REVIEW OF DIFFERENT METHODS FOR DERIVING WEIGHTS IN THE ANALYTIC HIERARCHY PROCESS

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ABSTRACT

The Analytic Hierarchy Process (AHP) is a multi-criteria decision making technique developed to solve both single and group decision making problems. The AHP approach that was proposed by Saaty to derive priority weights by using the eigenvector method (EVM) has its drawbacks, and since its publication various methods have been developed as alternatives. This paper presents an extensive review of various methods for deriving priority weights in AHP, including group decision making, and focuses on comparison of each method with the EVM. The results of the comparison reveal that each method has its advantages and disadvantages, and the decision of which method to use should be based on the application.

Keywords: Analytic Hierarchy Process, Review, Multicriteria Decision, Group Decision Making.

1. Introduction

The Analytic Hierarchy Process (AHP) is one of the most widely used tools to solve multi-criteria decision making problems. It was developed by Thomas Saaty in the 1970's while he was directing research projects for the Arms Control and Disarmament Agency at the U.S. Department of State. During his successive work, Saaty realized that there was a lack of common, easily understood, and easily applicable methodology to facilitate making complex decisions (Forman and Gass, 2001). Therefore, he attempted to develop a simple, practical, systematic way to help people make complex decisions, and the result was the AHP. Saaty criticized ignoring the human factors, at least in social sciences, by making simplified assumptions to suit the quantitative models. To make realistic models, all important tangible and intangible, quantitatively measurable, and qualitative factors must be included and measured, and this is what the AHP attempts to accomplish (Saaty, 1988).

The power and simplicity behind the AHP has led to its widespread acceptance and use throughout the world; a literature review revealed that the AHP model has been widely and successfully applied to a wide range of problem situations that include:

- The selection of one alternative from a given set of alternatives, where multiple decision criteria are involved.
- Ranking by ordering a set of alternatives from most to least desirable.

- Prioritization by determining the relative merit of members of a set of alternatives, as opposed to selecting a single one or merely ranking them.
- Allocation of scarce resources.
- Predicting outcomes.
- Measuring performance.

There are also numerous articles and doctoral dissertations that discuss the applications of AHP in many fields. For example, Zanakis, et al. (1995) studied over 100 applications of AHP within the service, social/manpower, natural resource/energy, education, and government sectors. Vaidya and Kumar (2006) reviewed about 32 articles that use AHP in selection problems, 26 articles in evaluation problems, 7 articles in benefit–cost analysis, 10 articles in allocation problems, 18 articles in the field of planning and development, 20 articles in priority and ranking problems, 21 articles in decision making problems, 4 articles in forecasting, and 5 articles in medicine and related fields.

In spite of its widespread use and applications, Saaty's approach for deriving weights and related consistency measure in AHP using the Eigenvector Method (EVM) has some drawbacks which have been discussed in the literature from different perspectives. Therefore, several attempts have been made to suggest different techniques to overcome these drawbacks; some of the proposed methods are deterministic while the others are stochastic ones. However, it is strongly noted that EVM is still the most used method to derive weights in the vast majority of the applications of AHP (Ishizaka and Labib, 2011; Mirhedayatian et. al., 2011; Grzybowski, 2010).

This paper aims to provide a detailed review of the basic, theoretical concepts as well as the methodological developments of the AHP, focusing on the method of deriving weights in both single and group decision making problems. It is hoped that this review will help researchers to be aware of the successive methodological developments of AHP since its appearance, and to take advantage of these developments in the applications. Although the main developments of the AHP have been briefly reviewed in Ishizaka and Labib (2011), the current paper tries to provide a wider and more detailed review about the methodological development of AHP in both single and group decision making problems. It is worth mentioning that the review of the AHP applications is not in the scope of this paper, and it has been discussed in other papers such as Subramanian and Ramanathan (2012), Ho, (2008), and Vaidya and Kumar (2006).

This paper consists of five sections and is organized as follows. In Section 2, we review the basics of AHP including structuring the decision problem, pairwise comparison, deriving priority weights, and synthesizing to get the final weights. Section 3 discusses the different methods for deriving priority weights. Section 4 discusses methods for deriving priority weights in group decision making problems, while section 5 gives some conclusions.

2. Basics of AHP

AHP is a structured technique for organizing and analyzing complex decision problems. The goal of the AHP is to find a unique vector of priority weights $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ of n alternatives with respect to a given criterion. The priority weights are chosen to be nonnegative and normalized to unity. The actual priority ratios $\frac{w_i}{w_j}$ which represent the pairwise comparison of each two elements can't be given precisely, but they can be translated into verbal expressions and then estimated by a decision maker or an expert who may make small errors in judgment (Saaty, 1994).

To obtain the weights for the alternatives using AHP, one should follow four primary steps: structuring the complex problem in a hierarchical form, constructing the pairwise comparison judgments, the prioritization procedures, and finally synthesizing through the structure to get the final weights for the alternatives.

2.1 Structuring the problem in a hierarchical form

Structuring a decision is the first step that should be taken to organize and represent any problem, particularly a decision problem. To deal with a problem, one attempts to identify the elements that relate to it, their connections and interactions, the cause(s) that give rise to the problem, and possible ways to a solution (Saaty and Shih, 2009). Structuring the decision problem in AHP is performed by decomposing the problem into its constituent parts and then presenting them in a hierarchy form (Figure 1). This structure comprises the main goal at the top level, criteria at the intermediate level, and finally the lowest level contains the options or alternatives. The criteria can be further broken down into subcriteria, sub-subcriteria, and so on, in as many levels as required. The elements at a particular level of the hierarchy are kept, independent of, but comparable to, the elements at the same level, and elements at any level are directly related to elements at the level immediately below them (Ramanujam and Saaty, 1981).

To build the hierarchy structure for the decision problem, Saaty (1994) suggested the following main steps:

- Identify the overall goal on the top level of the structure. The main goal can be identified by answering the following questions: What are we trying to accomplish? What is the main question?
- Identify criteria that must be satisfied to fulfill the overall goal.
- Identify sub-criteria under each criterion, if relevant.
- The options or alternatives are added at bottom of the structure.

Representing problems in a hierarchal form has two basic advantages; it provides an overall view of the complex system of the situation, and helps the decision maker assess the homogeneity of the issues in each level, so he can that compare such elements accurately. On the other hand, an element in a given level in the hierarchal form does not need to function as an attribute for all the elements in the level below (Saaty, 1990). Also, the decision maker can add or delete levels and elements to the hierarchy as necessary.

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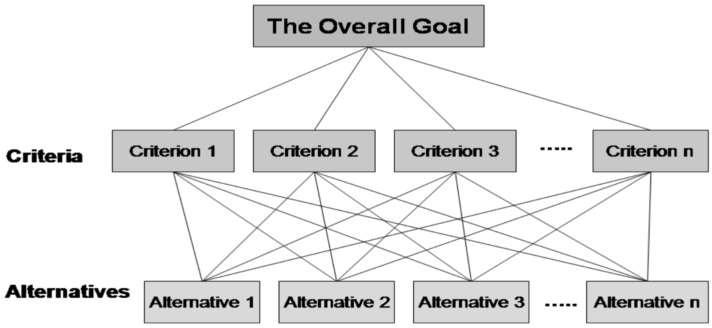

Figure 1. Hierarchy structure in AHP

2.2 Pairwise comparison judgments using ratio scale

The second step in the AHP is the construction of the matrix of comparison judgments given by a decision maker or an expert. The judgment or comparison is a numerical representation of a relationship between two elements that share a common parent in the hierarchy, and it represents the dominance or importance of an element over the other one in the same level. The set of all such judgments can be represented in a square matrix called a Pairwise Comparison Matrix (PCM), in which the set of elements are put in the rows and columns of the matrix where the set of elements is compared with itself. In the PCM, the decision maker is asked to evaluate the various elements of the problem; each element is evaluated with regard to the other one. The pairwise comparison process is strongly recommended by psychologists; they argue that, to express one's opinion on only two alternatives is easier and more accurate than to do it simultaneously on all the alternatives. Moreover, it allows consistency cross checking between the different pairwise comparisons (Ishizaka and Labib, 2001). To make pairwise comparisons, Saaty proposed a ratio scale measurement (called the Fundamental Scale) to be used by the policy maker or expert to indicate how many times more important or dominant one element is over the other one for each pair of elements (criteria or alternatives).

This ratio scale ranges from one (Equal Importance) to 9 (Extreme Importance) (Table 1). If, for example, element i is assigned 3 in comparison with element j, this means that element i is three times important compared to element j with regard to the overall goal. Saaty (1990) stated that the scale (1.1, 1.2 . . . . . 1.9) can be used when the elements being compared are closer together than indicated by the above scale.

Table 1
The fundamental scale used in AHP

Dominance of Importance	Definition
1	Equal Importance
2	Weak or slight
3	Moderate Importance
4	Moderate plus
5	Strong Importance
6	Strong plus
7	Very Strong Importance
8	Very, very strong
9	Extreme Importance

Source: (Saaty, 1990).

Saaty and Ozdemir (2003) suggested that, for the consideration of consistency of the information derived from relations among the compared elements, the number of compared elements should not be large (not more than seven). On the other hand, Saaty (2001) stated that, for the consistency and homogeneity condition required for stability of the eigenvector of priorities, the upper limit of the scale should not be more than 9. He argued that this condition is intrinsic to the way in which our brains actually operate; qualitatively, people can only deal with information involving a few facts simultaneously.

The set of all pairwise comparison judgments are presented in a square matrix in which the set of compared elements is put on the rows and columns of the matrix. This matrix is a "Reciprocal Symmetric Matrix" (RSM) and takes the following form:

$$A = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_n \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & & A_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} \\ a_{21} & a_{22} \ldots\ldots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & a_{n2} \ldots\ldots & a_{nn} \end{bmatrix} \end{array}$$

The RSM satisfies the following conditions:

- It is a matrix of non-zero elements,

$$a_{ij} > 0, \qquad i, j = 1,\ldots, n \tag{1}$$

- Comparing an element with itself is always assigned the value 1, so the main diagonal entries of the matrix are all 1,

$$a_{ii} = 1, \qquad i = 1,\ldots, n \tag{2}$$

- Elements below the diagonal are the reciprocal of the elements above the diagonal. This means that if p represents the dominance of element A over element B, then the dominance of B compared to A is $\frac{1}{p}$,

$$a_{ij}a_{ji} = 1 \ \ or \ a_{ij} = \frac{1}{a_{ji}}, \qquad for \ i \neq j \quad i, j = 1,\ldots, n \tag{3}$$

In AHP, a number of comparison matrices are constructed; one matrix for the criteria in terms of their importance to achieve the overall goal, and a number of matrices are constructed for the alternatives with regard to each criterion. In some cases when the number of alternatives is large, we face a problem of missing comparisons judgments, i.e. the comparison matrix is incomplete. In such cases, missing entries must be filled in with values that improve the consistency of the matrix. According to Harker (1987), there are three reasons for obtaining incomplete judgments. These reasons are first, the great time and effort needed to complete $n(n-1)/2$ comparisons; second, unwillingness of the decision maker/expert to make a direct comparison between two alternatives; and third, being unsure of some of the comparisons. Deriving priorities from incomplete comparison matrices has been discussed in the literature by several studies such as Harker (1987a; 1987b), Shiraishi et al. (1998), Nishizawa (2005), Fedrizzi and Giove (2007), and Gomez-Ruiz et al. (2010).

2.3 Prioritization procedures

As was mentioned before, the goal of AHP is to find a unique, nonnegative, and normalized natural vector of priorities $w_1, w_2, ..., w_n$ of n alternatives with respect to a number of criteria. The actual priority ratios $\frac{w_i}{w_j}$ which represent the pairwise comparison of each two elements cannot be given precisely, but they can be estimated by judgments of a decision-maker or an expert about alternatives and criteria that are used to judge the alternatives. So, in the conventional AHP, a decision-maker estimates these ratios of priorities, which form the PCM $A = \lfloor a_{ij} \rfloor_{n\times n}$.

In reality, we cannot expect that the elements of PCM give exactly priority ratios, as the human mind is not a perfect measurement device. Human judgments may depend on personal taste, experience, specific knowledge, the judge's temporary mood and temper and may vary with time. Based upon this, checking consistency is very important for the comparison matrix before using it to derive priority weights. Therefore, the values of the PCM (a_{ij}) are taken as estimates of the actual priority weight ratios $\frac{w_i}{w_j},\ i, j = 1,...,n$ for the compared elements in the matrix. Then, priority weights vector $\mathbf{w} = (w_1, w_2, ..., w_n)$ for these elements are derived based on the values of the PCM (**A**):

$$\mathrm{A} = \begin{array}{c|c} & \begin{matrix} A_1 & A_2 & \cdots & A_n \end{matrix} \\ \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{matrix} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \end{array} \approx \begin{array}{c|c} & \begin{matrix} A_1 & A_2 & \cdots & A_n \end{matrix} \\ \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{matrix} & \begin{bmatrix} \frac{w_1}{w_1} & \frac{w_1}{w_2} & \cdots & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & \frac{w_2}{w_2} & \cdots & \frac{w_2}{w_n} \\ \vdots & & & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \cdots & \frac{w_n}{w_n} \end{bmatrix} \end{array}$$

A necessary and sufficient condition for matrix **A** to be perfectly consistent is that

$$a_{ij} = \frac{w_i}{w_j}, \quad i, j = 1,...,n \qquad (4)$$

Saaty (1990) stated the following conditions for the positive PCM to be consistent:

1) Reciprocal: $a_{ij} = \frac{1}{a_{ji}}, \qquad for\ i \neq j \quad i, j = 1,...,n \qquad (5)$

2) Transitive: $a_{ij}\ a_{jk} = a_{ik}, \qquad for\ i, j, k = 1,...,n \qquad (6)$

Therefore, measuring the degree of inconsistency of the PCM is a fundamental issue in AHP. To derive priority weights from pairwise comparison matrices as well as to measure the degree of inconsistency of the PCM, Saaty proposed using the EVM. He stated that the priority vector should satisfy two conditions; the first is that it belongs to a ratio scale which means that it should remain invariant under multiplication by a positive constant, the second is that it must be unique which implies that it should be invariant under hierarchic composition for its own judgment matrix so that one does not keep getting new priority vectors from that matrix (Saaty, 2003). According to the EVM, the required priority vector **w** for a consistent matrix **A** must satisfy the relation $\mathbf{Aw} = n\mathbf{w}$, n> 0, where **A** has been multiplied on the right by the transpose of the vector of weights **w** as follows:

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$$Aw = \begin{array}{c} \\ A_1 \\ \vdots \\ A_n \end{array} \begin{array}{c} \begin{array}{ccc} A_1 & \cdots & A_n \end{array} \\ \begin{bmatrix} \frac{w_1}{w_1} & \cdots & \frac{w_1}{w_n} \\ \vdots & & \vdots \\ \frac{w_n}{w_1} & \cdots & \frac{w_n}{w_n} \end{bmatrix} \end{array} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = n \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = nw.$$

The result of this multiplication is n**w**. Thus, solving this system of homogeneous linear equations $\mathbf{Aw} = n\mathbf{w}$ or $(\mathbf{A} - \mathrm{nI})\mathbf{w} = 0$ results in the right eigenvector of the **A** matrix corresponding to the eigenvalue *n*. It is clear that matrix **A** has a rank one because every row is a constant multiple of the first row. Thus, all its eigenvalues except one are zero, and the sum of the eigenvalues of the matrix is equal to its trace, the sum of the diagonal elements. In this case, the trace of **A** is equal to n. Therefore, n is the largest, or principal, eigenvalue of **A** (Saaty, 1990). Then, the solution of $\mathbf{Aw} = n\mathbf{w}$ is the principal right eigenvector **w** corresponding to the principal eigenvalue n of **A**, and it consists of positive entries and gives the desired unique priority vector.

According to Saaty (1990, 2003), a necessary and sufficient condition for consistency is that the principal eigenvalue of **A** is equal to *n*, the order of that matrix. The inconsistent matrix is the derivation from a consistent one by perturbation of some or all the matrix components. Because the eigenvalues and eigenvectors of a matrix depend continuously on components of the matrix, small perturbations in the components will result in small changes in the eigenvalues and eigenvectors. Thus, when the perturbations of the components are small, the maximal eigenvalue is close to *n*, and the corresponding normalized eigenvector is close to the normalized eigenvector of the unperturbed consistent matrix. Therefore, Saaty argued that the normalized eigenvector is also valid for inconsistent matrices. From the perturbation theory, it is known that a small perturbation around a simple eigenvalue when **A** is consistent leads to an eigenvalue problem of the form $\mathbf{Aw} = \lambda_{Max}\mathbf{w}$ **,** and **A** may no longer be consistent but is still reciprocal (Saaty, 1990). Saaty proved that $\lambda_{Max} \geq n$, with the equality if and only if **A** is consistent.

The question now is to what extent the inconsistent comparison matrix can be used to derive priority weights. To answer this question it is necessary to measure the consistency level for comparison matrices. Therefore, Saaty proposed a measure of consistency, called the Consistency Ratio (CR). The CR is a ratio of the Consistency Index (CI) to the Random Index (RI).

$$CR = \frac{CI}{RI} \tag{7}$$

$$CI = \frac{\lambda_{Max} - n}{n-1} \tag{8}$$

where n is the dimension of the PCM, λ_{Max} is maximal eigenvalue, and RI is calculated as the average value of consistency indices of a 500 randomly generated reciprocal matrices from the scale 1 to 9. The average RI and the order of the matrix are shown in Table 2.

Table 2
Saaty's random index

n	1	2	3	4	5	6	7	8	9	10
RI	0	0	0.52	0.89	1.11	1.25	1.35	1.40	1.45	1.49

Source: (Saaty, 1990).

If the CR is less than or equal to 10%, the matrix can be considered to have an acceptable consistency. If one can't reach this acceptable level, one should gather more information or reexamine the framework of the hierarchy. Ishizaka and Labib (2011) stated that other researchers have run simulations with different numbers of matrices and their random indices were different but close to Saaty's. Grzybowski (2010) argued that even if the comparisons are done very carefully, PCM is inconsistent and we have to express the relation between the PCM elements and the priority weights in the form: $a_{ij} = e_{ij}\frac{w}{w_j}$, where e_{ij} is a perturbation factor which is expected to be near 1, and in the statistical approach this factor is interpreted as a realization of a random variable. Therefore, several researchers have attempted to propose new methods to measure consistency other than the CR. For example;
Crawford and Williams (1985) suggested a new measure for consistency based on the sum the difference between the ratio of calculated priorities $\frac{w_i}{w_j}$ and the given comparisons in the Geometric Consistency Index (GCI), where:

$$GCI = \frac{2\sum_{i<j}(\ln a_{ij} - \ln \frac{w_i}{w_j})^2}{(n-1)\ (n-2)} \tag{9}$$

Then, the smaller the value of GCI, the smaller the distance between the judgments a_{ij} and the ratios $\frac{w_i}{w_j}$ will be, and the better the fit between judgments and priorities.
Aguaron and Moreno-Jimenez (2003) proposed thresholds for the GCI which are analogous to that proposed by Saaty's CR. Dodd, et. al. (1993) proposed a statistical approach based upon the significance level concept (95% or 99% …). The level of confidence is selected by the decision maker(s) and differs according to the decision problems. The CI for a given comparison matrix is compared to the tolerance level at the selected confidence level, and then the decision is made to reject or accept the matrix.

Moreover, Zeshui and Cuiping (1999) proposed a convergent algorithm to modify consistency for a given comparison matrix using the following relationship:

$$a_{ij}^{(k+1)} = (a_{ij}^{(k)})^{\lambda}\left(\frac{w_i^{(k)}}{w_j^{(k)}}\right)^{1-\lambda} \tag{10}$$

where $w = (w_1, w_2, ..., w_n)$ is the principle eigenvector corresponding to the maximal eigenvalue λ_{Max}, $0 < \lambda < 1$. This process is repeated until the modified matrix satisfies the consistency condition.

Also, Peláez and Lamata (2003) proposed a method based on the determinant of the PCM. Alonso and Lamata (2006) proposed a new criterion for accepting/rejecting PCM in AHP using the average principal eigenvalues λ_{Max}. Stein and Mezzi (2007) proposed a new consistency measure called the Harmonic Consistency Index (HCI), based on the method of Additive Normalization (AN). Ergu et al.(2011) introduced a new method based on the theorem of matrix multiplication, vectors dot product, and the definition of consistent pairwise comparison matrix. Borodin et al. (2011) proposed two methods for correcting the elements of an inconsistent pairwise comparison matrix to get a consistent one using a correction Matrix $\mathbf{H}$ such that; $\|\mathbf{H}\|^2 \rightarrow \min,$ so that $(A + H)$ is consistent. The first method uses the Lagrange multipliers, while the second solves the correction problem by correcting the inconsistent set of linear equations with the spare structure.

2.4 Synthesizing through the structure to get the final weights

After computing priority weights for the criteria and alternatives, the final step is to obtain the overall priority weights. This is done for each alternative by multiplying the priority weight obtained from the alternatives' comparisons according to a specific criterion by the corresponding priority weight of this criterion, and then added over all the criteria. The alternative with the highest priority weight value should be taken as the best one. This synthesizing process can be illustrated as follows:

$$\mathrm{w}\,(\mathrm{A}_1) = \sum_{\mathrm{i}=1}^{\mathrm{n}} \left[\mathrm{w}(\mathrm{c_i}) \times \mathrm{w}(\mathrm{A}_1)|_{\mathrm{c_i}}\right] \tag{11}$$

where: $w\,(A_1)$ is the weight value of alternative A_1, $w(c_i)$ weight value of the ith criterion (c_i), $w(A_1)|_{c_i}$ is the weight value of alternative A_1 with regard to of the ith criterion (c_i), $i = 1, \ldots, n$. This process is repeated to all the alternatives to get the final priority weight vector **w**.

3. Different methods for deriving priority weights

A review of the literature revealed that using EVM to derive priority weights in AHP has some drawbacks including the following:

a) Condition of Order Preservation

Condition of Order Preservation (COP) means that the ranking process should not only indicate an order of preference among the alternatives, but also reflect the intensity or the cardinal preference among these alternatives as it is indicated by the ratios of the numerical values. For example, if there are four alternatives x_1, x_2, x_3, x_4 such that $x_1$strongly dominates x_2, and x_3 moderately dominates x_4, then whenever possible, the vector of priorities **w** be such that $\frac{w_1}{w_2} > \frac{w_3}{w_4}$, which indicates that the intensity of preference of x_1 over x_2 is higher than the intensity of preference of x_3 over x_4. Costa and Vansnick (2008) argued that using EVM to derive weights in AHP does not satisfy this property.

b) The Right and Left Eigenvector Inconsistency

The second drawback is related to the right and left eigenvector inconsistency; where Saaty used only right eigenvectors, the use of left eigenvectors should be equally justified. For ranking a set of alternatives in a decision problem, both eigenvector approaches should yield the same result. Unfortunately, it is often not the case in AHP, the solution of the right eigenvector $\mathbf{Aw} = n\mathbf{w}$ which gives the right

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eigenvector $\mathbf{w}$ is not necessarily the same as the left eigenvector solution $\mathbf{w'}^T\mathbf{A} = \mathbf{w'}^T\lambda$ (Ishizaka and Labib, 2011). This right and left inconsistency arises only for inconsistent matrices with a dimension greater than 3.

c) Rank Reversal

Rank reversal phenomenon refers to the changes of the relative rankings of the other alternatives after a new alternative is added or a current one is deleted. Such a phenomenon was first noticed and pointed out by Belton and Gear (1983). In order to avoid the rank reversal, they suggested normalizing the eigenvector weights of alternatives using their maximum rather than their sum, which was usually called B–G modified AHP. However, Saaty and Vargas (1985) provided a counterexample to show that B–G modified AHP was also subject to rank reversal. Barzilai and Golany (1994) showed that no normalization method could prevent rank reversal. Several other studies investigated the problem, and the literature shows that the rank reversal phenomenon has not been perfectly resolved, and debate still exists about the ways of avoiding rank reversals (Wang and Elhag, 2006).

In addition to the problems discussed above, other disadvantages of EVM have been discussed regarding its calculation difficulties and the lack of a practical statistical theory behind it. The eigenvalue method is a deterministic (non-stochastic) method to derive priority weights where the errors in judgments are assumed not to exist. Therefore, several researchers have attempted to present different methods to derive priority weights in AHP. Each method has its advantages and disadvantages where there is no specific standard to measure the superiority of one over the others. Some of these methods are deterministic, and others used the stochastic techniques. They are discussed in the following subsections.

3.1 Deterministic methods

1) **Mean of normalized values**

This is the oldest method introduced by psychologists who used the pairwise comparison matrices before Saaty (Ishizaka and Labib, 2001). This method can be, briefly explained in the following steps:

- Computing the sum of each column of the PCM **A**: $S_j = \sum_{i=1}^{n} a_{ij}, \ j = 1,...,n$.
- Normalizing the elements of matrix **A** by dividing each value by its column sum: $\tilde{a}_{ij} = \dfrac{a_{ij}}{S_j}, \ \forall \ i, j = 1,,...,n$.
- Computing the mean of each row: $w_i = \dfrac{\sum_{i=1}^{n} \tilde{a}_{ij}}{n}, \ i = 1,...,n$.
- The vector of mean values calculated in step three is taken as the priority weight vector **w**. For this method, no mathematical approach is available for dealing with inconsistent matrices.

2) **Method of Least Square**

The proposed Least Square Method (LSM) is based on minimizing the sum square of the differences (distances) between the actual priority weight ratios $\frac{w_i}{w_j}$ and the elements of the comparison matrix (a_{ij}) such that $a_{ij} \approx \frac{w_i}{w_j}$. For example:

Chu et al. (1979) utilized a weighted least-square method by solving the following minimizing problem:

$$\text{Min}\, S' = \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij}w_j - w_i)^2 + 2\varphi\sum_{i=1}^{n} w_i \tag{12}$$

where φ is the Lagrange multiplier.

Differentiating the previous equation with respect to w_m yields the following set of equations:

$$\sum_{i=1}^{n}(a_{im}w_m - w_i)\ a_{im} - \sum_{i=1}^{n}(a_{mj} - w_m) + \varphi = 0, m = 1,2,\ldots,n \tag{13}$$

These equations in addition to the one of $\sum_{i=1}^{n} w_i = 1$ form a set of $(n+1)$ inhomogeneous linear equations with $(n+1)$ unknowns and can be solved for $w_1, w_2, \ldots, w_n, \varphi$. Moreover, they stated that the weighted least-square method is conceptually easier to understand than the EVM. However, at the same time, they argued that the EVM guarantees that the obtained priority vector is nonnegative, while it is not known whether such a theorem exists for the weighted least-square method. Although the numerical results they obtained indicate that the weights obtained by this method are greater than zero and are comparable to those obtainable by the EVM.

Cogger and Yu (1985) introduced a new eigen-weight technique to derive priority weights. Unlike Saaty's approach, they were interested in the upper diagonal elements of the comparison matrix. They defined three matrices **T** = $[t_{ij}]$, **U** = $[U_{ij}]$, and **D** = $[d_{ij}]$

where $t_{ij} = \begin{Bmatrix} a_{ij} & \text{if } j \ge i \\ 0, & \text{O.W} \end{Bmatrix}$, $u_{ij} = \begin{Bmatrix} w_{ij} & \text{if } j \ge i \\ 0, & \text{O.W} \end{Bmatrix}$, $d_{ij} = \begin{Bmatrix} n-i+1 & \text{if } j = i \\ 0, & \text{O.W} \end{Bmatrix}$, and $w_{ij} = \dfrac{w_i}{w_j}$.

In addition to the column weight vector **w** with elements w_i, $i = 1,\ldots,n$, $w_i \ge 0$, and $\sum_{i=1}^{n} w_i = 1$.

It is clear from the above definitions that:

$$\mathbf{Uw} = \mathbf{Dw} \text{ or } \mathbf{D}^{-1}\mathbf{Uw} = \mathbf{w} \tag{14}$$

Thus, **w** may be interpreted as the normalized eigenvector of $\mathbf{D}^{-1}\mathbf{U}$ corresponding to the unit eigenvalue. Since **T** is an estimate of U, a reasonable estimate of **w** would be any nonzero solution $\hat{\mathbf{w}}$, if it exists, to the equation $(D^{-1}T - I)\hat{w} = 0$. They argued that the new eigen-weight method commands some advantages over the EVM that is used by Saaty, including less rigid assumptions on the error terms, robustness of solution, in addition to the computational advantage.

Crawford and Williams (1985) proposed the Logarithmic Least Squares Method (LLSM) to minimize the multiplicative error e_{ij} such that:

$$a_{ij} = e_{ij}\frac{w_i}{w_j},\ i, j = 1,\ldots,n \tag{15}$$

The objective in that model is to find the weight vector $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ that minimizes the sum of logarithmic values of these errors square, i.e. to minimize:

$$S = \sum_i \sum_j \left(\ln a_{ij} - \ln \left(\frac{w_i}{w_j}\right) \right)^2 \tag{16}$$

They proved that the S function is strictly convex and it has a unique minimum at the point w_i, where $w_i = \sqrt[n]{\prod_{j=1}^{n} a_{ij}}$. Thus, weights are derived by calculating the normalized values of the geometric mean of the comparison values a_{ij}. They stated that, compared with the EVM, the geometric mean method is better and easier to calculate.

Gao et al. (2009) argued the (LLSM) function is rather difficult to solve because the objective function S is nonlinear and usually non-convex, moreover, no unique solution exists, and the solutions are not easily computable. Therefore, they proposed three methods using the following definition of the error:

$$e_{ij} = a_{ij} w_j - w_i,\ i, j = 1,...,n \tag{17}$$

The three methods take the following forms:

- *New Least Squares method*

$$Min \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} w_j - w_i)^2 \tag{18}$$

- *Minimax Method*

$$Min \underset{1 \le i, j \le 1}{Max} \left| a_{ij} w_j - w_i \right| \tag{19}$$

- *Absolute Deviation Method*

$$Min \sum_{i=1}^{n} \sum_{j=1}^{n} \left| a_{ij} w_j - w_i \right| \tag{20}$$

In addition, some researchers have proposed another least squares method which aims to minimize sum of squares of additive error e_{ij} such that:

$$a_{ij} = \frac{w_i}{w_j} + e_{ij},\ i, j = 1,...,n \tag{21}$$

i.e. to minimize:

$$S = \sum_i \sum_j \left(a_{ij} - \left(\frac{w_i}{w_j}\right) \right)^2 \tag{22}$$

3) Method of linear programming

Chandran et al. (2005) developed a two-stage linear programming approach for generating priority weights. The proposed model aims to minimize the absolute logarithms values of the error term ω, where $\omega_{ij} = \frac{1}{e_{ij}},\ i, j = 1,..., n$, and e_{ij} as defined by Equation 15. They used three transformed decision variables:

$$\beta_i = \ln\ (w_i),\ \gamma_{ij} = \ln\ (\omega_{ij}),\ and\ z_{ij} = \left|\gamma_{ij}\right|.$$

First-stage:

$$\text{Min} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} z_{ij} \tag{23}$$

S.T.

$$\beta_i - \beta_j - \gamma_{ij} = \ln(a_{ij}), \quad i, j = 1,\dots,n; \; i \neq j \tag{24}$$

$$z_{ij} \geq \gamma_{ji}, \quad i, j = 1,\dots,n; \; i > j$$

$$z_{ij} \geq \gamma_{ji}, \quad i, j = 1,\dots,n; \; i > j \tag{25}$$

$$\beta_1 = 1 \tag{26}$$

$$\beta_i - \beta_j \geq 0, \quad i, j = 1,\dots,n; \; a_{ij} \geq 1 \tag{27}$$

$$\beta_i - \beta_j \geq 0, \quad i, j = 1,\dots,n; \; a_{ik} \geq a_{jk}, \; \textit{for all } k; \; a_{iq} > a_{jq}, \; \textit{for some } q \tag{28}$$

$$z_{ij} \geq 0, \quad i, j = 1,\dots,n \tag{29}$$

$$\beta_i, \gamma_{ij} \text{ unrestricted}, \quad i, j = 1,\dots,n \tag{30}$$

By defining z^* as the optimal value of the objective function in the first-stage; this objective function is, in some sense, a measure of the inconsistency, that is, the greater the value of the objective function, the more inconsistent is the matrix. In the case of perfect consistency, there is no error in the estimate and z^* is equal to zero. Therefore, they defined the (CI) within the linear programming framework as follows:

$$CI_{LP} = \frac{2z^*}{n(n-1)} \tag{31}$$

CI_{LP} is the average value of z_{ij} for elements above the diagonal in the comparison matrix. They showed that CI_{LP} and Saaty's CI are highly correlated.

Second stage

Due to the possibility of multiple optimal solutions to the first-stage model, another linear programming model has been solved to select from this set of alternatives optima the priority vector that minimizes the maximum of errors ω_{ij} as follows:

Min $z_{\max}$

Subject to the same constraints of the first stage model in addition to the following constraints:

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} z_{ij} = z^* \tag{32}$$

$$z_{\max} \geq z_{ij}, \quad i, j = 1,\dots,n; \; i > j \tag{33}$$

$$z_{ij} \geq 0, \quad i, j = 1,\dots,n \tag{34}$$

$$z_{\max} \geq 0, \quad i, j = 1,\dots,n \tag{35}$$

Wang and Chin (2011) proposed an approximate solution to the EVM using linear programming approach. The proposed model aims to maximize the principal eignvalue of the PCM ($\lambda_{\max}$), or equivalently to minimize $\sum_{i=1}^{n} o_i = 1/\lambda_{\max}$. They defined the Assurance Region (AR) Ψ such that:

$$\Psi = \min \left\{ \max_i \left(\frac{1}{r_i} \sum_{j=1}^{n} a_{ij} r_j \right), \; \min_i \left(\frac{1}{c_i} \sum_{j=1}^{n} a_{ij} c_j \right) \right\} \tag{36}$$

where $r_i, \; i = 1,2,\dots,n$ and $c_j, \; j = 1,2,\dots,n$ are the row sums and column sums of the comparison matrix **A** respectively. Then, they proved that:

$$\frac{w_j}{\Psi} \le o_j \le \frac{w_j}{n}, \quad j = 1,2,\ldots,n \tag{37}$$

It is evident that maximizing the eigenvalue λ_{Max} is equivalent to minimizing the sum of z_j, $(j = 1,\ldots,n)$. So, the linear approximate model to the EM can finally be formulated as follows:

$$\text{Min } \sum_{i=1}^{n} o_i \tag{38}$$

S.T

$$\sum_{j=1}^{n} a_{ij} o_j = w_i, \quad i = 1,2,\ldots,n \tag{39}$$

$$o_i - \frac{1}{n} w_i \le 0, \quad i = 1,2,\ldots,n \tag{40}$$

$$o_i - \frac{1}{\Psi} w_i \ge 0, \quad i = 1,2,\ldots,n \tag{41}$$

$$\sum_{i=1}^{n} w_i = 1 \tag{42}$$

$$w_i, \quad o_i \ge o \tag{43}$$

By using a numerical example, they showed that the proposed model turns out to be as good as the EVM in producing extremely close priorities to eigenvector weights, but it is easier to solve and more convenient to use.

4) Method of Goal Programming

Bryson (1995) introduced a Goal Programming Method (GPM) for generating a priority vector and the corresponding measure of consistency. At first, he defined the real numbers $\delta^{+}_{ij} \ge 1$ & $\delta^{-}_{ij} \ge 1$, such that $a_{ij} = (\frac{w_i}{w_j}) * (\frac{\delta^{+}_{ij}}{\delta^{-}_{ij}})$ where δ^{+}_{ij} & δ^{-}_{ij} cannot both be greater than 1. Then, the following relations are defined:

$$\left\{ \begin{array}{ll} \delta^{+}_{ij} = \delta^{-}_{ij} = 1 & \Leftrightarrow (\frac{w_i}{w_j}) = a_{ij} \\ \delta^{+}_{ij} = 1 & \Leftrightarrow (\frac{w_i}{w_j}) > a_{ij} \\ \delta^{-}_{ij} = 1 & \Leftrightarrow (\frac{w_i}{w_j}) < a_{ij} \end{array} \right\}$$

Therefore, a consistency can be found in case of $\delta^{+}_{ij} = \delta^{-}_{ij} = 1$. The objective in this model is to minimize the product $\prod_i \prod_j \delta^{+}_{ij} \delta^{-}_{ij}$, which can be translated to the following linear goal programming problem:

$$\log \Theta = \min \sum_i \sum_{j>i} (\log\ \delta^{+}_{ij} + \log\ \delta^{-}_{ij}) \tag{44}$$

S.T.

$$\log\ w_i - \log\ w_j + \log\ \delta^{+}_{ij} - \log\ \delta^{-}_{ij} = \log\ a_{ij}, \quad 1 \le i < j \le n \tag{45}$$

$$\log \delta^{+}_{ij}, \log\ \delta^{-}_{ij} \ge 0 \tag{46}$$

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The solution of this problem results in the un-normalized vector $\mathbf{w} = (w_1, w_2, ..., w_n)$ which can be further normalized to give the vector **w**.

Lin (2006) proved that, for the Bryson's GPM, when the minimal objective value is reached, a maximum of $n(n-1)+1$ equations exist, including the objective function, for determining the values of n^2 variables. Consequently, alternative optimal solutions are likely to exist, and the solution that is obtained is not necessarily the most consistent. He argued that both the GPM and the LLSM have their individual drawbacks. On one hand, the GPM performs better than the LLSM when outliers exist, but suffers from the problem of alternative optimal solutions. On the other hand, the LLSM gives a unique solution but is sensitive to outliers. To overcome these drawbacks, he proposed an Enhanced Goal Programming Method (EGPM) which combines the principle of the GPM and the (LLSM) as follows:

$$\min \log \Theta + \varepsilon \Delta \quad (47)$$

S.T

$$\log w_i - \log w_j + \log \delta_{ij}^{+} - \log \delta_{ij}^{-} = \log a_{ij} \quad 1 \le i < j \le n \quad (48)$$

$$\log \Theta = \sum_i \sum_{j>i} (\log \delta^{+}_{ij} + \log \delta^{-}_{ij}) \quad (49)$$

$$\Delta = \sum_i \sum_{j>i} ((\log \delta_{ij}^{+})^2 + (\log \delta_{ij}^{-})^2) \quad (50)$$

where ε denotes a sufficiently small positive number. According to Lin, this method gives a unique solution and is not sensitive to outlier.

Grzybowski (2010) proposed another goal programming approach to minimize the distance to the given PCM with respect to given criterion functions measuring the distance as follows:

$$\text{Min } GPCI = \frac{1}{n} \sum_{i=1}^{n} (d_i^{+} + d_i^{-}) \quad (51)$$

S.T.

$$d_i^{+} - d_i^{-} + \sum_{j=1}^{n} a_{ij} w_j = n w_i \quad (52)$$

$$\sum_{j=1}^{n} w_j = 1 \quad (53)$$

$$w_i, d_i^{+}, d_i^{-} \ge 0 \quad (54)$$

In a consistent case the Goal Programming Consistency Index (GPCI) is always equal to 0. For inconsistent PCMs the index takes positive values.

5) Method Data Envelopment Analysis

Some researchers have used the Data Envelopment Analysis (DEA) approach to derive priority weights in AHP. For example, Ramanathan (2006) used a DEA approach to derive priority weights for consistent matrices. His model, which is briefly called Data Envelopment AHP (DEAHP), takes the following form:

$$Max\ w_0 = \sum_j a_{0j} v_j \quad (55)$$

S.T

$$u_1 = 1 \quad (56)$$

$$\sum_j a_{ij} v_j - u_1 \le 0, i = 1,2,...,n \quad (57)$$

$$u_1, v_j \ge 0, \qquad j = 1,2,\ldots,n \quad (58)$$

where w_0 refers to the priority weight of the criterion or alternative under evaluation.

This model is solved for each criterion and alternative to obtain the priority weights vector. Wang et al. (2008) stated that Ramanathan's DEAHP model suffers from some drawbacks. On one hand, it may produce counterintuitive local weights for inconsistent comparison matrices. On the other hand, it may be overly insensitive to some comparisons in a comparison matrix. Therefore, they developed a new DEA model with Assurance Region (DEA\AR) as follows:

$$Max\ w_0 \tag{59}$$

S.T.

$$w_i = \sum_{j=1}^{n} a_{ij} v_j \le 1, i = 1,2,...,n \tag{60}$$

$$w_j / \Psi \le v_j \le w_j / n, \qquad j = 1,2,...,n \tag{61}$$

$$w_j, v_j \ge 0, \qquad j = 1,2,...,n \tag{62}$$

where subscript zero refers to the decision criterion or alternative under evaluation and Ψ is the assurance region defined in Equation 36. They proved that the DEA\AR model can produce true weights for perfectly consistent comparison matrices. For inconsistent ones, and due to the role of assurance region, it is able to produce rational, logical, and intuitive weights consistent with the decision makers' subjective judgments.

In addition, Wang and Chin (2009) and Mirhedayatian and Saen (2011) also developed different DEA models to overcome the drawbacks of DEAHP for deriving priorities from the pairwise comparison matrices.

6) Method of Interval Priority Estimation

Sugihara et al. (2004) suggested a method for estimating priority weights in intervals rather than crisp values. The estimated interval weights are denoted as $W_i = \left[\underline{w_i},\ \overline{w_i}\right]$ where $\underline{w_i}$ and $\overline{w_i}$ are the lower and upper bounds of the interval weight W_i. The estimated interval matrix can be defined as follows:

$$W_{ij} = \left[\frac{\underline{w_i}}{\overline{w_j}},\ \frac{\overline{w_i}}{\underline{w_j}}\right],\ \ \forall\ i, j\ (i \ne j) \tag{63}$$

Assuming that $\underline{w_i} \ge \varepsilon$, where ε is a very small positive number.

To estimate the interval weights, they used a linear programming model called "Possibilistic AHP for Crisp Data (PAHPC)", the model takes the following form:

$$\underset{\overline{w_i}, \underline{w_i}}{Min}\ j = \sum_i (\overline{w_i} - \underline{w_i}) \tag{64}$$

S.T.

$$\forall\ i, j\ (i \ne j)\ \ a_{ij} \overline{w_j} \ge \underline{w_i} \tag{65}$$

$$\forall\ i, j\ (i \ne j)\ \ a_{ij} \underline{w_j} \le \overline{w_i} \tag{66}$$

$$\forall j\ \sum_{i \in \Omega - j} \overline{w_i} + \underline{w_j} \ge 1 \tag{67}$$

$$\forall j \ \sum_{i\in\Omega-j} \underline{w}_i + \overline{w}_j \le 1 \tag{68}$$

$$\forall i \ \underline{w}_i + \overline{w}_j \le 1 \tag{69}$$

$$\forall i \ \underline{w}_i \ge \varepsilon \tag{70}$$

where Ω is a set $\{1,2,\ldots,n\}$.

They proved that the PAHPC gives an optimal solution (interval vector), and that the consistency between the given matrix and the model can be represented as the value of the objective function j, that is if (j = 0) it can be said that the given matrix is perfectly consistent. On the other hand, the estimated interval weights $W_i = \lfloor \underline{w}_i, \ \overline{w}_i \rfloor$ is preferred over $W_j = \lfloor \underline{w}_j, \ \overline{w}_j \rfloor$ if and only if $\underline{w}_i \ge \underline{w}_j$ and $\overline{w}_i \ge \overline{w}_j$.

Entani and Tanaka (2007) adopted the idea of interval weights not only to obtain the local priority intervals (weights for each matrix) but also the global priority intervals (overall priority weights). If the local weight of alternative A_i under criterion C_{ik} is denoted as $W_{ki} = [\underline{w}_{ki}, \overline{w}_{ki}]$ and the referenced priority weight of criterion C_{ik} is denoted as $P_{ki} = [\underline{p}_k, \overline{p}_k]$, then the global weight of alternative A_i is obtained as $W_i = \sum_k P_k W_{ki}$ by interval arithmetic.

7) Other Deterministic Methods

Grass and Rapcsak (2004) used the Singular Value Decomposition (SVD), Wang et al., (2007) proposed the method of Correlation Coefficient Maximization Approach (CCMA) by maximizing the correlation coefficients between the priorities themselves and each column of the comparison matrix. Also, Podinovski (2007) proposed an approach to obtain a unique point estimates for the priority weights from interval judgments. The proposed approach transforms the problem to a multi-criterial minimization problem with equally important criteria, and can be solved using the method of Symmetrical-Lexicographic problem (SL-problem).

3.2 Stochastic methods

The stochastic methods assume that the evaluation of the priority ratio may come from a range of possibilities, and the judge gives his most likely or best estimate of the required evaluation within a range of possibilities. A number of methods tried to employ probability distributions to describe the pairwise comparison of the alternatives. For example, in their earlier work**,** Lipovetsky and Tishler (1997) analyzed the AHP for five types of random variables distributions including: triangle, beta, normal, Laplace, and Cauchy. They showed that the probability density function (p.d.f) of $\frac{1}{a_{ij}}$ is not the same as the one of a_{ij} when the distribution of a_{ij} is normal, triangle, Laplace, or beta. However, if the density function of a_{ij} is Cauchy, the density function of $\frac{1}{a_{ij}}$ is also Cauchy. Then, the p.d.f. of these variables is used to estimate the elements of the comparison matrix which can be further used to derive the priorities weights using any of the deterministic methods.

On the other hand, several articles in the literature investigated the problem of extracting preference weights from interval judgments rather than point ones. For example, Haines (1998) argued that the decision maker may be uncertain about his preferences and therefore reluctant to assign single point scores to the pairwise comparisons. Instead, the decision maker may assign an interval with upper and lower bounds in an interval judgment matrix ($\mathbf{I}_{n\times n} = [l_{ij}, u_{ij}]$). Such a matrix is reciprocal in the sense that: $l_{ji} = \frac{1}{u_{ij}}$ and $u_{ji} = \frac{1}{l_{ij}}$. The interval pairwise comparison matrix is commonly assembled in a matrix of the form:

$$\mathbf{I} = \begin{bmatrix} 1 & (l_{12}, u_{12}) & (l_{13}, u_{13}) \ldots\ldots & (l_{1n}, u_{1n}) \\ (\frac{1}{u_{12}}, \frac{1}{l_{12}}) & 1 & (l_{23}, u_{23}) \ldots\ldots & (l_{2n\backslash}, u_{2n}) \\ \vdots & & & \vdots \\ (\frac{1}{u_{1n}}, \frac{1}{l_{1n}}) & (\frac{1}{u_{2n}}, \frac{1}{l_{2n}}) & \ldots\ldots & 1 \end{bmatrix}$$

The set of priority weights on the alternatives are defined by:

$$S = \left\{ (w_1, w_2, ..., w_n) : l_{ij} \le \frac{w_i}{w_i} \le u_{ij},\ w_i \ge 0, \sum_{i=1}^{n} w_i = 1,\ i = 1,2,...,n \right\} \tag{71}$$

Haines (1998) argued that the feasible region, S, is a polytope and any weight vector, **w**, in that region can therefore be expressed as a convex combination of its extreme vertices in the following way:

1. Generate (k – 1) points randomly on the interval, [0, 1], i.e. generate a random partition of [0, l] comprising L subintervals with lengths $d_1, d_2, ..., d_l$, where $d_i \ge 0,\ i = 1,....,l$ and $\sum_{i=1}^{l} d_i = 1$.
2. Form the random weight vector $w = \sum_{i=1}^{l} d_i v_i$, where the vectors, $v_1, v_2, ..., v_l$, denote the extreme vertices of S.

Various other simulation approaches have been reported for the stochastic interval AHP. For example, an early and much cited work of Arbel and Vargas (1993) applied a uniform distribution to the decision makers selected intervals. Banuelas and Antony (2004) used gamma and triangular distributions, and Zhang et al. (2003) adopted both normal and uniform distributions.

Moreover, Laininen and Hamalainen (2003) used the logarithmic linear regression approach. The model aims to minimize the sum of square of the multiplicative error term e_{ij} defined by Equation 15. By taking the logarithm for two sides of Equation 15:

$$\ln a_{ij} = \ln w_i - \ln w_j + \ln e_{ij} \tag{72}$$

Or equivalently,

$$y_{ij} = \beta_i - \beta_j + \varepsilon_{ij}, \qquad i \ne j, i = 1,...,n-1,\ j = 1,...,n \tag{73}$$

In matrix notation, the proposed linear model takes the following form:

$$\mathbf{y} = \mathbf{X\beta} + \boldsymbol{\varepsilon} \tag{74}$$

The vector of error term $\boldsymbol{\varepsilon}$ is assumed to be normally distributed with expectation $E(\varepsilon) = 0$ and with constant variance $Var(\varepsilon) = \sigma^2 I$, **y** is the $(n(n-1)/2) \times 1$ vector

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of dependent variable, $\boldsymbol{\beta}$ is the $(n \times 1)$ vector of parameters, and $\mathbf{X}$ is $n(n-1) \times n$ matrix of the dummy variables taking values (-1, 0,1) according to the values of i and j. To avoid problems of identification, the n[th] alternative is established as the benchmark alternative $(\beta_n = 0 \Leftrightarrow w_n = 1)$.

The priority vector $w_i,\ i = 1,...,n$ is obtained by the relationship:

$$w_i = \frac{\exp(\beta_i)}{\sum_{i=1}^{n} \exp(\beta_i)}, \qquad i = 1,...,n \tag{75}$$

In addition to the above, various other approaches have been reported for the stochastic interval AHP. For example, Arbel and Vargas (1993) and Hauser and Tadikamalla (1996) applied a uniform distribution to the selected intervals; Zhang et al. (2003) adopted both normal and uniform distributions; Banuelas and Antony (2004) used gamma and triangular distributions; while Jalao, et al. (2014) used beta distribution to model the linguistic imprecise pairwise comparisons, then a nonlinear programming model has been used to calculate priority weights from the beta stochastic pairwise comparisons.

4. Deriving priority weights in group decision making problems

In Group Decision Making (GDM) problems, a number of decision makers/experts express their preferences about the alternatives, and then these preferences are incorporated to obtain a common conclusion about the alternatives to support the final decision. GDM has received considerable interest in recent years, as many decision-making processes in the real world take place in group situations. Moreover, it is argued that using the opinions of several decision makers/experts when solving a decision making problem may produce better and more effective solutions to complex problems. The increasing complexity of the current socio-economic environment makes it less possible for a single decision maker/expert to consider all relevant aspects of a problem (Vanícek et al., 2009).

Reviewing the literature in GDM revealed that there are four different approaches to incorporate preferences of the multiple decision makers/experts. Three of these approaches deal with the judgments of individuals while the other one deals with the priorities derived from the judgments. These four approaches include:

- Consensus between the actors where group members give a single numeric value for each pair of compared elements resulting in a "consensus" matrix.
- Compromise or voting when consensus cannot be reached.
- Aggregation of Individual Judgments (AIJ) which refers to the aggregation of the elements of comparison matrices of the different policy makers/experts to provide a new pairwise comparison matrix for the group.
- Aggregation of Individual Priorities (AIP) which refers to the aggregation of the priority weights derived separately.

In consensus and voting options, group members agree upon a value for each pairwise comparison. However, this requires the presence of all group members and a considerable amount of discussions. In addition, these methods are usually strenuous and time consuming, especially if the hierarchy is large with many pairwise comparison matrices. Therefore, it is most appropriate to use aggregating approaches in group decision making problems.

The most widely used methods for aggregation process are the Geometric Mean Method (GMM) and the Arithmetic Mean Method (AMM). To show, simply, how these approaches can be used in the aggregation process for both AIJ and AIP, let $a_{ijk}, \quad i, j = 1,...,n \ ; k = 1,...,m$ represents the pairwise judgments in comparing alternative i with alternative j expressed by the kth member in the group decision making process, and w_{ik} represents the ith priority weight for the kth member in the group. Then, the composite judgment a_{ij}^{G} and the composite priority weight w_i^G for that group are:

- For AIJ:
  - Using the (GMM): $a_{ij}^{G} = \left(\prod_{k=1}^{m} a_{ijk} \right)^{(\frac{1}{m})}$
  - Using the (AMM): $a_{ij}^{G} = \dfrac{\left(\sum_{k=1}^{m} a_{ijk} \right)}{m}$
- For AIP:
  - Using the (GMM): $w_{i}^{G} = \left(\prod_{k=1}^{m} w_{ik} \right)^{(\frac{1}{m})}$
  - Using the (AMM): $w_{i}^{G} = \dfrac{\left(\sum_{k=1}^{m} w_{ik} \right)}{m}$

But which of these methods is better? The answer to this question has been a subject of debate between researchers. Some have argued that the GMM is appropriate for combining judgments because it preserves the reciprocal property of the comparison matrices; it is showed by Aczel and Saaty (1983) that the geometric mean is the only mathematically valid way to synthesize reciprocal judgments preserving the reciprocal condition. In addition, when equal importance is given to all the policy makers, GMM is the proper way of synthesizing judgments. Escobar and Moreno-Jime´nez (2007) argued that using GMM for aggregation of judgments preserves the acceptable inconsistency level, that is, if the individual decision makers have an acceptable inconsistency, then so does the group. On the other hand, Ramanathan and Ganesh (1994) argued that using GMM for the aggregation of judgments does not satisfy the *Pareto Optimality axiom* which means that if all individuals prefer A to B, then so should the group. The group may be homogeneous in some paired comparisons and heterogeneous in others, thus violating Pareto Optimality. However, Van den Honert and Lootsma (1996) argued that this violation of the Pareto Optimality is not related to any deficiency in GMM. Therefore, and based on the above discussions, GMM is appropriate for both (AIJ) and (AIP).

Also in GDM situations, when we are dealing with a group of decision makers or experts, it should be noted that their opinions are not necessarily equally important. Some of them have higher formal authority, better academic training, better socio-economic background, are greater experts in the considered field, or have greater influence. Such decision makers or experts can have a higher weight, which means that their judgments or priorities will have a larger influence on the group judgments or priorities. Determining the relative importance of the group's members is remarkable when considered in GDM problems.

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In addition to the aggregation process of the individuals' preferences (judgments or priorities) to obtain the group priority weights, several researchers attempted to develop methods to be applied directly to derive priority weights for group decision making situations. Some of these methods are generalizations of those which are used in the single case. Also, these methods can be divided into deterministic and stochastic methods:

4.1 Deterministic Methods

The LGP model is one of the deterministic methods proposed by Bryson and Joseph (1999) to derive priority weights in GDM AHP. LGP aims to minimize the product: $\prod_{k=1}^{m}\prod_{i=1}^{n}\prod_{j=1}^{n} p_{ij}^{k}\, q_{ij}^{k}$, where $p^{k}{}_{ij} \geq 1$ & $q^{k}{}_{ij} \geq 1$ are real numbers such that $a_{ij} = (\frac{w_i}{w_j})*(\frac{p_{ij}^{k}}{q_{ij}^{k}})$, and p_{ij}^{k} & q_{ij}^{k} cannot both be greater than 1, $i, j = 1,...,n$, and $k = 1,...,m$ (where k is the number of decision makers/experts). The (LGP) model takes the following form:

$$\log\Theta = Min\ (\frac{1}{M})\sum_{k}\log\Theta^{k} \tag{76}$$

S.T.

$$\log\ w_i - \log\ w_j + \log\ p_{ij}^{k} - \log\ q_{ij}^{k} = \log\ a_{ijk}\quad i, j = 1,...,n; k = 1,...,m \tag{77}$$

$$(\frac{1}{S})\sum_{i}\sum_{j}\log\ p_{ij}^{k} + \log\ q_{ij}^{k}) - \log\Theta^{k} = 0,\quad k = 1,...m \tag{78}$$

$$\log\delta^{+}{}_{ij}, \log\ \delta^{-}{}_{ij} \geq 0 \tag{79}$$

where $S = n(n-1)$, and all variables are nonnegative.

The solution of this model gives the un-normalized vector $\mathbf{w} = (w_1, w_1,..., w_n)$ which can be normalized to give the normalized consensus priority point vector $\mathbf{v} = (v_1, v_1,..., v_n)$.

Entani and Inuiguchi (2010) proposed three approaches based on the concept of interval regression analysis to derive the group interval priority weights $W_i = [\underline{w_i}, \overline{w_i}]$ using the aggregation of the individual priority intervals. First, the interval priority weights $W_{ik} = [\underline{w_{ik}}, \overline{w_{ik}}]$ of alternatives are obtained from the individually given pairwise comparison matrices. Then, these individual interval priority weights are aggregated in order to reach the group priority weights. The three methods are:

1. *Least Upper Approximation Model*
2. *Greatest Lower Approximation Model*
3. *Least Squares Model*

On the other hand, Bolloju (2001) argued that the application of the AHP to model the subjective preferences of individuals in large groups requires homogeneity of the group. Aggregating decision maker's preferences into a single model may not represent any single decision maker nor satisfy the majority of the decision makers. Therefore, he proposed a method of aggregating individual AHP models in homogeneous subgroups based on similarities in decision makers' preferences. This approach consists of four steps:

First, individual utility functions are derived from the AHP models that represent preferences of individual decision makers. A utility function, UF^i, corresponding to the AHP model of the k^{th} member of a group of m members can be defined as:

$UF^k = \sum_{i=1}^{n^k} w_j^k f_j^k$, where w_j^k is the weight associated with the factor (criterion) f_j^k for that member, and n^k is the number of factors used by that member.

Second, the differences among the individual utility functions are resolved and unified utility functions are produced through content analysis.

Third, the unified utility functions are divided into subgroups based on similarities in preferences. Several techniques such as cluster analysis, data envelopment analysis, and discriminate analysis can be used for identifying subgroups or clusters based on similarities in the unified decision models.

Fourth, the utility functions in each subgroup are aggregated to represent subgroup's aggregated utility functions using arithmetic mean.

On the other hand, and similar to the single decision making case, DEA has been used in the group decision making problems. For example, Wang and Chin (2009) extended their DEA method for deriving the priorities in the single case, for use with group decision problems (the method is briefly called DEAW&C) as follows:

$$Max\ w_0 = \sum_{j=1}^{n}\left(\sum_{k=1}^{m}\alpha_k a_{0j}^{(k)}\right)x_j \tag{80}$$

S.T.

$$\sum_{j=1}^{n}\left(\sum_{k=1}^{m}\sum_{i=1}^{n}\alpha_k a_{ij}^{(k)}\right)x_j = 1 \tag{81}$$

$$\sum_{j=1}^{n}\left(\sum_{k=1}^{m}\alpha_k a_{ij}^{(k)}\right)x_j \geq n x_i, \qquad i = 1,\ldots,n \tag{82}$$

$$x_j \geq 0,\ \ j = 1,2,\ldots,n \tag{83}$$

where m is the number of decision makers, α_k is the weight assigned to the k^{th} decision maker.

Hosseinian et al. (2009) proposed another DEA model called Data Envelopment Analysis for weight Derivation in Group Decision "DEA-WDGD". They argued that this model enables simpler derivation of weights than does the DEAW&C model. The DEA-WDGD takes the following form:

$$Max\ \sum_{i=1}^{n} w_i \tag{84}$$

S.T.

$$\sum_{k=1}^{m}\left(\sum_{j=1}^{n} a_{ij}^{(k)}\right)v_k - w_i = 0,\ \ i - 1,\ldots,n \tag{85}$$

$$Max\ \sum_{i=1}^{n} w_i \tag{86}$$

$$\alpha_k v_{k+1} - \alpha_{k+1} v_k = 0, \qquad k = 1,\ldots,m-1 \tag{87}$$

$$w_i, v_k \geq 0, \qquad i = 1,2,\ldots,n;\ k = 1,\ldots,m \tag{88}$$

Grošelj et al. (2011) proved that the DEA-WDGD model violates the reciprocal property which is required in the AHP model. Therefore, he proposed a new DEA model using the weighted geometric mean instead of the weighted arithmetic mean in the DEAW&C. The new model is called Weighted Geometric Mean DEA (WGMDEA) and takes the following form:

$$Max\ w_0 = \sum_{j=1}^{n}\left(\prod_{k=1}^{m}\left(a_{0j}^{(k)}\right)^{\alpha_k}\right)x_j \tag{89}$$

S.T.

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\prod_{k=1}^{m}\left(a_{ij}^{(k)}\right)^{\alpha_k}\right)x_j = 1 \tag{90}$$

$$\sum_{j=1}^{n}\left(\prod_{k=1}^{m}\left(a_{ij}^{(k)}\right)^{\alpha_k}\right)x_j \geq nx_i,\ i = 1,\ldots,n \tag{91}$$

$$x_j \geq 0,\ \ j = 1,2,\ldots,n \tag{92}$$

Angiz et. al. (2012) proposed another DEA model which manipulates both the priority weights and ranking aspect of each decision maker.

In addition, the nonlinear programming approach has been used in the GDM. For example, Chou et al (2007) proposed a nonlinear programming model to minimize the sum square of distances between the group weights and single weights as follows:

$$Min\ J_{r,c}\ (u,w) = \sum_{j=1}^{n}\left(u_k\right)^r\left(\left\|w - w_k\right\|^2 + c\right) \tag{93}$$

S.T.

$$\sum_{i=1}^{m}u_i = 1,\ \sum_{j=1}^{n}w_j = 1 \tag{94}$$

where u_k is the aggregation weight and $r \in (1,\infty], c \geq 0$ are two constants to control the aggregation process. They presented an iterative algorithm to derive the aggregated priority weights and aggregates pairwise comparison matrices into a consensus matrix.

4.2 Stochastic Methods

Some researchers proposed stochastic and statistical approaches for the GDM problems; for example**,** Ramanathan (1997) proposed a method of stochastic goal programming to derive the maximum likelihood point estimates for priority weights for GDM. According to Ramanathan, the judgments can be interpreted as stochastic when more than one value for the same judgment is considered.; this is the situation of group decision making. He argued that the judgment is captured on a semantic scale and is converted into a numerical integer value ζ_{ij} using a logarithmic scale. Then the numerical estimate of the preference ratio $\frac{w_i}{w_j}$ is defined as $\frac{w_i}{w_j} \approx a_{ij} = e^{\zeta a_{ij}}$, where ζ, is a scale parameter. The model takes the following form

$$Min \sum_{i=1}^{n}\sum_{j>i}^{n}\left(\frac{d_{ij}^{2}}{\zeta^{2}\ \sigma_{ij}^{2}}\right) \tag{95}$$

S.T

$$\beta_i - \beta_j + d_{ij} = \zeta * a_{ij}, \quad i = 1,\dots,n; \ \ j > i \tag{96}$$

$$\beta_i \geq 0, \ \ i = 1,2,\dots,n \tag{97}$$

where d_{ij} are the deviational variables, and σ_{ij}^{2} is the variance associated with ζ_{ij} .

Basak (2011) used a statistical model to derive composite priority weights of a number of decision makers. First, he defined the following relation:

$$t_{ik} = \frac{1}{1 + \sum_{j=1, i\neq j}^{n} \frac{1}{a_{ijk}}}, \qquad i = 1,\dots,n; \ \ k = 1,\dots,m \tag{98}$$

where a_{ijk} denotes the PCM entry denoting the intensity of preference of alternative A_i to alternative A_j (or of criterion C_i to criterion C_j) by the k^th^ decision maker. He argued that t_{ik} can be considered as an estimate for π_i in the following stochastic model:

$$t_{ik} = \pi_i + \eta_{ik} \tag{99}$$

where $\pi_i = (w_i / \sum_{i=1}^{n} w_i), \ i = 1,\dots,n$, and η_{ik} are random variables distributed as Multivariate Normal $N(0, \mathbf{\Sigma})$; then, t_{ik} are then distributed as multivariate Normal $N(\boldsymbol{\pi}, \mathbf{\Sigma})$, where $\pi' = (\pi_1, \pi_1, \dots, \pi_n)$.

Let $\mathbf{T}_k$ denotes the vector of random variables $(t_{1k}, t_{2k}, \dots, t_{nk})$ for the k^th^ individual in a given group. The model parameters π and $\mathbf{\Sigma}$ can be estimated by maximum likelihood method and are given by:

$$\hat{\pi}_i = \frac{\bar{t}_{im}}{\sum_{i=1}^{n} \bar{t}_{im}}, \qquad i = 1,\dots,n \tag{100}$$

$$\overline{T}_m = \frac{1}{m}\sum_{k=1}^{m} T_k = (\bar{t}_{1m},\dots,\bar{t}_{nm}) \tag{101}$$

$$\hat{\mathbf{\Sigma}} = \frac{1}{m-1}\sum_{k=1}^{m}(T_k - \overline{T}_m)\ \ (T_k - \overline{T}_m) \tag{102}$$

However, Basak did not give any logical explanation about the relation in Equation 98. Also, he did not prove his arguments about the relationship between t_{ik} and π_i which creates some vagueness with regards to his model.

Another method proposed by Altuzarra et al. (2007), who proposed a new prioritization procedure based on Bayesian approach, is as follows:

For m decision makers, let $\{A^{(k)} = (a_{ijk}),\ k = 1,2,\dots,m\}$ be the reciprocal judgment matrices for group decision makers, $(w_1^G, w_2^G, \dots, w_n^G),\ w_i \geq 0$ be the group's un-normalized priorities for the alternatives, and $(v_1^G, v_2^G, \dots, v_n^G),\ v_i \geq 0$ be their normalized distribution values: $v^G{}_i = \dfrac{w_i^G}{\sum_{j=1}^{n} w_j^G}$.

The proposed approach adopts a multiplicative model with log-normal errors, this model is given by:

$$a_{ijk} = \frac{w_i^G}{w_j^G} e_{ijk}, \qquad i, j = 1,\dots,n;\ \ k = 1,\dots,m, \tag{103}$$

with e_{ij}^k distributed as $LN(0, \sigma^{(k)2})$. By eliminating reciprocal judgments $(a_{ij}, i \geq j)$ and taking logarithms, a regression model with normal errors is obtained:

$$y_{ijk} = \beta_i^G - \beta_j^G + \varepsilon_{ijk}, \quad i = 1,\dots,n;\ \ j = i+1,\dots,n;\ \ k = 1,\dots,m, \tag{104}$$

where $y_{ijk} = \log(a_{ijk}),\ \beta_i^G = \log(w_i^G),\ and\ \varepsilon_{ijk} = \log(e_{ijk})$.

To avoid problems of identification, the n[th] alternative is established as the benchmark alternative $(\beta_n = 0 \Leftrightarrow v_n = 1)$.

In matrix notation, this model is expressed as:

$$\mathbf{y}^{(k)} = \mathbf{X}\boldsymbol{\beta}^G + \boldsymbol{\varepsilon}^{(k)} \tag{105}$$

They proved that, by taking a constant non-informative distribution as the prior distribution for the vector of log-priorities $\boldsymbol{\beta}^G$, the posterior distribution of $\boldsymbol{\beta}^G$ for complete and precise information is given by:

$\boldsymbol{\beta}^G \mid \mathbf{y}$ Follows $N_{n-1}(\hat{\beta}_B, \hat{\Sigma}_B)$, where:

$$\hat{\boldsymbol{\beta}}_B = \frac{\sum_{k=1}^{m} \tau^{(k)} \hat{\beta}^{(k)}}{\sum_{k=1}^{m} \tau^{(k)}},\ \hat{\boldsymbol{\Sigma}}_B = \left(\sum_{k=1}^{m} \tau^{(k)} \right)^{-1} (\mathbf{X}'\mathbf{X})^{-1} \tag{106}$$

$$(X'X)^{-1} = \begin{bmatrix} 2/n & 1/n \dots\dots & 1/n \\ 1/n & 2/n \dots\dots & 1/n \\ \vdots & & \vdots \\ 1/n & 1/n \dots\dots & 2/n \end{bmatrix}, \text{ and } \tau^{(k)} = \frac{1}{\sigma^{(k)2}} \tag{107}$$

For the Aggregation of individual judgments (AIJ), a matrix of aggregated judgments $A^G = (a_{ij}^G)$ is constructed using the geometric mean. Then, the model becomes:

$$y_{ij}^G = \log(a_{ij}^G) = \frac{1}{m}\sum_{k=1}^{m} y_{ijk} \tag{108}$$

$$= \beta_i^G - \beta_j^G + \sum_{k=1}^{m} \varepsilon_{ijk} = \beta_i^G - \beta_j^G + \bar{\varepsilon}_{ij} \tag{109}$$

In this case, the maximum likelihood estimator of the vector of the group's log priorities, $\boldsymbol{\beta}^G$, is obtained by the geometric mean (of judgments) estimator given by:

$$\hat{\beta}_{AIJ,i} = \bar{y}_i^G - \bar{y}_n^G \ and\ \bar{y}_i^G = \frac{1}{n}\sum_{j=1}^{G} y_{ij}^G, \qquad i = 1,\ldots,n-1 \tag{110}$$

For the Aggregation of individual priorities, they used the geometric mean as following:

$$\hat{\mathbf{w}}_{AIP} = \frac{1}{m}\sum_{k=1}^{m} \hat{\mathbf{w}}^{(k)} \tag{111}$$

Or;

$$\hat{\boldsymbol{\beta}}_{AIP} = \frac{1}{m}\sum_{k=1}^{r} \hat{\beta}^{(k)} \tag{112}$$

They proved that the proposed Bayesian estimator for complete and precise matrices with known variance $(\hat{\boldsymbol{\beta}}_B)$ is more efficient than the usual $\hat{\boldsymbol{\beta}}_{AIJ}$ and $\hat{\boldsymbol{\beta}}_{AIP}$ applied to aggregate judgments and priorities. They also extended the Bayesian approach in case the information provided by any of the decision makers is incomplete, which means that not all of the decision makers express the n(n-1)/2 possible judgments in the reciprocal pairwise comparison.

Also, Gargallo et al. (2007) used the Bayesian hierarchical model based on mixtures of normal distributions to describe the distribution of individual priorities and for the identification of existing groups of the decision makers' preferences.

In spite of its advantages as a powerful tool for parameter estimation problems and inferences, using Bayesian approach for deriving priority weights in group AHP has some disadvantages. On one hand, it works with a lot of assumptions especially for the prior distribution of the parameters; on the other hand, it often comes with a high computational cost, especially in models with a large number of parameters. Researchers may find it difficult to understand and apply this approach for decision making problems with these limitations.

5. Conclusion

Since its real appearance as a multi-criteria decision making technique in the 1980's, AHP has been greatly accepted among researchers as a simple and practical technique that can be applied to different fields for different objectives. In this paper, the main theoretical concepts of AHP were discussed including structuring of a decision problem, judgments, prioritization procedures, consistency indices, and synthesizing to obtain the final weights. The methodological developments of deriving weights in AHP, in both single and group decision making, are reviewed in detail.

Previous literature reviews revealed that using EVM for deriving weights in AHP, as proposed by Saaty, has some shortcomings. Therefore, several attempts have been made to propose new methods using different techniques to be used as alternatives to EVM. This paper reviewd several proposed methods, and concluded, first, many of these methods are complicated and not easy to be applied compared to EVM, second, the proposed methods did not provide real solutions for the claimed drawbacks of the EVM, particularly, the rank reversal problem which is still not fully resolved, and third, there is no measure for the superiority of these methods to tell determine which technique obtains the best results, especially for those who are not specialized. In addition, most of the applicants of AHP are unaware of its successive developments (Ishizaka and Labib, 2011). These conclusions may explain the limited use of those suggested methods to derive weights in AHP compared to EVM.

REFERENCES

Aczel, J. and T.L. Saaty (1983). Procedures for synthesizing ratio judgments. *Journal of Mathematical Psychology, 27*, 93–102.

Aguaron, J. and J. M. T. Moreno-Jimenez (2003). The geometric consistency index: Approximated thresholds. *In European Journal of Operational Research, 147,* 137−145.

Alonso, J. A. and M. T. Lamata (2006). Consistency in the Analytic Hierarchy Process: A new approach. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 14 (4),* 445−459.

Altuzarra, A., J. María, Moreno- Jiménez, and M. Salvador (2007). A Bayesian prioritization procedure for AHP-group decision making. *European Journal of Operational Research*, 182, 367–382.

Angiz, L. M. Z., A. Mustafa, N. A. Ghani, and A. A. Kamil (2012). Group decision via usage of Analytic Hierarchy Process and preference aggregation method. *SainsMalaysiana, 41(3),* 361–366.

Arbel, A. and L., G. Vargas (1993). Preference simulation and preference programming: Robustness issues in priority derivation. *European Journal of Operational Research, 69*, 200-209.

Banuelas, R. and J. Antony (2004). Modified analytic hierarchy process to incorporate uncertainty and managerial aspects. *International Journal of Production Research, 42,* 3851–3872.

Basak, I. (2011). An alternate method of deriving priorities and related inferences for group decision making in Analytic Hierarchy Process. *Journal of Multi-Criteria Decision Analysis, 18(5-6), 279-287.*

Belton, V. and T. Gear (1983). On a shortcoming of Saaty's method of Analytic Hierarchies. *Omega, 11 (3),* 228-230.

Bolloju, N. (2001). Aggregation of Analytic Hierarchy Process models based on similarities in decision makers preferences. *European Journal of Operations Research*, (128), 499-508.

Borodin, D., V. Gorelik, W. D. Bruyn, B. V. Vreckem (2011). Two methods for consistency-driven approximation of a pairwise comparison matrix. *Proceedings of the International Symposium on the Analytic Hierarchy Process 2011.*

Bryson, N. (1995). A goal programming method for generating priority vectors. *The Journal of the Operational Research Society, 46 (5),* 641-648.

Bryson, N. and A. Joseph (1999). Generating consensus priority point vectors: a logarithmic goal programming approach. *Computers & Operations Research, 26,* 637-643.

Chandran, B., B. Golden and E. Wasil (2005). Linear programming models for estimating weights in the Analytic Hierarchy Process. *Computers & Operations Research, 32,* 2235–2254.

Chou, M., H. lee, C. Chu and C. Cheng (2007). Synthesizing comparison matrices of AHP under group decision. *Knowledge-Based Intelligent Information and Engineering Systems*, 1323–1330.

Chu, A. T. W., R. E. Kalaba and K. Npingarn (1979). A comparison of two methods for determining the weights of belonging to fuzzy sets. *Journal of Optimization Theory and Application, 27((4),* 531-538.

Cogger, K. O. and P. L. YU (1985). Eigenweigh ectors and least-distance approximation for revealed preference in pairwise weight ratios. *Journal of Optimization Theory and Applications, 46 (4), 483-491.*

Costa, C.A.B. and J. Vansnick (2008). A critical analysis of the eigenvalue method used to derive priorities in AHP. *European Journal of Operations Research, 187,* 1422–1428.

Crawford, G. and C. Williams (1985). A note on the analysis of subjective judgment matrices. *Journal of Mathematical Psychology, 29(4)*, 387–405.

Entani, T. and H. Tanaka (2007). Interval estimations of global weights in AHP by upper approximation. *Fuzzy Sets and Systems, 158,* 1913 – 1921.

Entani, T. and M. Inuiguchi (2010). Group decisions in interval AHP based on interval regression analysis. *Integrated Uncertainty Management and Applications, 68,* 269-280.

Ergu, D., G. Kou, Y. Peng and Y. She (2011). A simple method to improve the consistency ratio of the pairwise comparison matrix in ANP. *European Journal of Operations Research, 213(1),* 246-259.

Escobar, M. T. and J. M. Moreno-Jime´nez (2007). Aggregation of individual preference structures in AHP-group decision making. *Group Decision and Negotiation, 16*, 287-301.

Fedrizzi, M. and S. Giove, (2007). Incomplete pairwise comparison and consistency optimization. *European Journal of Operational Research, 183(1),* 303–313.

Forman, E. H. and S. I. Gass, (2001). The Analytic Hierarchy Process: An exposition. *Operations Research, 49(4),* 469-486.

Gao, S., Z. Zhang, and C. Cao (2009). New methods of estimating weights in AHP. *International Symposium on Information Processing*, 201-204.

Gargallo, P., J. M. Moreno-Jime´nez and M. Salvador (2007). AHP-group decision making: A Bayesian approach based on mixtures for group pattern identification. *Group Decision and Negotiation, 16*, 485–506.

Gomez-Ruiz, J. A., M. Karanik, and J. I. Peláez (2010). Estimation of missing judgments in AHP pairwise matrices using a neural network-based model. *Applied Mathematics and Computation, 216*, 2959–2975.

Grass, S. I. and T. Rapcsak (2004). Singular value decomposition in AHP. *European Journal of Operational Research, 154,* 573-584.

Grošelj, P., Š. Pezdevšek, and P. Malovrh (2011). Methods based on data envelopment analysis for deriving group priorities in Analytic Hierarchy Process. *Central European Journal of Operations Research, 19(3),* 267-284.

Grzybowski A.Z. (2010). Goal programming approach for deriving priority vectors - some new ideas. *Scientific Research of the Institute of Mathematics and Computer Science, 1(9), 17-27.*

Haines, L. M. (1998). A statistical approach to the analytic hierarchy process with interval judgments. (I). Distributions on feasible regions. *European Journal of Operational Research, 110,* 112-125.

Harker, P.T. (1987a). Alternative modes of questioning in the analytic hierarchy process. *Mathematical Modeling, 9(3–5),* 353–360.

Harker, P.T. (1987b) Incomplete pairwise comparisons in the analytic hierarchy process. *Mathematical Modeling, 9(11),* 837–848.

Hauser, D. and P. Tadikamalla (1996). The Analytic Hierarchy Process in an uncertain environment: A simulation approach. *European Journal of Operational Research, 91*, 27-37.

Ho, W. (2008). Integrated Analytic Hierarchy Process and its applications- a literature review. *European Journal of Operations Research*, *186*, 211-228.

Hosseinian S.S., H. Navidi and A. Hajfathaliha (2009). A new method based on data envelopment analysis to derive weight vector in the group analytic hierarchy process. *Journal of Applied Sciences, 1(8), 3343-3349.*

Ishizaka, A. and A. Labib (2011). Review of the main developments in the Analytic Hierarchy Process. *Expert Systems with Applications, 38,* 14336–14345.

Jalao, E. R., T. Wu and D. Shunk (2014). A stochastic AHP decision making methodology for imprecise preferences. *Information Sciences, 270,* 192–203.

Laininen, P. and R. P. Hamalainen (2003). Analyzing AHP-matrices by regression. *European Journal of Operational Research, 148,* 514–524.

Lin, C. –C (2006). An enhanced goal programming method for generating priority vectors. *The Journal of the Operational Research Society 57 (12),* 1491-1496.

Lipovetsky, S. and A. Tishler (1997). Interval estimation of priorities in the AHP: Methodology and application. *Israel Institute of Business Research,* Working Paper No. 16/97, Faculty of Management, Tel Aviv University.

Mirhedayatian, S. M. and R. F. Saen (2011). A new approach for weight derivation using data envelopment analysis in the Analytic Hierarchy Process. *Journal of the Operational Research Society, 62,* 585-1 595.

Mirhedayatian, S. M., M. Jafarian, and R. F. Saen (2011). A multi-objective slack based measure of efficiency model for weight derivation in the Analytic Hierarchy Process. *Journal of the Applied Sciences, 11(19),* 3338-3350.

Nishizawa, K. (2005). Estimation of unknown comparisons in incomplete AHP and its compensation. *Report of the Research Institute of Industrial Technology*, Nihon University, No. 77.

Peláez, J. I. and M. T. Lamata (2003). A new measure of consistency for positive reciprocal matrices. *Computers and Mathematics with Applications, 46*, 1839-1845.

Podinovsky, V. V. (2007). Interval articulation of superiority and precise elicitation of priorities. *European Journal of Operational Research, 180,* 406-417.

Ramanathan, R. (1997). Stochastic decision making using multiplicative AHP. *European Journal of Operational Research, 97,* 543-549.

Ramanathan, R. (2006). Data envelopment analysis for weight derivation and aggregation in the Analytic Hierarchy Process. *Computers & Operations Research, 33*, 1289–1307.

Ramanathan, R. and L.S., Ganesh (1994). Group preference aggregation methods employed in AHP: An evaluation and an intrinsic process for deriving members' weightages. *European Journal of Operational Research*, *79,* 249-265.

Ramanujam, V. and T. L. Saaty (1981). Technological choice in the less developed countries: An analytic hierarchy approach. *Technological Forecasting and Social Change, 19*, 81–98.

Saaty, T. L. (1988). *Analytic Hierarchy Process: Planning, priority setting, and resource allocation*. Pittsburgh, PA: University of Pittsburgh Press.

Saaty, T. L. (1990). How to make a decision: Analytic Hierarchy Process. *European Journal of Operations Research (48),* 9-26.

Saaty, T. L. (1994). How to Make a Decision: Analytic Hierarchy Process. *Interfaces, 24*, 19-43.

Saaty, T. L. (2001). The seven pillars of the Analytic Hierarchy Process: Models, methods, concepts & applications of the Analytic Hierarchy Process. *International Series in Operations Research & Management Science, 34,* 27-46.

Saaty, T. L. (2003). Decision-making with the AHP: Why is the principal eigenvector necessary. *European Journal of Operations Research, 145,* 85-91.

Saaty, T. L. and M. S. Ozdemir (2003). Why the magic number seven plus or minus two. *Mathematical and Computer Modeling, 38,* 233-244.

Saaty, T.L. and L.G. Vargas (1985). The legitimacy of rank reversal. *Omega, 12(5), 513-516.*

Saaty, T. L. and H. Shih (2009). Structures in decision making: On the subjective geometry of hierarchies and networks. *European Journal of Operations Research, 199,* 867-872.

Shiraishi, S., T. Obata, and M. Daigo (1998). Properties of a positive reciprocal matrix and their application to AHP. *Journal of Operational Research Society of Japan, 41,* 404–414.

Stein, W. E., and Mizzi P. J., (2007). The harmonic consistency index for the Analytic Hierarchy Process. *European Journal of Operations Research, 177,* 488-497.

Subramanian, N. and R. Ramanathan (2012). A review of applications of Analytic Hierarchy Process in operations management. *International Journal of Production Economics, 138,* 215–241.

Sugihara, K., H. Ishii and H. Tanaka (2004). Interval priorities in AHP by interval regression analysis. *European Journal of Operational Research, 158,* 745–754.

Vaidya, O. S., and Kumar, S. (2006). Analytic Hierarchy Process: An overview of applications. *European Journal of Operations Research, 169,* 1-29.

Vanícek, J., I. Vrana and s. Ali (2009). Fuzzy aggregation and averaging for group decision making: A generalization and survey. *Knowledge-Based Systems, 22,* 79-84.

Van den Honert, R. C. and F.A. Lootsma (1996). Group preference aggregation in the multiplicative AHP ± the model of the group decision process and Pareto optimality. *European Journal of Operational Research, 96,* 363-370.

Wang, Y. and K. Chin (2009). New data envelopment analysis method for priority determination and group decision making in the Analytic Hierarchy Process. *European Journal of Operational Research* (*195),* 239–250.

Wang, Y. and K. Chin (2011). A linear programming approximation to the eigenvector method in the Analytic Hierarchy Process. *Information Sciences, 181,* 5240–5248.

Wang, Y. and T. M. S. Elhag (2006). An approach to avoiding rank reversal in AHP. *Decision Support Systems, 42(3), 1474-1480*.

Wang, Y., C. Parkan and Y. Luo (2007). Priority estimation in the AHP through maximization of correlation coefficient. *Applied Mathematical Modeling, 31*, 2711-2718.

Wang, Y., K. Chin and G. K. K. Poon (2008). A data envelopment analysis method with assurance region for weight generation in the Analytic Hierarchy Process. *Decision Support Systems, 45*, 913–921.

Zanakis, S., T. Mandakovic, S. Gupta, S. Sahay, and S. Hong, (1995). A review of program evaluation and fund allocation within the service and government sectors. *Socio-Economic Planning Sciences, 29*, 59-79.

Zeshui, X. and W. Cuiping (1999). A consistency improving method in the analytic hierarchy process. *European Journal of Operational Research,* 116, 443-449.

Zhang, Q., J. Ma, Z.P. Fan, and W.C. Chiang (2003). A statistical approach to multiple-attribute decision-making with interval numbers. *International Journal of Systems Science, 34*, 683–692.