AN ALGEBRAIC REPRESENTATION VIA DIFFERENTIAL EQUATIONS FOR PAIRWISE COMPARISONS OF AHP

Takafumi Mizuno
Faculty of Urban Science
Meijo University
tmizuno@urban.meijo-u.ac.jp

Eizo Kinoshita
Faculty of Urban Science
Meijo University
kinoshit@urban.meijo-u.ac.jp

ABSTRACT

We propose a simple algebraic representation for pairwise comparisons of AHP. The representation is an associative relation between the importances of elements and consists of basic arithmetic operations. First, we define a ratio, which is estimated by decision makers by comparing the importances of elements, as a partial differentiation of importances (Section 2). Then, we construct systems of differential equations. Algebraic representations of the importances are derived as formal solutions of the equations. We analyze pairwise comparisons and the construction of the importances from them with the representations (Section 3). The validity of using eigenvectors and C.I. in AHP is illustrated by deriving a particular solution of the equations. *https://doi.org/10.13033/ijahp.v9i1.278*

Keywords: Pairwise comparison method; AHP; partial differentiation

1. Introduction

Pairwise comparisons are primitive procedures in AHP (Saaty, 1977, 1980). Decision makers construct relative importances of elements from ratios of pairs of elements. Let a_1, $\cdots$, a_n be the elements, and x_i be an importance of an element a_i. Decision makers want to obtain x_i, but they can only estimate ratios x_i/x_j by pairwise comparisons for all pairs (a_i, a_j). There are many methods to derive importances from the set of ratios (Cogger & Yu, 1985). In the actual usage of AHP, relative importances are often obtained by applying the principal eigenvector method (Saaty, 1980). In this method, a ratio r_{ij} which is an estimation of x_i/x_j is arranged in the i-th row j-th column cell in the pairwise comparison matrix R, which is $n \times n$ square matrix. The importances which decision makers want are obtained as elements of the principal eigenvector of R; a detected relative importance $\widehat{x_i}$ is an element of vector $\widehat{\boldsymbol{x}} = [\widehat{x_1}, \cdots, \widehat{x_n}]^t$ which holds $R\widehat{\boldsymbol{x}} = \lambda_{max}\widehat{\boldsymbol{x}}$. Harker and Vargas (1987) discussed why we can regard the vector as the approximation of importances. Their illustrations, however, are correct but quite difficult because of

ISSN 1936-6744
https://doi.org/10.13033/ijahp.v9i1.278

their analyses of eigenvectors. In a decision making process, we have to make decision makers intuitively understand the usefulness of the methods. We also want to construct useful semantics which treat mental measurements and physical models with the same scheme. In this paper, we propose a representation which simply illustrates the validity of calculations for relative importances from pairwise comparison.

2. Hypotheses

We presume that the importance x_i of an element a_i can be represented in a multi-variable function whose arguments are $x_j, j \neq i$;

$$\mathrm{x}_i \equiv x_i(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n), \quad i = 1, \cdots, n. \tag{1}$$

In the pairwise comparisons of the AHP, for all pairs $(x_i,\ x_j)$, decision makers give an estimated ratio r_{ij} which means that x_i is r_{ij} times as large as x_j. We make the further assumption that the ratio is an estimation of the partial differentiation of these functions;

$$r_{ij} \equiv (n-1)\frac{\partial x_i}{\partial x_j}. \tag{2}$$

It means that if decision makers enlarge the estimate of the a_j, then the a_i will be larger, and the a_i growth rate of the estimate will be r_{ij} times larger than that of the a_j. There is a term $(n-1)$ in Equation (2), because decision makers estimate the ratio of x_i as a single-variable function whose argument is x_j in spite of the former assumption that the function is an $(n-1)$-variable function.

3. An analysis of the pairwise comparison method

With the hypotheses in the previous section, we can write the pairwise comparison matrix R as follows:

$$R = \begin{bmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} & \cdots & r_{nn} \end{bmatrix} \equiv (n-1)\begin{bmatrix} \frac{\partial x_1}{\partial x_1} & \cdots & \frac{\partial x_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial x_1} & \cdots & \frac{\partial x_n}{\partial x_n} \end{bmatrix} = (n-1)\partial\boldsymbol{x}\underline{\boldsymbol{\partial x}}^t \tag{3}$$

where $\partial\boldsymbol{x} = [\partial x_1, \cdots, \partial x_n]^t$, and $\underline{\boldsymbol{\partial x}} = [1/\partial x_1, \cdots, 1/\partial x_n]^t$.

Let $d\boldsymbol{x} = [dx_1, \cdots, dx_n]^t$, and let us consider a product $Rd\boldsymbol{x}$. Combining the formula of total differentiation, we obtain a relation

$$d\boldsymbol{x} = \left(\partial\boldsymbol{x}\,\underline{\boldsymbol{\partial x}}^t - I\right)d\boldsymbol{x} = \frac{1}{(n-1)}(R-I)d\boldsymbol{x}, \tag{4}$$

$$dx_i = \frac{1}{(n-1)}\left[r_{i1}dx_1 + \cdots + r_{i,i-1}dx_{i-1} + r_{i,i+1}dx_{i+1} + \cdots + r_{in}dx_n\right]. \tag{5}$$

ISSN 1936-6744
https://doi.org/10.13033/ijahp.v9i1.278

where I is the identity matrix. Notice that the total differentiation of x_i is $dx_i = \partial x_i/\partial x_1 dx_1 + \cdots + \partial x_i/\partial x_{i-1} dx_{i-1} + \partial x_i/\partial x_{i+1} dx_{i+1} + \cdots + \partial x_i/\partial x_n dx_n$. We can represent the importances $\boldsymbol{x}$ as a system of total differential equations.

We obtain an algebraic representation of x_i by integrating Equation (5).

$$\begin{aligned} x_i &= \int dx_i = \frac{1}{(n-1)}\left[\int r_{i1}\, dx_1 + \cdots + \int r_{in}\, dx_n\right] \\ &= \frac{1}{(n-1)}\left[r_{i1}x_1 + \cdots + r_{i,i-1}x_{i-1} + r_{i,i+1}x_{i+1} + \cdots + r_{in}x_n\right] - d_i, \end{aligned} \tag{6}$$

$$\boldsymbol{x} = \frac{1}{n-1}(R-I)\boldsymbol{x} - \boldsymbol{d} \tag{7}$$

where $\boldsymbol{d} = [d_1, \cdots, d_n]^t$ is a constant of integration. To determine the constant, we reformulate Equation (7).

$$R\boldsymbol{x} = n\boldsymbol{x} + (n-1)\boldsymbol{d}. \tag{8}$$

This is an algebraic representation for importances. It has a degree of freedom caused by the constant of integration $\boldsymbol{d}$.

To find particular solutions by determining the constant of integration, let $\hat{\boldsymbol{x}}$ be an eigenvector of R, and λ its corresponding eigenvalue. Thus the representation can be transformed to:

$$R\hat{\boldsymbol{x}} = \lambda\hat{\boldsymbol{x}} = n\hat{\boldsymbol{x}} + (n-1)\boldsymbol{d}, \tag{9}$$

$$\boldsymbol{d} = \frac{\lambda - n}{(n-1)}\hat{\boldsymbol{x}}. \tag{10}$$

We obtain a representation of importances as the system of equations:

$$\boldsymbol{x} = \frac{1}{n-1}(R-I)\boldsymbol{x} - \frac{\lambda - n}{n-1}\hat{\boldsymbol{x}}, \tag{11}$$

$$\frac{1}{n-1}(R-nI)\boldsymbol{x} = \frac{\lambda - n}{n-1}\hat{\boldsymbol{x}}. \tag{12}$$

If the null-space of the matrix $(R - nI)$ has the same dimensions, then the solution will be

$$\boldsymbol{x} = \boldsymbol{y} + \hat{\boldsymbol{x}}. \tag{13}$$

A vector $\boldsymbol{y}$ is the solution of the equation $(R - nI)\boldsymbol{y}$=0. We can confirm that $\hat{\boldsymbol{x}}$ is also the solution of the Equation (12).

ISSN 1936-6744
https://doi.org/10.13033/ijahp.v9i1.278

4. Conclusions

We propose an algebraic representation for the pairwise comparisons of the AHP. A key idea is that we regard ratios of importances as partial differentiations of them. Relations between importances are derived directly from these differentiations. In Section 3, we also naturally introduced why eigenvectors are needed and what C.I. the term $(\lambda - n)/(n - 1)$, is. Eigenvectors are particular solutions of the system of differential equations, and C.I. is a coefficient of the nonhomogeneous term of the equations.

In this paper, we demonstrate that estimated ratios can be regarded as differentials of importances without any fault. This means that we can include physical models in the pairwise comparisons of the AHP. We expect that mental measurements, which are obtained using ordinary pairwise comparisons, and physical models are treated using the same scheme. And we can apply the semantics to machine learnings, or can retrieve importances of any element automatically to put in the AHP. In real databases of physical models, there are many numeric calculations for extracting differentiations.

ISSN 1936-6744
https://doi.org/10.13033/ijahp.v9i1.278

REFERENCES

Saaty, T. (1980). *The Analytic Hierarchy Process*, New York: McGraw-Hill. Doi: http://dx.doi.org/10.1080/00137918308956077

Saaty, T. (1977). A scaling method for priorities in hierarchical structure. *Journal of Mathematical Psychology, 15*, 234-281. Doi: http://dx.doi.org/10.1016/0022-2496(77)90033-5

Cogger, K. O. & Yu., P. L. (1985). Eigenweight vectors and least-distance approximation for revealed preference in pairwise weight ratios. *Journal of Optimization Theory and Applications, 46(4)*, 483-491. Doi: 10.1007/BF00939153

Harker, P. and Vargas, L. (1987). The theory of ratio scale estimation: Saaty's analytic hierarchy process. *Management Science, 33(11)*, 1383–1403. Doi: http://dx.doi.org/10.1287/mnsc.33.11.1383