), x(-t)\} & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0 \end{cases} .$$

Proposition 19. [26] The equations $a + x = \overline{0}$ and $ax = \overline{1}$ have unique solutions if and only if a is crisp.

Definition 20. [16] A partial ordering on $\mathbb{R}(I)$ is defined by

$$x \leq y \text{ if } a^1_\alpha \leq a^2_\alpha \text{ and } b^1_\alpha \leq b^2_\alpha \ , \ (\forall)\alpha \in (0,1] \ ,$$

where $[x]_\alpha = [a^1_\alpha, b^1_\alpha]$ and $[y]_\alpha = [a^2_\alpha, b^2_\alpha]$.

Proposition 21. [26] If $[a_\alpha, b_\alpha]$, $0 < \alpha \leq 1$, are the α-level sets of a fuzzy real number x, then:

1. $[a_{\alpha_1}, b_{\alpha_1}] \supseteq [a_{\alpha_2}, b_{\alpha_2}]$, $(\forall) 0 < \alpha_1 \leq \alpha_2$;
2. $[\lim_{k\to\infty} a_{\alpha_k}, \lim_{k\to\infty} b_{\alpha_k}] = [a_\alpha, b_\alpha]$, where $\{\alpha_k\}$ is an increasing sequence in $(0,1]$ converging to α.

Conversely, if $[a_\alpha, b_\alpha]$, $0 < \alpha \leq 1$, is a family of non-empty intervals which satisfy the conditions (1) and (2), then the family $[a_\alpha, b_\alpha]$ represents the α-level sets of a fuzzy real number.

Remark 22. As α-level sets of a fuzzy real number is an interval, there is a debate in the nomenclature of fuzzy real numbers. In [15], D. Dubois and H. Prade suggested to call this fuzzy interval. They developed a different notion of fuzzy real number by considering it as a fuzzy element of the real line.

5 Fuzzy topological spaces

From the notion of fuzzy set, to the notion of fuzzy topological space, there was one more step to be taken. Thus, in 1968, C.L. Chang [10] introduced the notion of fuzzy topological space. The definition is a natural translation to fuzzy sets of the ordinary definition of topological space. Indeed, a fuzzy topology is a family $\mathcal{T}$, of fuzzy sets in X, such that $\mathcal{T}$ is closed with respect to arbitrary union and finite intersection and $X, \emptyset \in \mathcal{T}$.

Definition 23. [10] Let X be an arbitrary set. A fuzzy topology on X is a family $\mathcal{T} \subset \mathcal{F}(X)$ satisfying the following axioms:

1. $\emptyset, X \in \mathcal{T}$, where $\emptyset$ is characterized by the membership function $\mu(x) = 0, (\forall) x \in X$ and X is characterized by the membership function $\mu(x) = 1, (\forall) x \in X$;
2. If $\mu_1, \mu_2 \in \mathcal{T}$, then $\mu_1 \wedge \mu_2 \in \mathcal{T}$;
3. If $\{\mu_i\}_{i\in I} \subset \mathcal{T}$, then $\bigvee_{i\in I} \mu_i \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ will be called fuzzy topological space. The elements of $\mathcal{T}$ will be called open fuzzy sets.

Definition 24. [10] Let $(X, \mathcal{T})$ be a fuzzy topological space. A fuzzy set μ_1 is a neighborhood of a fuzzy set μ_2 if there exists an open fuzzy set μ such that $\mu_2 \subseteq \mu \subseteq \mu_1$.

Theorem 5.1. Let $(X, \mathcal{T})$ be a fuzzy topological space. A fuzzy set μ is an open fuzzy set if and only for each fuzzy set $\mu_2 \subseteq \mu$, we have that μ is a neighborhood of μ_2.

Definition 25. [10] Let X, Y be arbitrary sets and $f : X \to Y$. If μ is a fuzzy set in Y, then the inverse of μ, denoted as $f^{-1}(\mu)$, is a fuzzy set in X defined by

$$f^{-1}(\mu)(x) := \mu(f(x)), (\forall) x \in X .$$

Conversely, if μ is a fuzzy set in X, the image of μ, denoted as $f(\mu)$, is a fuzzy sets in Y defined by

$$f(\mu)(y) = \begin{cases} \sup\limits_{x\in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases} .$$

Definition 26. A function f from a fuzzy topological space $(X, \mathcal{T})$ to a fuzzy topological space $(Y, \mathcal{G})$ is said to be fuzzy continuous if the inverse of each open fuzzy set is an open fuzzy set.

In 1976, R. Lowen [33] remarked that with Chang's definition constant functions between fuzzy topological spaces are not necessarily continuous. Thus R. Lowen suggested an alternative and more natural definition replacing the condition $X, \emptyset \in \mathcal{T}$ with every constant function belong to $\mathcal{T}$.

Let $(X, \mathcal{T})$ be a topological space. We recall that a function $f : X \to \mathbb{R}$ is said to be lower semi-continuous if for all $a \in \mathbb{R}$, $\{x \in X \ : \ f(x) > a\}$ is an open set in X.

Example 27. [33] Let $(X, \mathcal{T})$ be a topological space. The lower semi-continuous fuzzy topology on X associated with $\mathcal{T}$ is

$$\omega(\mathcal{T}) := \{\mu : X \to [0,1] \ : \ \mu \text{ is lower semi-continuous}\} .$$

The usual fuzzy topology on $\mathbb{K}$ is the lower semi-continuous fuzzy topology generated by the usual topology of $\mathbb{K}$.

Remark 28. [34] If $(X, \mathcal{T}_i)_{i\in I}$ is a family of topological spaces and $\mathcal{T}$ is the product topology on $X = \prod_{i\in I} X_i$, then $\omega(\mathcal{T})$ is the product of fuzzy topologies $\omega(\mathcal{T}_i), i \in I$.

Definition 29. [33] The closure and the interior of a fuzzy set μ in a fuzzy topological space $(X, \mathcal{T})$ are defined by

$$\overline{\mu} = \inf\{\mu_1 \ : \ \mu \subseteq \mu_1 \text{ and } \mathcal{C}(\mu_1) \in \mathcal{T}\}$$

$$\overset{\circ}{\mu} = \sup\{\mu_1 \ : \ \mu_1 \subseteq \mu \text{ and } \mu_1 \in \mathcal{T}\} .$$

We must note that, in paper [37], J. Michálek defined and studied another concept of fuzzy topological space which is quite different from the classic Chang's definition. In paper [35] it is shown the divergences between these two types of fuzzy topological spaces.

In paper [58], it is shown that the fuzzy continuous functions can be characterized by the closure of fuzzy sets, a subbasis of a fuzzy topology, and a fuzzy neighborhood.

In [53] a more consistent approach to the use of ideas of fuzzy mathematics in general topology has been developed.

Definition 30. [53] A fuzzy topological space is a pair $(X, \mathcal{T})$, where X is an arbitrary set and $\mathcal{T} : \mathcal{F}(X) \to [0,1]$ is a map satisfying the following axioms:

1. $\mathcal{T}(0) = \mathcal{T}(1) = 1$;
2. $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2), (\forall)\mu_1, \mu_2 \in \mathcal{F}(X)$;
3. $\mathcal{T}\left(\bigvee_{i\in I} \mu_i\right) \geq \bigwedge_{i\in I} \mathcal{T}(\mu_i), (\forall)\{\mu_i\}_{i\in I} \subseteq \mathcal{F}(X)$.

A nice survey concerning fuzzy topological spaces was written by A.P. Shostak [54]. This survey contains: various approaches to the definition of fuzzy topology, fundamental interrelations between the categories of fuzzy topology and the category of topological spaces, the notion of a fuzzy point, the convergence structure in fuzzy spaces, important topological properties for fuzzy spaces etc.

6 Fuzzy metric spaces

One of the important problems concerning the fuzzy topological spaces is to obtain an adequate notion of fuzzy metric space. Many authors have investigated this question, and several notions of fuzzy metric space have been defined and studied. We mention that the concept of fuzzy metric was introduced by I. Kramosil and J. Michálek [9] in 1975. Their notion is equivalent, in certain sense, with that of statistical metric. We note that the statistical metrics were studied many years before, and a brief survey on them was made by B. Schweizer and A. Sklar in paper [52]. We also note that, in 1994, A. George and P. Veeramani [18] modified the definition of fuzzy metric in order to obtain a Hausdorff topology on a fuzzy metric space.

Definition 31. [52] A binary operation

$$* : [0,1] \times [0,1] \to [0,1]$$

is called triangular norm (t-norm) if it satisfies the following condition:

1. $a * b = b * a, (\forall) a, b \in [0,1]$;
2. $a * 1 = a, (\forall) a \in [0,1]$;
3. $(a * b) * c = a * (b * c), (\forall) a, b, c \in [0,1]$;
4. If $a \le c$ and $b \le d$, with $a, b, c, d \in [0,1]$, then $a * b \le c * d$.

Example 32. Three basic examples of continuous t-norms are $\wedge, \cdot, *_L$, which are defined by $a \wedge b = \min\{a, b\}$, $a \cdot b = ab$ (usual multiplication in $[0,1]$) and $a *_L b = \max\{a + b - 1, 0\}$ (the Lukasiewicz t-norm).

Definition 33. [9] The triple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy metric, i.e. a fuzzy set in $X \times X \times [0, \infty)$ which satisfies the following conditions:

(M1) $M(x, y, 0) = 0, (\forall) x, y \in X$;

(M2) $[M(x, y, t) = 1, (\forall) t > 0]$ if and only if $x = y$;

(M3) $M(x, y, t) = M(y, x, t), (\forall) x, y \in X, (\forall) t \ge 0$;

(M4) $M(x, z, t + s) \ge M(x, y, t) * M(y, z, s), (\forall) x, y, z \in X, (\forall) t, s \ge 0$;

(M5) $(\forall) x, y \in X$, $M(x, y, \cdot) : [0, \infty) \to [0,1]$ is left continuous and $\lim_{t \to \infty} M(x, y, t) = 1$

Remark 34. In the definition of the fuzzy metric space, I. Kramosil and J. Michálek have imposed another condition: "$M(x, y, \cdot)$ is nondecreasing, for all $x, y \in X$". M. Grabiec [12] showed that this statement derives from the other axioms.

Indeed, for $0 < t < s$, we have

$$M(x, y, s) \ge M(x, x, s - t) * M(x, y, t) = 1 * M(x, y, t) = M(x, y, t).$$

Example 35. [18] Let (X, d) be a metric space. Let

$$M_d : X \times X \times [0, \infty), M_d(x, y, t) = \begin{cases} \frac{t}{t + d(x,y)} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}.$$

Then $(X, M_d, \wedge)$ is a fuzzy metric space. M_d is called standard fuzzy metric.

Theorem 6.1. [18] Let $(X, M, *)$ be a fuzzy metric space. For $x \in X, r \in (0,1), t > 0$ we define the open ball

$$B(x, r, t) := \{y \in X \ : \ M(x, y, t) > 1 - r\}.$$

Let

$$\mathcal{T}_M := \{T \subset X \ : \ x \in T \text{ iff } (\exists) t > 0, r \in (0,1) \ : \ B(x, r, t) \subseteq T\}.$$

Then $\mathcal{T}_M$ is a topology on X.

Proposition 36. [18] Let (X, d) be a metric space and M_d be the corresponding standard fuzzy metric on X. Then the topology $\mathcal{T}_d$ induced by the metric d, and the topology $\mathcal{T}_{M_d}$ induced by the standard fuzzy metric M_d are the same.

Definition 37. [18] Let $(X, M, *)$ be a fuzzy metric space and (x_n) be a sequence in X. The sequence (x_n) is said to be convergent if there exists $x \in X$ such that $M(x_n, x, t) = 1, (\forall)t > 0$. In this case, x is called the limit of the sequence (x_n) and we write $\lim\limits_{n\to\infty} x_n = x$, or $x_n \to x$.

Remark 38. [18] Let $(X, M, *)$ be a fuzzy metric space. A sequence (x_n) is convergent to x if and only if (x_n) is convergent to x in topology $\mathcal{T}_M$.

Indeed,

$$x_n \to x \text{ in topology } \mathcal{T}_M \Leftrightarrow$$

$$\Leftrightarrow (\forall)r \in (0,1), (\forall)t > 0, (\exists)n_0 \in \mathbb{N} \;:\; x_n \in B(x, r, t), (\forall)n \geq n_0 \Leftrightarrow$$

$$\Leftrightarrow (\forall)r \in (0,1), (\forall)t > 0, (\exists)n_0 \in \mathbb{N} \;:\; M(x_n, x, t) > 1 - r, (\forall)n \geq n_0 \Leftrightarrow$$

$$\Leftrightarrow \lim_{n\to\infty} M(x_n, x, t) = 1, (\forall)t > 0 \;.$$

Definition 39. [18] Let $(X, M, *)$ be a fuzzy metric space and (x_n) be a sequence in X. The sequence (x_n) is said to be a Cauchy sequence if

$$(\forall)r \in (0,1), (\forall)t > 0, (\exists)n_0 \in \mathbb{N} \;:\; M(x_n, x_m, t) > 1 - r, (\forall)n, m \geq n_0 \;.$$

A fuzzy metric in which every Cauchy sequence is convergent is called complete fuzzy metric space.

Definition 40. [18] Let $(X, M, *)$ be a fuzzy metric space. A subset A of X is said to be fuzzy bounded if there exist $r \in (0, 1)$ and $t > 0$ such that $M(x, y, t) > 1 - r$, for all $x, y \in A$.

Remark 41. If $(X, M, *)$ is a fuzzy metric space induced by a metric d on X, then $A \subseteq X$ is fuzzy bounded if and only if A is bounded.

We say that a topological space $(X, \mathcal{T})$ is fuzzy metrizable if the topology is generated by a fuzzy metric. V. Gregori and S. Romaguera [22] proved that a topological space is fuzzy metrizable if and only if it is metrizable.

In paper [21], the fuzzy metric $M^*(x, y, t) := \frac{\min\{x,y\}+t}{\max\{x,y\}+t}$ and other fuzzy metrics related to it were studied. This fuzzy metric is useful for measuring perceptual colour differences between colour samples.

7 Fuzzy topological vector spaces

The starting point of the theory of fuzzy topological vector spaces was a series of papers of A.K. Katsaras (see [27], [28], [29]).

Let X be a vector space over a field $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$).

Definition 42. [27] Let $\mu_1, \mu_2, \cdots, \mu_n$ be fuzzy sets in X. Then $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$ is a fuzzy set in X^n defined by

$$\mu(x_1, x_2, \cdots, x_n) = \mu_1(x_1) \wedge \mu_2(x_2) \wedge \cdots \wedge \mu_n(x_n) \;.$$

Let $f : X^n \to X \;,\; f(x_1, x_2, \cdots, x_n) = \sum\limits_{k=1}^{n} x_k$. The fuzzy set $f(\mu)$ is called the sum of fuzzy sets $\mu_1, \mu_2, \cdots, \mu_n$ and it is denoted by $\mu_1 + \mu_2 + \cdots + \mu_n$. In fact

$$(\mu_1 + \mu_2 + \cdots + \mu_n)(x) = \vee\{\mu_1(x_1) \wedge \mu_2(x_2) \wedge \cdots \wedge \mu_n(x_n) \;:\; x = \sum_{k=1}^{n} x_k\} \;.$$

Let μ be a fuzzy set in X and $\lambda \in \mathbb{K}$. The fuzzy set $\lambda\mu$ is the image of μ under the map $g : X \to X$, $g(x) = \lambda x$. Thus,

$$(\lambda\mu)(x) = \begin{cases} \mu\left(\frac{x}{\lambda}\right) & \text{if } \lambda \neq 0 \\ 0 & \text{if } \lambda = 0, x \neq 0 \\ \vee\{\mu(y) \ : \ y \in X\} & \text{if } \lambda = 0, x = 0 \end{cases} .$$

Definition 43. [28] A fuzzy topological vector space is a vector space X over $\mathbb{K}$ equipped with a fuzzy topology such that the mappings

$$+ : X \times X \to X \ , \ (x, y) \mapsto x + y$$

$$\cdot : \mathbb{K} \times X \to X \ , \ (\lambda, x) \mapsto \lambda \cdot x$$

are fuzzy continuous when $\mathbb{K}$ has the fuzzy usual topology and $X \times X$ and $\mathbb{K} \times X$ have the corresponding product fuzzy topologies.

In paper [28], the fuzzy vector topologies were characterized in terms of the corresponding families of neighborhoods of zero.

Theorem 7.1. [29] Let X be a vector space over $\mathbb{K}$, and $\mathcal{T}$ be a topology on X. Then $(X, \mathcal{T})$ is a topological vector space if and only if $(X, \omega(\mathcal{T}))$ is a fuzzy topological vector space.

8 Fuzzy normed linear spaces

Studying fuzzy topological vector spaces, A.K. Katsaras [29], introduced in 1984 for the first time, the notion of fuzzy norm on a linear space. In 1992, C. Felbin [17] introduced another concept of fuzzy norm by assigning a fuzzy real number to each element of the linear space. In 1994, S.C. Cheng and J.N. Mordeson [5] introduced another idea of fuzzy norm on a linear space such that their corresponding fuzzy metric was of Kramosil and Michálek type. Following S.C. Cheng and J.N. Mordeson, in 2003, T. Bag and S.K. Samanta [2] introduced a new concept of fuzzy norm, and studied the properties of finite dimensional fuzzy normed linear spaces. A comparative study on fuzzy norms introduced Katsaras, Felbin and Bag and Samanta was made in paper [4]. Other approaches for fuzzy normed linear spaces can be found in [1,7,10,44,48,51,60]. Recently, S. Nădăban introduced the concepts of fuzzy pseudo-norm and fuzzy F-space [11].

Different types of fuzzy bounded linear operators and the relation between fuzzy continuity and fuzzy boundedness were studied in [3], in the context of Bag-Samanta's type fuzzy normed linear spaces. The study of fuzzy continuous mappings and fuzzy bounded linear operators in fuzzy normed linear spaces initiated by T. Bag and S.K. Samanta in [3] was continued by I. Sadeqi and F.S. Kia [51] as well, as S. Nădăban [45] in a more general setting.

Fuzzy bounded linear operators in Felbin's type fuzzy normed linear space were introduced by M. Itoh and M. Cho in [23]. J.Z. Xiao and X.H. Zhu [59,60] gave a new definition for fuzzy norm of bounded operators. In [5], different definitions of strongly fuzzy bounded linear operators and weakly fuzzy bounded linear operators were given and a new idea of their fuzzy norm were introduced. In [25], some properties of the space of all weakly fuzzy bounded linear operators were studied.

In 2006, R. Saadati and J.H. Park introduced the notion of intuitionistic fuzzy Euclidean normed space (see [49], [50]). In paper [42] some special fuzzy norms on $\mathbb{K}^n$ were introduced, and in this way, fuzzy Euclidean normed spaces were obtained, . In order to introduce this concept it is proved that the cartesian product of a finite family of fuzzy normed linear spaces is a fuzzy normed linear space.

Definition 44. [27] A fuzzy set ρ in X is said to be:

1. convex if $t\rho + (1-t)\rho \subseteq \rho, (\forall)t \in [0,1]$;
2. balanced if $\lambda\rho \subseteq \rho, (\forall)\lambda \in \mathbb{K}, |\lambda| \leq 1$;
3. absorbing if $\bigvee_{t>0} t\rho = 1$;
4. absolutely convex if it is both convex and balanced.

Proposition 45. [27] Let ρ be a fuzzy set in X. Then:

1. ρ is convex if and only if
$$\rho(tx + (1-t)y) \geq \rho(x) \wedge \rho(y), (\forall)x, y \in X, (\forall)t \in [0,1];$$
2. ρ is balanced if and only if $\rho(\lambda x) \geq \rho(x), (\forall)x \in X, (\forall)\lambda \in \mathbb{K}, |\lambda| \leq 1$.

Definition 46. [29] A Katsaras fuzzy semi-norm on X is a fuzzy set ρ in X which is absolutely convex and absorbing.

Proposition 47. [31] Let ρ be a Katsaras fuzzy semi-norm on X. Let
$$p_\alpha(x) := \inf\{t > 0 \ : \ \rho\left(\frac{x}{t}\right) > \alpha\}, \alpha \in (0,1)\ .$$
Then $\mathcal{P} = \{p_\alpha\}_{\alpha\in(0,1)}$ is an ascending family of semi-norms on X.

Definition 48. [10] A fuzzy semi-norm ρ on X will be called Katsaras fuzzy norm if
$$\rho\left(\frac{x}{t}\right) = 1, (\forall)t > 0 \ \Rightarrow \ x = 0\ .$$

Remark 49. a) It is easy to see that
$$\left[\rho\left(\frac{x}{t}\right) = 1, (\forall)t > 0 \Rightarrow x = 0\right] \Leftrightarrow \left[\inf_{t>0} \rho\left(\frac{x}{t}\right) < 1, \text{for } x \neq 0\right]\ .$$
b) The condition $\left[\rho\left(\frac{x}{t}\right) = 1, (\forall)t > 0 \Rightarrow x = 0\right]$ is much weaker than that one imposed by A.K. Katsaras [29],
$$\left[\inf_{t>0} \rho\left(\frac{x}{t}\right) = 0, \text{for } x \neq 0\right]\ .$$

Proposition 50. [10] Let ρ be a Katsaras fuzzy semi-norm and
$$p_\alpha(x) := \inf\{t > 0 \ : \ \rho\left(\frac{x}{t}\right) > \alpha\}, \alpha \in (0,1).$$
Then the family of semi-norms $\mathcal{P} = \{p_\alpha\}_{\alpha\in(0,1)}$ is sufficient if and only if ρ is a Katsaras fuzzy norm.

Definition 51. [17] Let X be a vector space over $\mathbb{R}$, let $||\cdot|| : X \to \mathbb{R}^*(I)$ and let the mappings $L, R : [0,1]\times[0,1] \to [0,1]$ be symmetric, nondecreasing in both arguments and satisfy $L(0,0) = 0$ and $R(1,1) = 1$. We write $[||x||]_\alpha = [||x||_1^\alpha, ||x||_2^\alpha]$, for $x \in X, \alpha \in (0,1]$.

We suppose that $(\forall)x \in X, x \neq 0$ there exists $\alpha_0 \in (0,1]$ independent of x such that for all $\alpha \leq \alpha_0$ we have
$$(A)\ ||x||_2^\alpha < \infty\ ,$$
$$(B)\ \inf ||x||_1^\alpha > 0\ .$$
The quadruple $(X, ||\cdot||, L, R)$ is called fuzzy normed linear space and $||\cdot||$ a fuzzy norm, if

1. $||x|| = 0$ if and only if $x = 0$;
2. $||rx|| = |r| \cdot ||x||, (\forall)x \in X, r \in \mathbb{R}$;
3. for all $x, y \in X$,
   (a) whenever $s \leq ||x||_1^1, t \leq ||y||_1^1$ and $s + t \leq ||x + y||_1^1$,
   $$||x + y||(s + t) \geq L(||x||(s), ||y||(t)) ,$$
   (b) whenever $s \geq ||x||_1^1, t \geq ||y||_1^1$ and $s + t \geq ||x + y||_1^1$,
   $$||x + y||(s + t) \leq R(||x||(s), ||y||(t)) .$$

Remark 52. C. Felbin [17] proved that, if $L(x, y) = \min\{x, y\}$ and $R(x, y) = \max\{x, y\}$, then the triangle inequality (3) in previous definition is equivalent to $||x + y|| \leq ||x|| + ||y||$. Further $|| \cdot ||_\alpha^i$ are crisp norms on X, for each $\alpha \in (0, 1]$ and $i = 1, 2$.

Remark 53. In paper [5], Felbin's definition of fuzzy normed linear space is slightly modified in the sense that:

1. the value of the fuzzy norm is taken to be a fuzzy real number in the sense of J.Z. Xiao and X.H. Zhu [59];
2. the condition (A) and (B) of Felbin's definition are relaxed by the condition
$$(A\prime)\ x \neq 0 \Rightarrow ||x||(t) = 0, (\forall)t \leq 0 .$$

Definition 54. [10] Let X be a vector space over a field $\mathbb{K}$ and $*$ be a continuous t-norm. A fuzzy set N in $X \times [0, \infty)$ is called a fuzzy norm on X if it satisfies:

(N1) $N(x, 0) = 0, (\forall)x \in X$;

(N2) $[N(x, t) = 1, (\forall)t > 0]$ if and only if $x = 0$;

(N3) $N(\lambda x, t) = N\left(x, \frac{t}{|\lambda|}\right), (\forall)x \in X, (\forall)t \geq 0, (\forall)\lambda \in \mathbb{K}^*$;

(N4) $N(x + y, t + s) \geq N(x, t) * N(y, s), (\forall)x, y \in X, (\forall)t, s \geq 0$;

(N5) $(\forall)x \in X$, $N(x, \cdot)$ is left continuous and $\lim_{t\to\infty} N(x, t) = 1$.

The triple $(X, N, *)$ will be called fuzzy normed linear space (briefly FNL-space).

Remark 55. a) T. Bag and S.K. Samanta [2], [3] gave a similar definition for $* = \wedge$, but in order to obtain some important results they assumed that the fuzzy norm satisfies also the following conditions:

(N6) $N(x, t) > 0, (\forall)t > 0 \Rightarrow x = 0$;

(N7) $(\forall)x \neq 0, N(x, \cdot)$ is a continuous function and strictly increasing on the subset $\{t : 0 < N(x, t) < 1\}$ of $\mathbb{R}$.

The results obtained by T. Bag and S.K. Samanta can be found in these more general settings [10].

b) I. Goleţ [7], C. Alegre and S. Romaguera [1] gave also the same definition in the context of real vector spaces.

Remark 56. $N(x, \cdot)$ is nondecreasing, $(\forall)x \in X$.

Theorem 8.1. [10] If $(X, N, *)$ is a FNL-space, then

$$M : X \times X \times [0, \infty) \to [0, 1], M(x, y, t) = N(x - y, t)$$

is a fuzzy metric on X, which is called the fuzzy metric induced by the fuzzy norm N. Moreover, we have:

1. M is a translation-invariant fuzzy metric;
2. $M(\lambda x, \lambda y, t) = M\left(x, y, \frac{t}{|\lambda|}\right), (\forall)x \in X, (\forall)t \geq 0, (\forall)\lambda \in \mathbb{K}^*$.

Corollary 57. [10] Let $(X, N, *)$ be a FNL-space. For $x \in X, r \in (0, 1), t > 0$ we define the open ball

$$B(x, r, t) := \{y \in X \; : \; N(x - y, t) > 1 - r\} .$$

Then

$$\mathcal{T}_N := \{T \subset X \; : \; x \in T \text{ iff } (\exists)t > 0, r \in (0, 1) \; : \; B(x, r, t) \subseteq T\}$$

is a topology on X.

Moreover, if the t-norm $*$ satisfies $\sup_{x \in (0,1)} x * x = 1$, then $(X, \mathcal{T}_N)$ is Hausdorff.

Theorem 8.2. [10] Let $(X, N, *)$ be a FNL-space. Then $(X, \mathcal{T}_N)$ is a metrizable topological vector space.

9 Conclusions

Lotfi A. Zadeh, born on February 4, 1921, is a famous mathematician, electrical engineer, computer scientist, and Professor Emeritus at the University of California, Berkeley, United State of America. He is father of fuzzy sets, fuzzy logic and computing with words. His pioneering paper, entitled "Fuzzy Sets" (1965, [61]), is cited over 58,540 time in many prestigious journals, and all his papers are cited over 151,300 time.

Some scientists, especially philosophers and mathematicians, had attempted to formalize the process of logical deduction. Their work culminated in the invention of the programmable digital computer, a machine based on the abstract essence of mathematical reasoning. This machine and the ideas behind it inspired a handful of scientists to begin seriously discussing the possibility of building an artificial brain.

In this survey paper we mentioned some fuzzy mathematical structures as fuzzy real line, fuzzy topological spaces, fuzzy metric spaces, fuzzy topological vector spaces, fuzzy normed linear spaces and fuzzy F-space.

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(a) Lotfi A. Zadeh & Ioan Dzitac at ICCCC 2008 (b) Ed. by L.A. Zadeh, D. Tufis, F.G. Filip, I. Dzitac

Figure 1: Meeting with Professor Lotfi A. Zadeh (Agora University of Oradea, Romania, 2008)

Bibliography

[1] Alegre, C., Romaguera, S. (2010); Characterizations of fuzzy metrizable topological vector spaces and their asymmetric generalization in terms of fuzzy (quasi-)norms, *Fuzzy Sets and Systems*, 161: 2182–2192.

[2] Bag, T., Samanta, S.K. (2003); Finite dimensional fuzzy normed linear spaces, *J. of Fuzzy Mathematics*, 11(3): 687–705.

[3] Bag, T., Samanta, S.K. (2005); Fuzzy bounded linear operators, *Fuzzy Sets and Systems*, 151: 513–547.

[4] Bag, T., Samanta, S.K. (2008); A comparative study of fuzzy norms on a linear space, *Fuzzy Sets and Systems*, 159: 670–684.

[5] Bag, T., Samanta, S.K. (2008); Fuzzy bounded linear operators in Felbin's type fuzzy normed linear spaces, *Fuzzy Sets and Systems*, 159: 685–707.

[6] Beg, I. (1999); Continuity of fuzzy multifunctions, *J. of Applied Mathematics and Stochastic Analysis*, 12(1): 17–22.

[7] Beg, I. (2001); Linear fuzzy multivalued operators, *J. of Fuzzy Mathematics*, 9(1): 127–137.

[8] Beg, I. (2001); Vector-valued fuzzy multifunctions, *J. of Applied Mathematics and Stochastic Analysis*, 14(3): 275–282.

[9] Chalco-Cano, Y., Rojas-Medar, M.A., Osuna-Gómez, R. (2004); s-Convex fuzzy processes, *Computers and Mathematics with Applications* 47: 1411–1418.

[10] Chang, C.L. (1968); Fuzzy topological spaces, *J. Math. Anal. Appl.*, 24: 182–190.

[11] Cheng, S.C., Mordenson, J.L. (1994); Fuzzy linear operator and fuzzy normed linear spaces, *Bull. Calcutta Math. Soc.*, 86: 429–436.

[12] Cho, Y.J., Grabiec, M., Radu, V. (2006); On nonsymmetric topological and probabilistic structures, *Nova Science Publishers*, Inc., New York.

[13] Das, R.K., Mandal, B. (2002); Fuzzy real line structure and metric space, *Indian J. pure appl. Math.*, 33(4): 565–571.

[14] Dubois, D., Prade, H. (1980); *Fuzzy Sets and Systems: Theory and Applications*, Academic Press, Inc.

[15] Dubois, D., Prade, H. (2005); Fuzzy elements in a fuzzy set, *Proc. of the 10th Intl. Fuzzy Systems Association (IFSA) Congres, Beijing, China.*

[16] Eklund, P. , Gahler, W. (1988); Basic notions for fuzzy topology, *Fuzzy Sets and Systems* 26: 333–356.

[17] Felbin, C. (1992); Finite dimensional fuzzy normed liniar space, *Fuzzy Sets and Systems*, 48: 239–248.

[18] George, A., Veeramani, P. (1994); On some results in fuzzy metric spaces, *Fuzzy Sets and Systems*, 64: 395–399.

[19] Goguen, J.A. (1973); The fuzzy Tychonoff theorem, *J. of Math. Analysis and Applications*, 43: 734–742.

[20] Goleţ, I. (2010); On generalized fuzzy normed spaces and coincidence point theorems, *Fuzzy Sets and Systems*, 161: 1138–1144.

[21] Gregori, V., Miñana, J.-J., Morillas, S. (2012); Some questions in fuzzy metric spaces, *Fuzzy Sets and Systems*, 204: 71–85.

[22] Gregori, V., Romaguera, S. (2000); Some properties of fuzzy metric space, *Fuzzy Sets and Systems*, 115: 485–489.

[23] Itoh, M., Cho, M. (1998); Fuzzy bounded operators, *Fuzzy Sets and Systems*, 93: 353–362.

[24] Jain, R. (1976); Tolerance analysis using fuzzy sets, *Int. J. Syst*, 7(12): 1393–1401.

[25] Janfada, M., Baghani, H., Baghani, O. (2011); On Felbin's-type fuzzy normed linear spaces and fuzzy bounded operators, *Iranian Journal of Fuzzy Systems*, 8(5): 117–130.

[26] Kaleva, O., Seikkala, S. (1984); On fuzzy metric spaces, *Fuzzy Sets and Systems*, 12: 215–229.

[27] Katsaras, A.K., Liu, D.B. (1977); Fuzzy vector spaces and fuzzy topological vector spaces, *Journal of Mathematical Analysis and Applications*, 58: 135–146.

[28] Katsaras, A.K. (1981); Fuzzy topological vector spaces I, *Fuzzy Sets and Systems*, 6: 85–95.

[29] Katsaras, A.K. (1984); Fuzzy topological vector spaces II, *Fuzzy Sets and Systems*, 12: 143–154.

[30] Kerre, K.K. (2011); A tribute to Zadeh's extension principle, *Scientia Iranica D*, 18(3): 593–595.

[31] Krishna, S.V., Sarma, K.M. (1991); Fuzzy topological vector spaces - topological generation and normability. *Fuzzy Sets and Systems*, 41: 89–99.

[32] Kramosil, I., Michálek, J. (1975); Fuzzy metric and statistical metric spaces, Kybernetica, 11: 326–334.

[33] Lowen, R. (1976); Fuzzy topological spaces and fuzzy compactness, *Journal of Mathematical Analysis and Applications*, 56: 621–633.

[34] Lowen, R. (1977); Initial and final topologies and the fuzzy Tychonoff theorem, *J. of Math. Analysis and Applications*, 58: 11–21.

[35] Lupiáñez, F.G. (2001); On Mich alek's fuzzy topological spaces, *Kybernetika*, 37(2): 159–163.

[36] Matloka, M. (2000); Convex fuzzy processes, *Fuzzy Sets and Systems*, 110: 109–114.

[37] Michálek, J. (1975); Fuzzy topologies, *Kybernetika*, 11(5): 345–354.

[38] Mizumoto, M., Tanaka, J. (1979); Some properties of fuzzy numbers, in M.M. Gupta et all, Editors, *Advances in Fuzzy Set Theory and Applications* (North-Holland, New York), 153–164.

[39] Nădăban, S., Dzitac, I. (2014); Atomic Decompositions of Fuzzy Normed Linear Spaces for Wavelet Applications, *Informatica*, 25(4): 643-662.

[40] Nădăban, S. (2014); Fuzzy pseudo-norms and fuzzy F-spaces, *Fuzzy Sets and Systems*, doi: 10.1016/j.fss.2014.12.010.

[41] Nădăban, S., Dzitac, I. (2014); Special types of fuzzy relations, *Procedia Computer Science*, 31: 552–557.

[42] Nădăban, S. (2015); Fuzzy euclidean normed spaces for data mining applications, *International J. of Computers Communications & Control*, 10(1): 70–77.

[43] Nădăban, S. (2015); Some fundamental properties of fuzzy linear relations between vector spaces, *Filomat* (in press).

[44] Nădăban, S., Bînzar, T., Pater, F., Ţerei, C., Hoară, S. (2015); Katsaras's type fuzzy norm under triangular norms, *Theory and Applications of Mathematics & Computer Science*, 5(2): 148–157.

[45] Nădăban, S. (2015); Fuzzy continuous mappings in fuzzy normed linear spaces, *International J. of Computers Communications & Control*, 10(6): 835-843.

[46] Papageorgiou, N.S. (1985); Fuzzy topology and fuzzy multifunctions, *J. of Mathematical Analysis and Applications*, 109: 397–425.

[47] Qiu, D., Yang, F., Shu, L. (2010); On convex fuzzy processes and their generalizations, *International J. of Fuzzy Systems*, 12: 268–273.

[48] Saadati, R., Vaezpour, S.M. (2005); Some results on fuzzy Banach spaces, *J. Appl. Math. & Computing*, 17(1-2): 475–484.

[49] Saadati, R., Park, J.H. (2006); Intuitionistic fuzzy Euclidean normed spaces, *Communications in Mathematical Analysis*, 1(2): 85–90.

[50] Saadati, R., Park, J.H. (2006); On the intuitionistic fuzzy topological spaces, *Chaos, Solitons and Fractals*, 27: 331–344.

[51] Sadeqi, I., Kia, F.S. (2009); Fuzzy normed linear space and its topological structure, *Chaos, Solitons and Fractals*, 40: 2576–2589.

[52] Schweizer, B., Sklar, A. (1960); Statistical metric spaces, *Pacific J. Math.*, 10: 314–334.

[53] Shostak, A.P. (1985); On a fuzzy topological structure, *Suppl. Rend. Circ. Mat. Palermo*, Ser. II, 11: 89–103.

[54] Shostak, A.P. (1989); Two decades of fuzzy topology: basic ideas, notions, and results, *Russian Math. Surveys*, 44(6): 125–186.

[55] Syau, Y.-R., Low, C.-Y., Wu, T.-H. (2002); A note on convex fuzzy processes, *Applied Mathematics Letters* 15: 193–196.

[56] Tsiporkova-Hristoskova, E., De Baets, B., Kerre, E. (1997); A fuzzy inclusion based approach to upper inverse images under fuzzy multivalued mappings, *Fuzzy Sets and Systems*, 85: 93–108.

[57] Tsiporkova, E., De Baets, B., Kerre, E. (1998); Continuity of fuzzy multivalued mapping, *Fuzzy Sets and Systems*, 94: 335–348.

[58] Warren, R.H. (1978); Neighborhoods, bases and continuity in fuzzy topological space, *Rocky Mountain Journal of Mathematics*, 8(3): 459–470.

[59] Xiao, J.Z., Zhu, X.H. (2002); On linearly topological structures and property of fuzzy normed linear space, *Fuzzy Sets and Systems*, 125(2): 153–161.

[60] Xiao, J.Z., Zhu, X.H. (2003); Fuzzy normed spaces of operators and its completeness, *Fuzzy Sets and Systems*, 133(3): 389–399.

[61] Zadeh, L.A. (1965); Fuzzy Sets, *Informations and Control*, 8: 338–353.

[62] Zadeh, L.A. (1975); The concept of a linguistic variable and its application to approximate reasoning, parts 1,2 and 3, *Inf. Sciences* 8: 199–249, 8: 301–357, 9: 43–80.

[63] Zadeh, L.A. (2011); My Life and Work - A Retrospective View, Special Issue on Fuzzy Set Theory and Applications, Dedicated to the 90th Birthday of prof. Lotfi A. Zadeh in *Aplied and Computational Mathematics*, 10(1):4-9, 2011.

[64] Zadeh, L.A. (2011); Foreword, Special Issue on Fuzzy Sets and Systems, Dedicated to the 90th Birthday of Prof. Lotfi A. Zadeh in *International J. of Computers Communications & Control*, 6(3): 385-386.

[65] Zadeh, L.A., Tufis, D., Filip, F.G., Dzitac I. (2008); *From Natural Language to Soft Computing: New Paradigms in Artificial Intelligence* , Ed. House of Romanian Academy.

[66] Zimmermann, H.-J. (2001); *Fuzzy set theory - and its applications*, 4th Edition, Kluwer Academic Publisher.