DECISIONS: STRUCTURE, JUDGMENT, AND NATURAL LAW

Thomas L. Saaty
University of Pittsburgh
Pittsburgh, PA 15260
saaty@katz.pitt.edu

ABSTRACT

The object here is to show that our thinking processes and our physical forms and those of all things that exist, are a result of response in nature to influences as stimuli, brought about by natural occurrences. The ideas are developed through a generalization of the role judgment plays in decision making. Judgment serves as the basic link between our conscious awareness and the stimuli of the natural world. The mathematics used to represent natural laws is derived from stimulus-response theory and this in turn from the representation of judgment as it is used in decision-making. The representation of discrete judgment as a principal eigenvalue problem is generalized to the continuous case through Fredholm theory. Solving the resulting fundamental functional equation, which is a necessary condition for the existence of a solution, gives rise to damped periodic oscillation. The Fourier transform of the real valued solution has a perturbed inverse square representation that poses a question raised on occasion in science about the full accuracy of exact inverse square laws of gravitation, optics and of electric charges. The Fourier transform of the complex valued solution is a linear combination of Dirac type distribution of impulsive functions representing how the brain must operate to respond to external stimuli. A generalization is made to a functional equation in operator form with its solution. These solutions describe all forms that exist in nature as anything that responds to influences. These considerations that originate in the mathematics of judgment, serve as a unifying approach to our understanding and to creating tools for modeling and solving complex physical and behavioral problems.

Keywords: Judgment, stimulus-response, natural law, response function, operator, Fundamental Equation

http://dx.doi.org/10.13033/ijahp.v1i1.13

1. Introduction

There are two basic characteristics of a decision; the structure that represents the factors and the alternatives of that decision and their connections, and the judgments and their syntheses that determine the best outcome of the decision. We can think of the world and of human experience in terms of internal and external influences to which we respond in our conscious experience. Our response characterizes the fundamental internal nature of how our brains must operate, and the law they satisfy to respond to external stimuli. In addition, in nature itself influences are responded to in physical and biological ways that create forms like particles, stones, plants, animals and people. They are characterized as particular forms of response to influences. It appears that stimulus-response theory, an offshoot of the judgments process and of decision making, serves to unify our understanding of the world.

To structure a decision requires use of the various steps normally recognized to belong to creative thinking. It requires understanding of the problem and of specialized thinking and expert knowledge in the area in which the decision is made. A structure can be a hierarchy whose levels and elements are arranged by the experts according to their understanding of the downward flow of influence. It can also be a more general network whose interdependencies capture the forward and backward directions of flows, their branching and confluence that work together to shape the outcome. A single decision, depending on its complexity can involve a number of structures whose outcomes must be brought together into a single overall outcome.

Many people think that measurement demands a physical scale with a zero and a unit of measure. That is not true. Surprisingly, we can derive accurate and reliable relative scales that do not have a zero or a unit by using our understanding and judgments which are, after all, the fundamental determinants of why we want to measure something. In reality we do this all the time without thinking about it. Physical scales are useful for the things we know how to measure. Even after we obtain readings from a physical scale, they still need to be interpreted with our judgment. And the number of things we do not know how to measure, the so-called intangibles, is infinitely larger than the things we know how to measure. Judgment followed by decision and action is the link between our inner understanding and the physical world that we depend on to satisfy our needs.

Not only all measurement needs to be interpreted in terms of our value system, but especially readings on a linear scale like the meter and the yard that are the simplest kind of measurements need interpretation to determine what the numbers actually mean. It is true that the readings denote quantity represented by a number, but how important that number is depends on the purpose we intend to use it for. Thus measurement provides information that is the basis of different kinds of judgments in different decisions. In the end it is judgment that is essential for our understanding.

Only when judgment is quantitative that we are able make subjective measurements and tradeoffs. How do we do that in a meaningful way that can be validated in practice? Judgment is always contextual, relating things among themselves or to standards we store in memory. It is expressed as comparisons among things. To be meaningful and relevant comparisons need to be reduced to pairwise comparisons. Pairwise comparisons give rise to priorities that is a cardinal way of determining rank order among things and representing that order numerically. The study of order belongs to the field of order topology in mathematics and is in contrast to metric topology used to determine the closeness of measurements in science. An important property of order is transitivity that is often imposed as an axiom. But when we need judgments, transitivity may be violated and order becomes intransitive. When cardinal numbers are used, we can have transitivity but still be inconsistent. To derive priorities requires that we solve the principal eigenvalue problem to derive priorities. To derive priorities from inconsistent judgments order and transitivity require that we solve the principal eigenvalue problem to derive priorities.

In decision making comparisons are discrete and can be represented by matrices which is not true when we automatically and without thinking take in a lot of sensory information that also requires comparison of a continuous kind. We learn from the solution to the generalization of the eigenvalue representation that our minds are constantly responding to stimuli in a way represented by a damped periodic function with two parameters. The Fourier transform of the complex valued solution is the sum of functions that are Dirac distributions implying that our responding mechanism, that is our brain, must respond with impulsive firings as the brain actually does. We also learn from the Fourier transform of the real solution that what we think natural law is, our response always takes the form of an inverse square law that is not precisely a square as assumed in science, but sufficiently close that it is difficult to detect small values of the parameters.

Let us note that the human mind cannot respond to stimuli in linear proportion to their intensity. As the intensity of the stimulus increases, our senses become duller and gradually level off to a point where we cannot distinguish between an explosion and a much larger explosion, a distant hill and a mountain or among very large numbers ranging from the millions to the billions. We can describe this dampening effect of response by a mathematical function that we derive from an equation that describes the relationship between for example the response at a distance from a stimulus that must be proportional to what that response would be at the origin of the stimulus. We learn from generalizing the continuous formulation to operators, that all forms of creation that respond to influences of which sense stimuli are only one kind have only certain possible general forms with which they can make their response.

2. Structuring—the need for creative thinking

This section is dedicated to observations we have made about the relationship between decision making and creative thinking as a process. A purposeful system is generally characterized in terms of its purpose or goal; its physical or geometric abstract structure of elements collected into subsystems; the functions of these subsystems carried out to fulfill the goal; and the flows that take place through the connections of the subsystems to perform the functions. In decision making the purpose of a system serves the satisfaction of human needs and values structured and measured in terms of benefits (B), opportunities (O), costs (C) and risks (R) referred to collectively as BOCR. They are a critical part of decision making.

It takes little effort to realize that creative thinking is an essential part of decision making. One can relate the four tenets of creativity to the steps taken in making a decision. They are to brainstorm all the factors that go into a decision. They are then related (synectics) by putting them into groups clusters or components of homogeneous (close identity) elements with respect to the property or criterion being considered (one axiom of the AHP) and pairwise compared with reciprocal values (another axiom of the AHP). The factors are then arranged into a hierarchic or network structures (morphological analysis in creativity that is also the third axiom of the AHP) that can include the goal, objectives, criteria, influential actors and their objectives and the alternatives of the decision (see later). Finally these structures are constantly polished, revised and expanded as needed in the process of arriving at the final best decision (lateral thinking which is the fourth axiom of the AHP with regard to expectations of the outcome and its reliability). We have applied these four operations of creativity on numerous occasions in individual decisions and in groups gathered for a special purpose and even by correspondence with the parties involved.

But even with creative thinking there is no better substitute to expert knowledge and understanding except through the use of the prioritization process. To create priorities needs creativity. Conversely, to be effective with creativity needs priorities. Two seemingly different occupations of the mind are fundamentally interdependent. One would like to think that making decisions needs creativity much more than being creative involves decision making, but that is a subject that awaits more exploration.

Creative thinking and decision making are linked together with a feedback relation. Our representation of the relationship between decision making and creative thinking is roughly represented in the alternative forms of Figures 1 and 2.

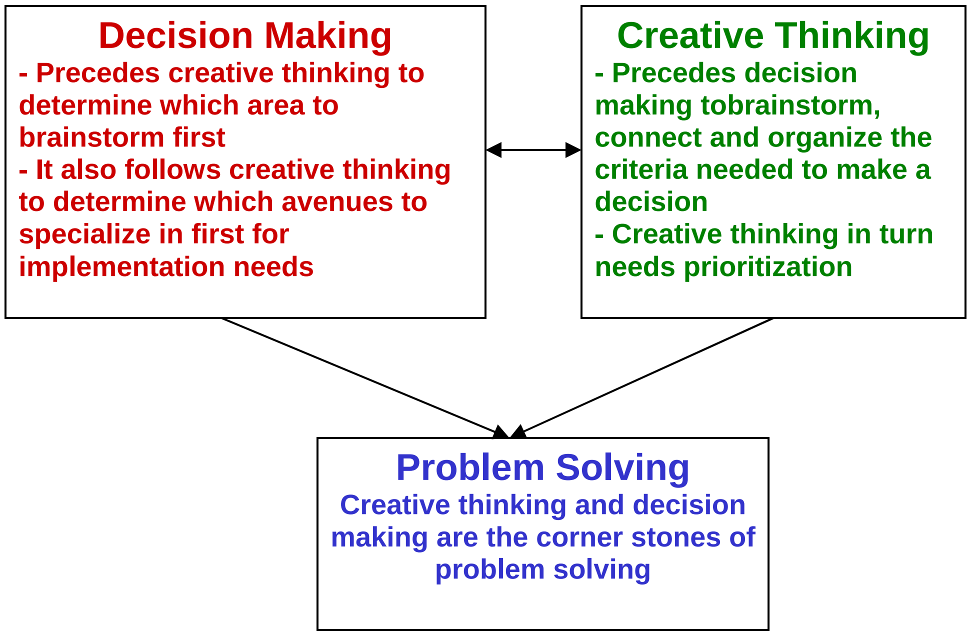


Figure 1 Relationship among creative thinking, decision making and problem solving

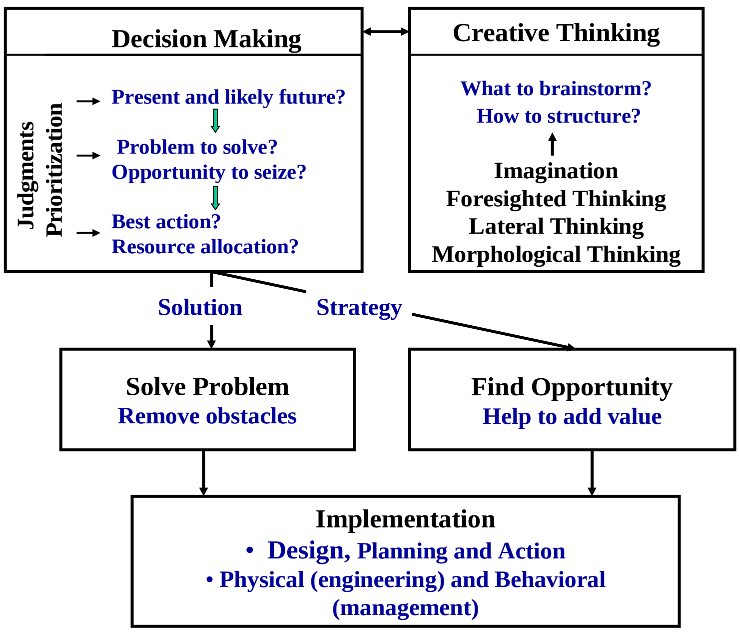


Figure 2 Relationship between creative thinking and decision making

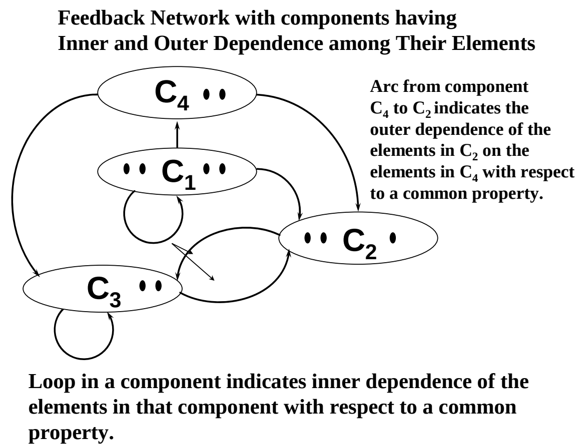
There are many creativity techniques that can be used to generate alternatives for an AHP/ANP model of a decision. The process of structuring the model and making the factors explicit can trigger thinking about what the alternatives should be. Thus with the AHP/ANP the very process of defining and structuring the problem is integrated with designing a solution. After the process is completed, reflection may lead the group back to refining the problem's definition. The AHP/ANP does not impose limits on how groups structure their thinking. A decision making method is essentially about eliciting tacit preferences from the decision makers. The AHP/ANP does not require physical measurements as inputs although such information can be used if it is available.

Not only is devising a set of alternatives essential, but also encouraging creativity makes a breakthrough decision more likely. Brainstorming enables a group to generate more alternatives. Brainstorming means that any judgment which may inhibit creativity must be deferred. Despite its wide use, the technique does have limitations and has been modified over the years. One of its modifications is brain writing or idea writing because the use of writing is considered to be better than presenting ideas orally as there is less danger of domination by certain participants. It also encourages people to participate who have trouble expressing their ideas orally. Participants have a chance to phrase their ideas clearly in writing beforehand or allow them to be recorded. The method will not work, however, if people are unwilling to express their ideas in writing. It works best with small groups, so big groups need to be broken into smaller groups in parallel sessions. After a proper introduction is given and a stimulating question is asked, group members write their initial response on a given form. They then react in writing to each other's forms. After each participant reads the comments, the small group discusses the principal ideas that emerge from the written interactions and summarizes the discussion in writing.

Other modifications of brainstorming include bug lists and negative brainstorming (generating complaints to identify weaknesses), the Crawford blue slip method (brainstorming in response to a number of questions that are related to a problem), and free discussion among group participants. Brainstorming has been used in complex problems to generate questions rather than solutions. The outcome is a list of questions that the group decides to pursue to move the process forward.

Synthesizing Priorities in Networks and in Hierarchies

The priority vectors derived from pairwise comparison matrices are each entered as a part of some column of a supermatrix. The supermatrix represents the influence priority of an element on the left of the matrix on an element at the top of the matrix. A supermatrix along with an example of one of its general entry (i,j) block is shown in Figure 3. The component C_i alongside the supermatrix includes all the priority vectors derived for nodes that are "parent" nodes in the C_i cluster. Figure 4 gives the structure of a hierarchy along with its supermatrix. The entry in the last row and column of the supermatrix of a hierarchy is the identity matrix I.

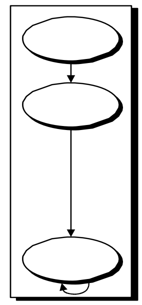


$$W = \begin{array}{cc} & \begin{array}{cccc} C_1 & C_2 & \cdots & C_N \\ e_{11}e_{12}\cdots e_{1n_1} & e_{21}e_{22}\cdots e_{2n_2} & & e_{N1}e_{N2}\cdots e_{Nn_N} \end{array} \\ \begin{array}{cc} C_1 & \begin{array}{c} e_{11} \\ e_{12} \\ \vdots \\ e_{1n_1} \end{array} \\ C_2 & \begin{array}{c} e_{21} \\ e_{22} \\ \vdots \\ e_{2n_2} \end{array} \\ \vdots & \\ C_N & \begin{array}{c} e_{N1} \\ e_{N2} \\ \vdots \\ e_{Nn_N} \end{array} \end{array} & \begin{bmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ W_{N1} & W_{N2} & \cdots & W_{NN} \end{bmatrix} \end{array}$$

$$W_{ij} = \begin{bmatrix} W_{i1}^{(j_1)} & W_{i1}^{(j_2)} & \cdots & W_{i1}^{(jn_j)} \\ W_{i2}^{(j_1)} & W_{i2}^{(j_2)} & \cdots & W_{i2}^{(jn_j)} \\ \vdots & \vdots & & \vdots \\ W_{in_i}^{(j_1)} & W_{in_i}^{(j_2)} & \cdots & W_{in_i}^{(jn_j)} \end{bmatrix}$$

Figure 3 Structure and supermatrix of a network and detail of a matrix in it

A holarchy, illustrated in Figure 5 is a hierarchy of two or more levels in which the goal is eliminated and what was the second level that used to depend on the goal, now depends on the bottom level of alternatives, thus as a whole the hierarchy is a cycle of successively dependent levels. We have encountered such a form in the analysis of the turn-around of the US economy in which the importance of the primary factors is determined in terms of the time periods at the bottom.



$$W = \begin{bmatrix} 0 & 0 & 0 & \dots & \bullet & 0 & 0 \\ W_{21} & 0 & 0 & \dots & \bullet & 0 & 0 \\ 0 & W_{32} & 0 & \dots & \bullet & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \bullet & \bullet & \bullet & \dots & W_{n-1,n-2} & \bullet & \bullet \\ 0 & 0 & 0 & \dots & \bullet & W_{n,n-1} & I \end{bmatrix}$$

Figure 4 Structure and supermatrix of a hierarchy

Figure 5 The U.S. holarchy of factors for forecasting turnaround in economic stagnation

In a network, the system of components may be regarded as elements that interact and influence each other with respect to a criterion or attribute with respect to which the influences occurs and control the thinking and judgments about them whether they are economic influences, political influence or social influences, for example. Each of these influences is an attribute in the decision that must be of a higher order of complexity than the components and a fortiori of higher order than the elements contained in the components. We call such an attribute a control criterion. Thus even in a network, there is a hierarchic structure that lists control criteria above the networks. For each of the four BOCR merits we have a system of control criteria that we use to assess influence. The result is that such control criteria and/or their subcriteria serve as the basis for all comparisons made under them, both for the components and for the elements in these components. In a hierarchy one does not compare levels according to influence because they are arranged linearly in a predetermined order from which all influence flows downwards. In a network, the effect of the influence of different clusters of elements can differ from cluster to another cluster and hence they need to be weighted to incorporate the proportionality of their contributions. The criteria for comparisons are either included in a level, or more often implicitly replaced by using the idea of "importance, preference or likelihood" with respect to the goal, without being more finely detailed about what kind of importance it is. The control criteria for comparisons in a network are intended to be explicit about the importance of influence that they represent.

Finally, we consider how to combine the four ranking outcomes of the alternatives for the BOCR. The top ranked alternative for each of the BOCR priorities is rated rather than compared with respect to strategic criteria. These are the overall criteria that serve the decision maker's goals in making any decision. The outcome of this rating is four priorities, one for each of the BOCR. These priorities are then used to weight the overall priorities of the alternatives for the corresponding BOCR and adding the result for the benefits and opportunities and subtracting the sum for the costs and the risks. The final priorities of some or all the alternatives can be negative.

3. Judgments are our unique conscious link to the physical world

An Intuitive Factual Observation

How we respond to stimuli is closely related to decision making. We note that in order to sense objects accurately in the environment, our brains miniaturize them within our system of neurons so that we have a proportional relationship between what we perceive and what is out there. Without proportionality we cannot coordinate our thinking with our actions with the accuracy needed to control the environment. Proportionality with respect to a single stimulus requires that our response to a proportionately amplified or attenuated stimulus should be proportional to what our response would be to the original value of that stimulus. If $w(s)$ is our response to a stimulus of magnitude s, then the foregoing observation leads us to consider the functional equation $w(as) = bw(s)$. This equation can also be obtained more rigorously as the necessary condition for solving Fredholm's equation of the second kind as we shall see next.

Formal Derivation from Judgments

We know from the Analytic Hierarchy Process (AHP) that because of inconsistency in judgments and their possible intransitivity, priorities need to be derived by solving for the principal eigenvalue $Aw = \lambda_{\max} w$. In the continuous case this is written as

$$\int_a^b K(s,t)\, w(t)\, dt = \lambda_{\max} w(s)$$

Where, instead of the positive reciprocal matrix A in the principal eigenvalue problem, we have a positive kernel, $K(s, t) > 0$, with $K(s, t)\, K(t, s) = 1$ that is also consistent; that is, $K(s, t)\, K(t, u) = K(s, u)$, for all $s, t,$ and u .

Our problem of extracting the principal eigenvector takes the familiar form of Fredholm's equation of the second kind

$$\int_a^b K(s,t) w(t) dt = \lambda_{\max} w(s)$$

Classically treated by using the form

$$\lambda \int_a^b K(s,t) w(t) dt = w(s)$$

with the normalization condition

$$\int_a^b w(s) ds = 1$$

It is easy to show that a consistent kernel has the form $K(s, t) = k(s)/k(t)$ from which what I call the "response" eigenfunction $w(s)$ solution can be shown to be

$$w(s) = \frac{k(s)}{\int_s k(s) ds}$$

Since the denominator is a constant, we can write $w(s) = ak(s)$.

Theorem 1 *$K(s,t)$ is consistent if and only if it is separable of the form:*

$$K(s,t) = k(s)/k(t)$$

Proof: (Necessity) $K(t, u_0) \neq 0$ for some $u_0 \varepsilon S$, otherwise $K(t, u_0) = 0$ for all u_0 would contradict $K(u_0, u_0) = 1$ for $t = u_0$. We obtain

$$K(s,t)\, K(t, u_0) = K(s, u_0)$$

$$K(s, t) = \frac{K(s, u_0)}{K(t, u_0)} = \frac{k(s)}{k(t)}$$

for all $u_0 \varepsilon S$ and the result follows.

(Sufficiency) If $K(s,t) = k(s)/k(t)$ holds, then it is clear that $K(s,t)$ is consistent.

We now prove that as in the discrete case of a consistent matrix, where the eigenvector is given by any normalized column of the matrix, an analogous result is obtained in the continuous case.

Theorem 2. *If $K(s,t)$ is consistent, the solution of* $\lambda \int_a^b K(s,t)w(t)dt = w(s)$ *is given by*

$$w(s) = \frac{k(s)}{\int_S k(s)ds}$$

Generalizing on the discrete approach in which the consistent matrix A has rank one, we assume that the kernel $K(s,t)$ is homogeneous of order 1. Thus, we have:

$$K(as,at) = aK(s,t) = k(as)/k(at) = a[k(s)/k(t)].$$

It follows that $w(as) = \alpha k(as) = \alpha a k(s) = \beta k(s) = bw(s)$.

To prove that $w(as) = bw(s)$ from $w(s) = \frac{k(s)}{\int k(s)ds}$ and $\frac{k(as)}{k(at)} = a\frac{k(s)}{k(t)}$, we first show that $\frac{w(as)}{w(at)} = \frac{w(s)}{w(t)}$. Integrating both terms of $\frac{k(as)}{k(at)} = a\frac{k(s)}{k(t)}$ first over *s*, we have $\frac{\int k(as)ds}{k(at)} = a\frac{\int k(s)ds}{k(t)}$. Next, we rearrange the terms and integrate over *t*, to obtain

$\int k(as)ds \int k(t)dt = a\int k(s)ds \int k(at)dt$ and this implies that $a\frac{\int k(s)ds \int k(at)dt}{\int k(as)ds \int k(t)dt} = 1$.

Thus,

$$\frac{w(as)}{w(at)} = \frac{k(as)/\int k(as)ds}{k(at)/\int k(at)dt} = a\frac{k(s)/\int k(s)ds}{k(t)/\int k(t)dt}\frac{\int k(at)dt}{\int k(as)ds}\frac{\int k(s)ds}{\int k(t)dt} = \frac{k(s)/\int k(s)ds}{k(t)/\int k(t)dt} = \frac{w(s)}{w(t)}$$

.

Assuming that the domain of integration is bounded or at least measurable, by integrating $w(as) = \frac{w(s)}{w(t)}w(at)$ over t we have $w(as)\int_\Omega dt = w(s)\int_\Omega \frac{w(at)}{w(t)}dt$ and letting $b = \frac{1}{\int_\Omega dt}\int_\Omega \frac{w(at)}{w(t)}dt$, we have the Fundamental Equation $\boldsymbol{w(as) = bw(s)}$.

The solution of this functional equation in the real domain was derived for the purposes of this author by Janos Aczel, a leading mathematician in the field of functional equations:

$$w(s) = Ce^{\log b \frac{\log s}{\log a}} P\left(\frac{\log s}{\log a}\right)$$

where P is a periodic function of period 1 and $P(0)$ = 1. One of the simplest such examples with $u = \log s / \log a$ is $P(u)$ = cos (u/2B) for which $P(0)$ = 1. It is a basic from that we use for $P(u)$.

4. Stimulus-Response —how the human mind works in response to stimuli

A consequence of the foregoing solution is the well-known logarithmic law of response to stimuli which can be obtained as a first-order approximation to this solution through series expansions of the exponential and the cosine functions as:

$$v(u) = C_1 e^{-\beta u} P(u) \approx C_2 \log s + C_3$$

where $\log ab \equiv -\beta, \beta > 0$. The expression on the right is known as the Weber-Fechner law of logarithmic response, $M = a \log s + b, a \neq 0$, to a stimulus of magnitude s. This law was empirically established and tested in 1860 by Gustav Theodor Fechner who used a law formulated by Ernest Heinrich Weber regarding discrimination between two nearby values of a stimulus. We have now shown that that Fechner's version can be derived by starting with our functional equation for stimulus response.

The integer-valued fundamental scale of response used in making paired-comparison judgments in the AHP can be derived from the logarithmic response function as follows. For a given value of the stimulus, the magnitude of response remains the same until the value of the stimulus is increased sufficiently large in proportion to the value of the stimulus, thus preserving the proportionality of relative increase in stimulus for it to be detectable for a new response. This suggests the idea, well known in psychology, of just noticeable differences (jnd). Thus, starting with a stimulus s_0, successive magnitudes of the new stimuli take the form

$$s_1 = s_0 + \Delta s_0 = s_0 + \frac{\Delta s_0}{s_0} s_0 = s_0(1+r)$$

$$s_2 = s_1 + \Delta s_1 = s_1(1+r) = s_0(1+r)^2 \equiv s_0\alpha^2$$

$$\vdots$$

$$s_n = s_{n-1}\alpha = s_0\alpha^n \; (n = 0,1,2,...)$$

We consider the responses to these stimuli to be measured on a ratio scale (b = 0). A typical response has the form $M_i = a \log \alpha^i, i = 1,..., n,$ or one after another they have the form $M_1 = a \log \alpha; M_2 = 2a \log \alpha,..., M_n = na \log \alpha.$ We take the ratios $M_i / M_1, i = 1,..., n,$ of these responses in which the first is the smallest and serves as the unit of comparison, thus we obtain the integer values 1, 2,…,n of the Fundamental Scale of the AHP. It appears that numbers are intrinsic to our ability to make comparisons, and were not just an invention by our primitive ancestors.

The upshot of this approach is to observe that our responses fall into categories involving just noticeable differences from one category into another. Within each category we are unable to tell the difference between a certain value and a slightly larger value. At the very beginning we can compare an object with itself and obtain the value 1 for its dominance over itself with respect to a property. We then compare it with an object that is a little larger. Because of the jnd syndrome, we would decide whether the slightly larger object is equal to it or falls in the next category, which is twice its size, and so on, thus obtaining the numbers 1, 2, 3, and so on. It would seem that if fuzziness has any real justification it lies in this psychophysical phenomenon and hence the AHP does not need further theoretical "fuzzifying". But as we said before we cannot go on with very large numbers because we are unable to compare the object with something that is too large. If we do, we will make such an error that our estimate in the comparison will be very inconsistent and therefore inaccurate and our result will be unreliable. Essentially it amounts to assigning values from the positive integers, by dividing our ability to sense things into high, medium, and low and then dividing each one into three categories so we would get for the largest value (high, high), followed by (high, medium), (high, low), (medium, high), (medium, medium), (medium, low), (low, high), (low, medium), (low, low). The numerical values we assigned to them would range from 9 for the (high, high) pair and so on down to 1 for the (low, low) pair. When we compare the smaller object, with the larger object we use the reciprocal value. If the large apple is three times bigger than the small orange, then the orange is automatically one-third as large as the apple. One can prove mathematically that small changes in the numbers lead to small changes in the final answers that we call priorities.

5. The inverse square law of physics in optics, gravity and electricity

The solution of Fredholm's equation derived above is defined in the frequency domain or transform domain in Fourier analysis as it is based on the flow of electric charge in the brain. We must now take its transform to derive the solution in the spatial or time domain. Thus our solution of Fredholm's equation here is given as the Fourier transform,

$$f(\omega) = \int_{-\infty}^{+\infty} F(x)\, e^{-2\pi i\omega x}\, dx = Ce^{\beta\omega}\; P(\omega)$$

whose inverse transform is the inverse Fourier transform of a convolution of the two factors in the product. We have:

$$F(x) = \int_{-\infty}^{+\infty} f\;(\omega)\; e^{2\pi i\omega x}\, d\omega$$

Since our solution is a product of two factors, the inverse transform can be obtained as the convolution of two functions, the inverse Fourier transform of each of which corresponds to just one of the factors.

Now the inverse Fourier transform of $e^{-\beta u}$ is given by

$$\frac{\sqrt{(2/\pi)}\beta}{\beta^2+\xi^2}$$

Also because a periodic function has a Fourier series expansion we have

$$P(u) = \sum_{k=-\infty}^{\infty} \alpha_k \, e^{2\pi iku}$$

This has the inverse Fourier transform:

$$\sum_{k=-\infty}^{\infty} \alpha_k \delta(\xi - 2\pi k)$$

The product of the transforms of the two functions is equal to the Fourier transform of the convolution of the two functions themselves which we just obtained by taking their individual inverse transforms. We have, to within a multiplicative constant:

$$\int_{-\infty}^{+\infty} \sum_{k=-\infty}^{\infty} \alpha_k \, \delta(\xi - 2\pi k) \frac{\beta}{\beta^2 + (x-\xi)^2} \, d\xi = \sum_{k=-\infty}^{\infty} \alpha_k \frac{\beta}{\beta^2 + (x-2k\pi)^2}$$

Let us consider the simple case where the periodic part of our solution is given by

$$P(u)= \cos u/2\pi=(1/2)(e^{iu/2\pi} + e^{-iu/2\pi}).$$

The inverse Fourier transform of $w(u)= C\,e^{-\beta u} \cos u/2\pi$, $\beta > 0$ is:

$$C\frac{\beta}{\sqrt{2\pi}}\left[\frac{1}{\beta^2+\left(\frac{1}{2\pi}+\xi\right)^2} + \frac{1}{\beta^2+\left(\frac{1}{2\pi}-\xi\right)^2}\right]$$

When the constants in the denominator are small relative to ξ we have C_1/ξ^2 which we believe is why experiments show that in optics, gravitation, electric charge, and sound intensity, forces act according to inverse square laws. This is the same law of nature in which an object responding to a force field must decide to follow that law by comparing infinitesimal successive states through which it passes. If the stimulus is constant, the exponential factor in the general response solution is constant, and the solution in this particular case would be periodic of period one. When the distance ξ is very small, the result varies inversely with the parameter $\beta>0$.

Now a word about problems concerning the exactness of the inverse square law observed in studies using gravitation.

Precession is a change in the direction of the axis of a rotating object. Precession happens to orbits of many planets but its effect is most noticeable for planets and other objects near the Sun with highly elliptical orbits. Seen from the Earth, Mercury's orbit appears to have a precession of 5600 seconds of arc per century (one second of arc=1/3600 degrees). Taking into account the effects of the other planets and the sun, Newton's equations predict a precession of 5557 seconds of arc per century, a discrepancy of 43 seconds of arc per century. This discrepancy was thought to be resolved in 1915, when the theory of general relativity predicted an additional perihelion precession of exactly 43 arc seconds per year. However, James Constant, in his 2006 article on the Internet, *Precession of perihelia; Le Verrier's and Einstein's predictions compared*, refers to the works of Louis Brillouin (Relativity Reexamined, Academic Press 1970 page 99) and Steven Weinberg (Gravitation and Cosmology, John Wiley & Sons, Inc. 1972 page 198) to conclude that there are "doubts and caveats that relativity can predict correct values for Mercury and other planets that meet requirements of elliptic orbits set by Newtonian theory." Elsewhere he demonstrates in a book that Einstein's theory of gravitation, expressed as Riemannian geometry, can not be reconciled with Newton's theory of gravitation. So the story is by no means settled. Some researchers, notably the Harvard astronomers Asaph Hall and Simon Newcomb in the 1800s, thought that perhaps Newtonian theory was at fault, and gravity isn't exactly an inverse square law. Hall thought that he could account for the precession of Mercury if instead of using the power 2 in $1/r^2$ one were to use 2.00000016. Some people find this proposal in conflict with basic conservation laws.

The May 19, 2007, issue of the *Economist* magazine reported that "Either Newton or Einstein was wrong, or there is something missing from the universe. The reason for this is that galaxies do not behave as the laws of gravity predict they should." In this article it is reported that dark matter and relativity are used to account for what happens after enormous collisions take place, but the fact that physicists sometimes question the form of the law of gravity makes one wonder about its universal accuracy as a strictly inverse square law.

6. The brain as a system of firing neurons

Solution to the Fundamental Equation in the complex domain has the form:

$$w(z) = z^{\ln b/\ln a} P(\ln z / \ln a)$$

Here $P(u)$ with $u = \ln z/\ln a$, is an arbitrary multivalued periodic function in u of period 1. Even without P being multivalued, the function $w(z)$ could be multivalued because $\ln b/\ln a$ is generally a complex number. If P is single-valued and $\ln b/\ln a$ turns out to be an integer or a rational number, then $w(z)$ is a single-valued or finitely multivalued function, respectively. This generally multivalued solution is obtained in a way analogous to the real case.

We now show that the space-time Fourier transform of the complex valued solution is a combination of Dirac distributions. Our solution of Fredholm's equation here is given as the Fourier transform,

$f(\omega) = \int_{-\infty}^{+\infty} F(x)\, e^{-2\pi i\omega x}\, dx = Ce^{\beta\omega}\, P(\omega)$ with its inverse transform given by:

$$(1/2\pi)\log a \sum_{-\infty}^{\infty} a'_n \left[\frac{(2\pi n + \theta(b) - x)}{\log a\,|\,b\,| + (2\pi n + \theta(b) - x)}\ i \right] \delta(2\pi n + \theta(b) - x)$$

where ∗(2Bn+2(b)-x) is the Dirac delta function. Thus response is expressed as a combination of impulsive functions.

The Fourier transform of the analytic solution $w(z) = \sum a_j^k z^{jk+q}, q < k$ is given by:

$$2\pi \sum_{j=1}^{\infty} (-1)^{jk+q} a_j \frac{d^{jk+q}\delta(x)}{dx^{jk+q}}$$

again involving impulsive functions. We note that the number of nonzero terms in this sum is finite.

7. Synthesis of stimuli from different senses

We need to synthesize different responses that have the form of the eigenfunction solution:

$$w_k(z_k) = (b_k)^{[\log|z_k|/[\log|a_k|]} P_k([\log|z_k| / \log|a_k|]), k = 1,...,n$$

where k refers to different neural response dimensions such as sound, "feeling" (which is a mixture of sensations), and so on.

Their product is a function of several complex variables and is the solution of the equation

$$\prod_{k=1}^{n} w_k(a_k z_k) = \prod_{k=1}^{n} b_k w_k(z_k)$$

The product of solutions of $w_k(a_k z_k) = b_k w_k(z_k)$ satisfies such an equation with the new $b = b_k$. Since the product of periodic functions of period 1 is also a periodic function of period one, the result of taking the product has the same form as the original function: a damping factor multiplied by a periodic function of period 1.

If we multiply n solutions in the same variable *z*, in each of which *b* and *W* are allowed to be different we obtain:

$$(b_1...b_n)^{[\log|z|/\log|a|]} W_1(z / a^{[\log|z|/\log|a|]})...W_n(z / a^{[\log|z|/\log|a|]}) =$$
$$b^{(z / a^{[\log|z|/\log|a|]})} V(z / a^{[\log|z|/\log|a|]})$$

$\cos u / 2\pi$ is the generic form we could adopt for the periodic component of period one. We also have the functional equation $w(a_1 z_1, \ldots, a_n z_n) = b\, w(z_1, \ldots, z_n)$

whose solution for the real variable case with $b > 0$, $a_k > 0$, and $z_k > 0$, (k = 1, . . . , n), is given by

$$w(z_1,...,z_n) = b^{\sum_{k=1}^{n} \log z_k / \log a_k)/n} P(\frac{\log z_1}{\log a_1},...,\frac{\log z_n}{\log a_n})$$

where P is an arbitrary periodic function of period one of n variables, that is,

$$P(t_1+1,...,t_n+1) = P(t_1,...,t_n).$$

For a continuum number of stimuli let $K(\mathbf{X},\mathbf{Y})$ be a compact reciprocal kernel i.e. $K(x,y)K(y,x)=1,$ for all $x\varepsilon\mathbf{X}$ and $y\varepsilon\mathbf{Y}$, where **X** and **Y** are compact subsets of the reals. We have the equation $w(\mathbf{X}) = \lambda\int_{\Omega} K(\mathbf{X};\mathbf{Y})w(\mathbf{Y})$. Formally, we write the general solution in the form

$$w(\mathbf{X}) = [\exp\int_{\Omega}(\log x/\log a)dx]P(\frac{\log \mathbf{X}}{\log a})$$

which involves a product integral.

We now give a fascinating theorem that illuminates the difference between mathematical functions of several variables and the firing of neurons each described by a function of a single variable.

8. Approximations and density

Our thoughts and representations are linear combinations of the response functions derived above and therefore, everything can be represented as accurately as desired by linear combinations of these functions or their transforms both the real and complex valued.

Approximation is about how general functions can be decomposed into more simple building blocks: polynomials, splines, wavelets, and the like. One needs to guarantee specified rates of convergence when the smoothness of the kind of functions being approximated is specified, such as in Sobolev or Lipschitz spaces (Sobolev space is the space of distributions in the sense of Schwartz, in $L_p(W)$ whose derivatives of order k also belong to the space $L_p(W)$, where W is an open subset of R_n. A function $f(x)$ belongs to a Lipschitz class Lip[α] if there exists a constant C > 0 such that $\left|f(x)-f(y)\right| \le C\left|x-y\right|^{\alpha}$.

The subset A in a metric space X is dense in X if every x in X is a limit of a sequence of elements in A. The celebrated theorem of Weierstrass proved in 1885 asserts that a set of algebraic polynomials is dense in the space of continuous functions on a compact set in R_d. A continuous function in a finite close interval can be approximated with any desired accuracy by polynomials and often also by trigonometric polynomials, and some mathematicians have recently considered approximation by nonlinear homogeneous polynomials on star-like origin-symmetric surfaces (of which convex surfaces are a special case by a pair of homogeneous polynomials as a conjecture). Their work is closely related to the findings of the general solution for the operator formulation of the Fundamental Equation examined in the next section.

- In 1937 Marshall Stone generalized the idea of approximations and density in what is known as the Stone–Weierstrass theorem: If K is a compact Hausdorff space (A set *A* of functions defined on *K* is said to separate points – is a Hausdorff space – if, for every two different points *x* and *y* in *K* there is a function *f* in *A* with *f*(*x*) not equal to *f*(*y*), and *A* is a sub algebra of *C*(*K,R*) which contains a non-zero constant function, then *A* is dense in *C*(*K*,*R*) if and only if it separates points. The original statement of Weierstrass is a special case because polynomials on [*a,b*] form a sub algebra of *C*[*a,b*] which separates points.

Runge's approximation theorem in complex analysis says that: If *K* is a compact subset of *C* (the set of complex numb)rs), *A* is a set containing at least one complex number from every bounded connected component of C\K, and f is a holomorphic function on *K*, then there exists a sequence (r_n) of rational functions with poles in *A* such that the sequence (r_n) approaches the function *f* uniformly on *K*.

The world is made up of many stimuli of different kinds. Our brain responds to each kind that it detects with different neurons each specialized for only one aspect of a stimulus. These responses are then synthesized into a complete response. Both real and complex valued solutions are dense in very general spaces because we respond to everything we sense or think about with such functions. We expect those spaces to be the most general conceivable.

The fact that we think mathematically of the world in terms of functions of several variables is only an abstraction of what actually happens with our brain constructions. In 1957, the Russian mathematician A. N. Kolmogorov in responding to Hilbert's 13th problem (that not only one cannot express the solution of higher order algebraic equations in terms of basic algebraic operations, but no matter what functions of one or two variables we add to these operations, we still would not be able to express the general solution) actually showed Hilbert to have been wrong and proved that an arbitrary continuous function $f(x_1,...,x_n)$ on an n-dimensional cube (of arbitrary dimension n) can be represented as a superposition and composition of continuous functions of only one variable. Here is the remarkable theorem of Kolmogorov:
Theorem (Kolmogorov, 1957): For every integer dimension $d \geq 2$, there exist continuous real functions $h_{ij}(x)$ defined on the unit interval $U[0,1]$, such that for every continuous

real function $f(x_1,...,x_d)$ defined on the *d*-dimensional unit hypercube U^d , there exist real continuous functions $g_i(x)$ such that $f(x_1,...,x_d) = \sum_{i=1}^{2d+1} g_i(\sum_{j=1}^{d} h_{ij}(x_j))$.

Because of their relation to the brain, we conjecture that the resulting one dimensional functions always have the same form as the solution to the Fundamental Equation involved in the firing of neurons.

Now we come to the most important concern we have that also derives from a generalization of judgments and it is about the mathematical form that response to influences in nature takes. For that purpose we use a general operator form of our Fundamental Equation.

9. The operator equation for stimulus-response

Julian Huxley wrote, “Something like the human mind might exist in lifeless matter.” Huxley suggested that all natural occurrences involve mental activity, although the mental happenings are at such a low level of intensity that they cannot be detected. In higher animals mental activity is reinforced through an organized system like the brain to reach a high level of intensity; therefore we become aware of it. All nature has a degree of awareness and solves problems. Let us now see what we can learn about the characteristics of things, all things in general respond to stimuli. Our assertion here is that all quantitative formulas used in science to study nature, physical or biological, including the brain itself, take the form of one of the few solutions of our operator equation.

We map the normed linear space *E* to another normed linear space G over *K, K* is either *R* (real) or *C* (complex), and α and β are given scalars in K with

$$\boldsymbol{W(\alpha x) = \beta W(x)}$$

Replace x by x/α to get

$$\boldsymbol{W(\frac{x}{\alpha}) = \frac{1}{\beta}W(x)}$$

or

$$W(x) = \beta W(\frac{x}{\alpha}) = \beta^2 W(\frac{x}{\alpha^2}) = ... = \beta^p W(\frac{x}{\alpha^p}) = ... = \beta^{2p} W(\frac{x}{\alpha^{2p}})$$

$$= ... = \beta^{(n-1)p} W(\frac{x}{\alpha^{(n-1)p}})$$

The equation has solutions derived by Nicole Brillouet-Belluot of Nantes, France, in response to an inquiry we made in 1998 about this operator equation to Janos Aczel who lectured on the solution of the simpler equation given above at a meeting on functional equations. The solutions are shown in Table 1 below. It is clear that a non-zero solution exists depending on whether α and β are or are not roots of one.

Table 1
Different solutions of $w(ax)=bw(x)$.

	$W: E \to G$
	General Solution
α root of 1 of order n $\beta^n \neq 1$	1) $W \equiv 0$
$\beta^n = 1$	2) $W(x) \begin{cases} \beta^p W_0(\alpha^{-p}x), & \text{if } x \in \alpha^p B \text{ and } p \in \{0, .., n-1\}, \\ b, & \text{if } x = 0. \end{cases}$ where b = $\begin{cases} 0, & \text{if } \beta \neq 1, \\ \text{is an arbitrary element of } G, & \text{if } \beta = 1, \end{cases}$ B subset of E
α not root of 1	Same as 2)
	Continuous Solutions
α root of 1 of order n $\beta^n = 1$	Same as 2) with limit conditions
α not root of 1, | α|=1	$W \equiv 0$

$\lvert\beta\rvert \neq 1$	
$\lvert\beta\rvert = 1$	β root of 1 and not equal to one $W \equiv 0$
	β root of 1 and equal to one W (x) = W_0 (π (x)) where W_0 is an arbitrary continuous mapping and $\pi(x)$ is the continuous natural mapping
	β not root of 1 and $\beta = \alpha^p$ for all p If E = C, $W(x) = \begin{cases} \left(\frac{x}{\lvert x\rvert}\right)^p W_0(\lvert x\rvert), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}$ where $W_0(0,+\infty) \to G$
	and if E is a separable inner product space, by: $W(x) = \left(\frac{x_{i(x)}}{\lvert x_{i(x)}\rvert}\right)^p \; g_{i(x)} \left(\lvert x_{i(x)}\rvert, \frac{\lvert x_{i(x)}\rvert}{x_{i(x)}}\left(x - x_{i(x)} e_{i(x)}\right) \quad (x \neq 0)\right)$ and $W(0) = 0$, where the functions g_n are continuous and subject to some conditions.
$\lvert\alpha\rvert > 1$ $\lvert\beta\rvert \leq 1$ $\beta \neq 1$	$W \equiv 0$
$\beta = 1$	Constant functions
$\lvert\beta\rvert > 1$	Same as in 2) above with b=0 and p = $\left[\frac{\ln\lVert x\rVert}{\ln\lvert\alpha\rvert}\right]$ and W_0 arbitrary continuous function satisfying a limit condition.
	Differentiable Solutions
α root of 1 of order n $\beta^n = 1$	In this case, there exists a unique p in {*0, 1, . . . , n − 1*} such that $\beta = \alpha^p$ $W(x_1,...,x_q) = \sum_{(n_1,...,n_q) \in J} a_{n_1...n_q} x_1^{n_1}...x_q^{n_q}$ where J ={(n_1, .., n_q) $\in N^q$: n_1 + .. +

	$n_q = p + j\,n$ *for some* $j \in N\}$ and $a_{n_1 \ldots n_q}$ are arbitrary elements of G
α not root of 1, $\|\alpha\|=1$ $\|\beta\| \neq 1$	$W \equiv 0$
$\|\beta\| = 1$	If β is root of 1 and $\beta \neq 1$ $W \equiv 0$
	If β is root of 1 and $\beta = 1$ the solution is the constant functions
	If β is not a root of 1 and $\beta \neq \alpha^{\mathrm{p}}$ for all nonnegative integers p $W \equiv 0$
	If β is not a root of 1 and $\beta = \alpha^{\mathrm{p}}$ for all nonnegative integers p , the homogeneous polynomials of degree p
$\|\alpha\|>1$ $\|\beta\| \leq 1$ $\beta \neq 1$	$W \equiv 0$
$\beta = 1$	Constant functions
$\|\beta\| > 1$	$\beta \neq \alpha^k$ for all positive integers k $W \equiv 0$
	$\beta = \alpha^k$ for some k W(x) = L(x, x, . . . , x) (x $\in$ C^q), where L is an arbitrary k-linear symmetric continuous mapping from $(C^q)^k$ into G.
	Analytic Solution
	$W(z) = \sum_{j=-\infty}^{\infty} c_j z^{jk+q}$ $\alpha^k = 1$, $a^q = \beta$

A mapping is a correspondence. It is said to be a **natural mapping** when it transforms a structure through arithmetic operations; that is, it applies an arithmetic operation to make the correspondence such as in multiplying the positive integers by two to get the even numbers and thus obtaining a 1-1 correspondence rather than by removing the odd numbers which would not be an arithmetic operation.

A multilinear form is a map $f : V^n \to K$ where V is a vector space over the field K that is linear in each of its n variables. The term multilinear map is used when the map is linear in all its arguments. For n = 2, f is called a bilinear form. A k-linear from is a multilinear form with k arguments. An important type of multilinear forms is alternating multilinear forms which change their sign under exchange of two arguments. They are called symmetric multilinear (k-linear) forms when unchanged on interchanging two arguments.

Homogeneous polynomials of degree p

$$\phi\left(x_1,\ldots,x_q\right)=\sum_{\left(n_1,\ldots,n_q\right)\in J} a_{n_1\ldots n_q}\, x_1^{n_1}\ldots x_q^{n_q}$$

where $J = \{ (n_1, .., n_q) \in N^q : n_1 + .. + n_q = p + j\,n \;\; \textit{for some } j \in N\}$ and a $a_{n_1\ldots n_q}$ are arbitrary elements of G.

In his book *A Mathematician Grappling with His Century*, Springer, 2001, Laurent Schwartz writes on page 134, "I tried to define Fourier transforms of operators, and that is where I encountered a total obstacle. The very special role of convolution in the definition of operators leaves absolutely no room for a Fourier transform." But there are works on Fourier transforms of operators. We are not sure whether the transform can be defined and used for practical purposes.

10. Geometric representations

Our purpose has been to show that the expression we obtain for neural firing can be validated in practice by showing that linear combinations are dense in the most general spaces that can be used to represent nearly everything we do with electric firings of neurons. But how can we do that?

While a function of a real variable depends on a single argument, a function of a complex variable depends on two independent variables one real and one imaginary and thus cannot be drawn in the plane. Further, complex valued functions cannot be drawn as one does ordinary functions in three dimensions because they depend on an imaginary variable. Nevertheless, one can make a plot of the modulus or absolute value of such a function. The basic assumption we make to represent the response to a sequence of individual stimuli is that all the layers in a network of neurons are identical, and each stimulus value is represented by the firing of a neuron in each layer to synthesize in stages specialized information from each receptor neuron into a more complex package that characterizes more closely what is perceived. This representation it is not invariant with respect to the order in which the stimuli are fed into the network. It is known in the case of vision that the eyes do not scan pictures symmetrically if they are not symmetric, and hence our representation must satisfy some order invariant principle. Taking into account this principle would allow us to represent images independently of the form in which stimuli are input into the network. For example, we recognize an image even if it is subjected to a rotation, or to some sort of deformation. Thus, the invariance principle must include affine and similarity transformations. This invariance would allow the network to recognize images even when they are not identical to the ones from which it recorded a given concept, e.g., a bird. The next step would be to use the network representation given here with additional conditions to uniquely represent patterns from images, sounds and perhaps other sources of stimuli such as smell. Our representation focuses on the real part of the magnitude rather than the phase of the Fourier transform. Tests have been made to see the effect of phase and of magnitude on the outcome of a representation of a complex valued function. There is much more blurring due to change in magnitude than there is to change in phase. Thus we focus on representing responses in terms of Dirac functions, sums of such functions, and on approximations to them without regard to the coefficients in the linear combination.

The rest of this section on geometric representations in the plane to give an idea how density works was developed in collaboration with my colleague Luis Vargas. The most significant observation about the brain, which consists of many individual neurons, is that it is primarily a synthesizer of the firings of individual neurons into clusters of information and these in turn into larger clusters and so on, leading to an integrated whole. Due to their sequential nature, the firings of a neuron that precede other neurons would be lost unless there is something like a field in which all the firings fit together to form a cohesive entity which carries information. Is there a field in the brain? No. We believe that the process of analytic continuation in the theory of functions of a complex variable provides insight into how neurons seem to know one another. On page 373 in their book From Neuron to Brain,, Kuffler and Nicholls say, "The nervous system appears constructed as if each neuron had built into it an awareness of its proper place in the system." That is what analytic continuation does. It conditions neurons to fall on a unique path to continue information that connects with information processed by adjacent neurons with which it is connected. The uniqueness of analytic continuation has the striking consequence that something happening on a very small piece of a connected open set completely determines what is happening in the entire set, at great distances from the small piece.

By raising the hypermatrix to powers one obtains transitive interactions. This means that a neuron influences another neuron to fire or not through intermediate neurons. All such two step interactions are obtained by squaring the matrix. Three step interactions are obtained by cubing the matrix and so on. By raising the matrix to sufficiently large powers, the influence of each neuron on all the neurons with which one can trace a connection, yields the transient influence of neurons in the original hypermatrix. Multiplying the hypermatrix by itself allows for combining the functions that represent the influence from pre-to post- synaptic neurons to accumulate all the transitive influences from one neuron to another and allow for feedback. The Fourier transform that takes place as a result of firing and the density of the resulting firings give us the desired synthesis. Depending on what parts of the brain are operational and participating in the synthesis, different physical and behavioral attributes are observed to take place, including consciousness related to the Fourier transform of the single valued sensory functions.

The functions

$$\{t^{\alpha} e^{-\beta t},\ \alpha,\ \beta \geq 0\}$$

result from modeling the neural firing as a pairwise comparison process in time and are used to approximate pulses. It is assumed that a neuron compares neurotransmitter-generated charges in increments of time.

We created a 2-dimensional network of neurons consisting of layers. For illustrative purposes, we assume that there is one layer of neurons corresponding to each of the stimulus values. Thus, if the list of stimuli consists of n numerical values, we created n layers with a specific number of neurons in each layer. Under the assumption that each numerical stimulus is represented by the firing of one and only one neuron, each layer of the network must also consist of n neurons with thresholds varying between the largest and the smallest values of the list of stimuli. We also assumed that the firing threshold of each neuron had the same width. Thus, if the perceptual range of a stimulus varies between two values θ_1 and θ_2, and each layer of the network has n neurons, then a neuron in the *i*th position of the layer will fire if the stimulus value falls between

$$\theta_1 + (i-1)\frac{\theta_2 - \theta_1}{n-1} \quad \text{and} \quad \theta_1 + i\frac{\theta_2 - \theta_1}{n-1}$$

Sound experiment

In the sound experiment we first recorded with the aid of the Mathematica software the first few seconds of Haydn's symphony No.102 in B-flat major and Mozart's symphony No. 40 in G minor. The result is a set of numerical amplitudes between -1 and 1. Each of these amplitudes was used to make neurons fire when the amplitude falls within a prescribed threshold range. Under the assumption that each neuron fires in response to one stimulus, we would need the same number of neurons as the sample size, i.e., 117,247 in Haydn's symphony and 144,532 in Mozart's symphony. Our objective was to approximate the amplitude using one neuron for each amplitude value, and then use the resulting values in Mathematica to play back the music. A small sample of the numerical data for Mozart’s symphony is displayed in Figure 6.

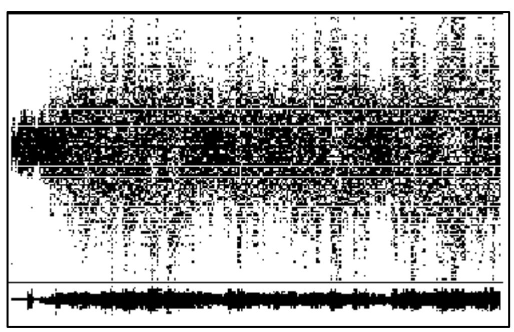


Figure 6 Mozart Symphony No. 40

This task is computationally demanding even for such simple geometric figures as the bird and the flower shown in Figures 7 and 8. For example, for the bird picture, the stimuli list consists of 124 values, and we would need 1242=15376 neurons, arranged in 124 layers of 124 neurons each The network and the data sampled to form the picture given in Figure 6, were used to create a 124 by 124 network of neurons consisting of 124 layers with 124 neurons in each layer. Each dot in the figure is generated by the firing of a neuron in response to a stimulus falling within the neuron's lower and upper thresholds. The sound experiment required 1000 times more data points than the picture experiment. Once the (x, y) coordinates of the points were obtained, the x-coordinate was used to represent time and the y-coordinate to represent response to a stimulus

Picture experiment

In the graphics experiment the bird and rose pictures required 124 and 248 data points, respectively. The numerical values associated with the drawings in Figures 7 and 8 were tabulated and the numbers provided the input to the neurons in the networks built to represent the bird and the rose.

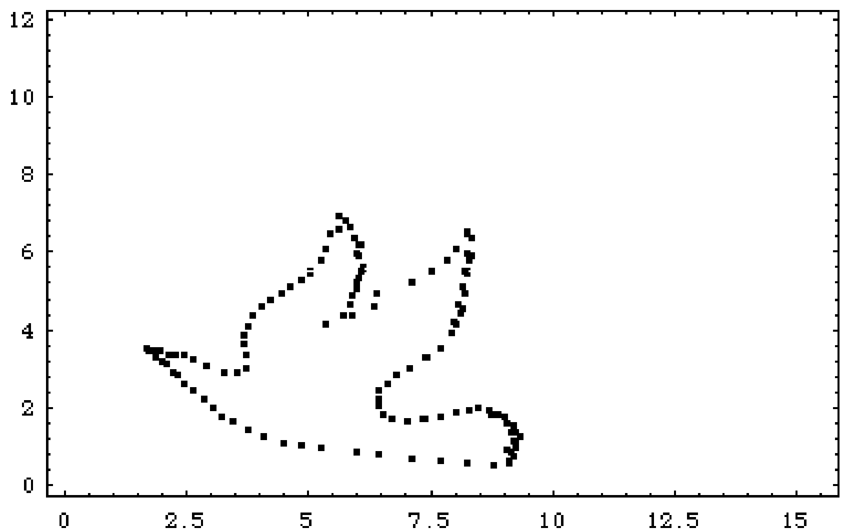


Figure 7 Bird

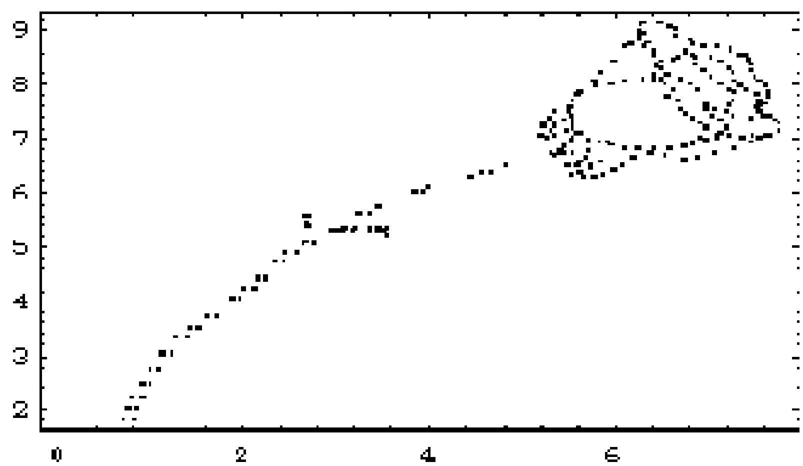


Figure 8 Rose

11. Conclusions

There is still much work to be done on the detailed interpretation of the solutions of the operator equation as they correspond to responsive forms in the real world.

I am grateful to three people for their help in the development of some of the ideas in this work. I mention them in the order and extent of help: my friend Luis Vargas for our close friendship and for early work on Fredholm's equation (first derived in a paper I coauthored with my student Hassan Ait-Kaci) and for the numerical representation of sounds and images; my friend Janos Aczel for his help with the solution of the functional equation I derived as necessary condition for the existence of a solution to Fredholm's equation and to Nicole Brillouet-Belluot for solving the generalization of that equation to operator form.

REFERENCES

Aczél, J. D. and M. Kuczma. Generalizations of a Folk-Theorem. *Vol. 19, Results in Mathematics 5-21*. Basel: Birkhäuser Verlag (1991).

Brillouet-Belluot, N. "On a Simple Linear Functional Equation on Normed Linear Spaces." Ecole Central de Nantes, (1999).

Huxley, J.: *Man in the Modern World*, New York: Mentor Books, 1953.

Kolmogorov, A.N. (1957). "On the Representation of Continuous Functions of Many Variables by Superposition of Real Functions of One Variable and Addition. *American Marthematical Society Translations Issue Series* 2, 28:55-59 (1963).

Saaty, T.L.: *The Brain, Unraveling the Mystery of How it Works, the Neural Network Process*, Pittsburgh, Pennsylvania: RWS Publications, 4922 Ellsworth Avenue, Pittsburgh, PA 15213 (2000).

Saaty, T.L. and L.G. Vargas: A Model of Neural Impulse Firing and Synthesis. *J. of Math. Psych. 200-219*, 2 (1993).

Saaty, T.L., and L.G. Vargas: Representation of Visual Response to Neural Firing. *Math. and Comp. Mod. 17-23*, 18/7 (1993).