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RICCI SOLITONS IN (ε, δ)-TRANS-SASAKIAN MANIFOLDS

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ABSTRACT. We study Ricci solitons in (ε, δ)-trans-Sasakian manifolds. It is shown that a symmetric parallel second order covariant tensor in a (ε, δ)-trans-Sasakian manifold is a constant multiple of the metric tensor. Using this it is shown that if $L_V g + 2S$ is parallel where V is a given vector field, then (g, V) is Ricci soliton. Further, by virtue of this result, Ricci solitons for 3-dimensional (ε, δ)-trans-Sasakian manifolds are obtained. Also an example of Ricci solitons in 3-dimensional (ε, δ)-trans-Sasakian manifold is provided in the region where trans-Sasakian manifold is expanding (shrinking) the Lorentzian trans-Sasakian manifold is shrinking (expanding).

1. INTRODUCTION

In [12], Gray-Harvella classification of almost Hermitian manifolds appears as a class W_4 of Hermitian manifolds which are closely related to locally conformal Kähler manifolds. A class of almost contact metric manifolds known as trans-Sasakian manifolds was introduced by J.A. Oubina [24] in 1985. An almost contact metric structure on a manifold M is called a trans-Sasakian structure [24] of the product manifold $M \times R$ belongs to the class W_4. The class $C_5 \oplus C_6$ [24] coincides with the class of trans-Sasakian structure of (α, β). This class consists of both Sasakian and Kenmotsu structures. If $\alpha = 1$, $\beta = 0$ then the class reduces to Sasakian where as if $\alpha = 0$, $\beta = 1$ then it reduces to Kenmotsu. The above manifolds are studied by many authors like D.E. Blair and J.A. Oubina [6], J.C. Marrero [20], K. Kenmotsu [17], C.S. Bagewadi and Venkatesha [2], U.C. De and M.M. Tripathi [9], A.A. Shaikh et. al. [25] etc.

The study of manifolds with indefinite metrics is of interest from the standpoint of physics and relatively. Manifolds with indefinite metrics have been studied by several authors. The concept of (ε)-Sasakain manifolds was initiated by Bejancu and Duggal [4] and further investigation was taken up by X. Xufeng and C. Xiaoli [31] and Rakesh Kumar et. al. [18]. The historical background of (ε)-Sasakain manifolds can be traced back to the classification of Sasakian manifolds with indefinite metrics which play crucial role in physics [10]. U.C. De and A. Sarkar [8] studied the notion of (ε)-Kenmotsu manifolds with indefinite metric. S.S. Shukla and D.D. Sing [28] extended the study to (ε)-trans-Sasakian manifolds with indefinite metric which are natural generalization of both (ε)-Sasakian and (ε)-Kenmotsu manifolds. The authors H.G. Nagaraja et. al. [22] studied (ε, δ)-trans-Sasakian manifolds which are extensions of (ε)-trans-Sasakian manifolds.

Ricci solitons move under the Ricci flow introduced by Hamilton [14] simply by diffeomorphisms of the initial metric that is they are stationary points of the Ricci flow: $\frac{\partial g}{\partial t} = -2Ric(g)$, (in this paper we use $Ric = S$) in the space of metrics on M.

Definition 1.1. *A Ricci soliton (g, V, λ) on a Riemannian manifold is defined by*

$$\mathcal{L}_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where S is the Ricci tensor, $\mathcal{L}_V$ is the Lie derivative along the vector field V on M and λ is a real. Ricci soliton is said to be shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$.

In 1923, L.P. Eisenhart [11] proved that if a positive definite Riemannian manifold (M, g) admits a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor,

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then it is reducible. In 1925, Levy [19] obtained the necessary and sufficient conditions for the existence of such tensors. Later, R. Sharma [26] studied second order parallel tensor in real and complex space forms.

In [27], R. Sharma initiated the study of Ricci solitons in Contact geometry. Thereafter Ricci solitons in contact metric manifolds have been studied by various authors such as M.M. Tripathi [29], Constantin Calin and Mircea Crasmareanu [7], Amadendu Ghosh and Ramesh Sharma [1], Mircea Crasmareanu [21], U.C. De et al. [30], H.G. Nagaraja et al. [23], C.S. Bagewadi et al. ([3], [13]) and many others.

Motivated by the above results we studied Ricci solitons in (ε, δ)-trans-Sasakian manifolds.

2. **Preliminaries**

Let M be an almost contact metric manifold of dimension n equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0. \tag{2.1}$$

Almost contact metric manifold M is called (ε)-almost contact metric manifold if there exists a semi-Riemannian metric g,

$$g(\xi, \xi) = \varepsilon, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y), \quad \eta(X) = \varepsilon g(X, \xi), \tag{2.3}$$

where ε is 1 or -1 according as ξ is space like or time like. In particular, if the metric g is positive definite, then an (ε)-almost contact metric manifold is the usual almost contact metric manifold [5].

An (ε)-almost contact metric manifold is called an (ε)-trans-Sasakian manifold [28] if

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \varepsilon\eta(Y)X) + \beta(g(\phi X, Y)\xi - \varepsilon\eta(Y)\phi X), \tag{2.4}$$

holds for some smooth functions α and β on M. In (ε)-trans-Sasakian manifolds ξ is never light like as per the definition of characteristic vector field ξ.

According to the characteristic vector field ξ we have two classes of (ε)-trans-Sasakian manifolds. When $\varepsilon = -1$ and index of g is odd, then M is a time-like trans-Sasakian manifold and when $\varepsilon = 1$ and index of g is even, then M is a space-like trans-Sasakian manifold. Further, M is usual trans-Sasakian manifold for $\varepsilon = 1$ and index of g is 0 and M is a Lorentzian trans-Sasakian manifold for $\varepsilon = -1$ and index of g is 1.

An (ε)-almost contact metric manifold is said to be (ε, δ)-trans-Sasakian manifold if it satisfies,

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \varepsilon\eta(Y)X) + \beta(g(\phi X, Y)\xi - \delta\eta(Y)\phi X), \tag{2.5}$$

holds for some smooth functions α and β on M, where ε is 1 or -1 according as ξ is space like or time like and δ is alike ε.
From (2.5), we have

$$\nabla_X \xi = -\varepsilon\alpha\phi X - \beta\delta\phi^2 X, \tag{2.6}$$

$$(\nabla_X \eta)Y = \beta\delta[\varepsilon g(X, Y) - \eta(X)\eta(Y)] - \alpha g(\phi X, Y). \tag{2.7}$$

Remark 2.1. *From (2.5), we have the following Remarks:*

1. $\varepsilon = \delta$, (ε, δ)*-trans-Sasakian manifold of type* (α, β) *reduces to* (ε)*-trans-Sasakian manifold of type* (α, β).
2. $\varepsilon = \delta = 1$, (ε, δ)*-trans-Sasakian manifold of type* (α, β) *reduces to trans-Sasakian manifold of type* (α, β).
3. $\alpha \neq 0$, $\beta \neq 0$, *and* $\varepsilon = -1$, $\delta = -1$, (ε, δ)*-trans-Sasakian manifold of type* (α, β) *reduces to the form Lorentzian trans-Sasakian manifold of type* (α, β).
4. $\alpha \neq 0$, $\beta \neq 0$, *and* $\varepsilon = 1$, $\delta = -1$, (ε, δ)*-trans-Sasakian manifold of type* (α, β) *reduces in the form* α*-Sasakian Lorentzian* β*-Kenmostu manifold of type* (α, β).
5. $\alpha \neq 0$, $\beta \neq 0$, *and* $\varepsilon = -1$, $\delta = 1$, (ε, δ)*-trans-Sasakian manifold of type* (α, β) *reduces in the form Lorentzian* α*-Sasakian* β*-Kenmostu manifold of type* (α, β).
6. $\alpha \neq 0$, $\beta = 0$ *and* $\varepsilon = 1$ *or* $\varepsilon = -1$, *the* (ε, δ)*-trans-Sasakian manifold reduces to* α*-Sasakian manifold or Lorentzian* α*-Sasakian manifold respectively.*

(7) $\alpha=0$, $\beta\neq 0$ *and* $\delta=1$ *or* $\delta=-1$, *the* (ε,δ)*-trans-Sasakian manifold reduces to* β*-Kenmostu manifold or Lorentzian* β*-Kenmotsu manifold respectively.*

(8) *If* α *and* β *are scalars and* $\alpha=1$ *and* $\beta=0$ *or* $\alpha=0$ *and* $\beta=1$ *then the* (ε,δ)*-trans-Sasakian manifold reduces to* (ε)*-Sasakian manifolds and* (δ)*-Kenmostu manifolds.*
   (a) *Again, if in* (ε)*-Sasakian manifolds* ε *is* 1 *or* -1 *then the* (ε)*-Sasakian manifolds reduces to Sasakian manifolds or Lorentzian Sasakian manifolds.*
   (b) *Further, if in* (δ)*-Kenmostu manifolds* δ *is* 1 *or* -1 *then the* (δ)*-Kenmostu manifolds reduces to Kenmotsu manifold or Lorentzian Kenmotsu manifold.*

Note: If (M,ϕ,ξ,η,g) is a Lorentzian almost contact manifolds then it satisfies:

$$\phi^2X = -X+\eta(X)\xi, \qquad \eta(\xi)=1, \tag{2.8}$$
$$g(\phi X,\phi Y) = g(X,Y)+\eta(X)\eta(Y), \quad \eta(X)=-g(X,\xi). \tag{2.9}$$

2.1. **Example for trans-Sasakian manifolds.** The 3-dimensional manifold $M=\{(x,y,z)\in R^3; z\neq 0\}$ is a trans-Sasakian manifold: If $\{E_1,E_2,E_3\}$ is a global frame field given by

$$E_1=z\left(\frac{\partial}{\partial x}+y\frac{\partial}{\partial z}\right), \quad E_2=z\frac{\partial}{\partial y}, \quad E_3=\frac{\partial}{\partial z}. \tag{2.10}$$

The (ϕ,ξ,η,g) is given by $\eta=dz-ydx$, $\xi=E_3=\frac{\partial}{\partial z}$, $\phi E_1=E_2$, $\phi E_2=-E_1$, $\phi E_3=0$, $g=\frac{1}{z^2}[(1-y^2z^2)dx\otimes dx+dy\otimes dy+z^2dz\otimes dz]$, $g(E_1,E_2)=g(E_2,E_3)=g(E_1,E_3)=0$, $g(E_1,E_1)=g(E_2,E_2)=g(E_3,E_3)=1$. The (ϕ,ξ,η,g) is trans-Sasakian structure for

$$\nabla_X\xi=-\alpha\phi X+\beta(X-\eta(X)\xi),$$

with $\alpha=-\frac{1}{2}z^2\neq 0$ and $\beta=-\frac{1}{z}\neq 0$.

2.2. **Example for (ε)-trans-Sasakian manifolds.** The 3-dimensional manifold $M=\{(x,y,z)\in R^3; z\neq 0\}$ is an (ε)-trans-Sasakian manifold: If $\{E_1,E_2,E_3\}$ is a global frame field given by

$$E_1=\frac{x}{z}\frac{\partial}{\partial x}, \quad E_2=\frac{y}{z}\frac{\partial}{\partial y}, \quad E_3=\varepsilon\frac{\partial}{\partial z}. \tag{2.11}$$

Then (ϕ,ξ,η,g) is given by $\phi E_1=E_2$, $\phi E_2=-E_1$, $\phi E_3=0$, $\eta=\varepsilon dz$, $\xi=E_3=\frac{\partial}{\partial z}$, $g=\frac{z^2}{x^2}dx\otimes dx+\frac{z^2}{y^2}dy\otimes dy+\varepsilon dz\otimes dz$, $g(E_1,E_2)=g(E_2,E_3)=g(E_1,E_3)=0$, $g(E_1,E_1)=g(E_2,E_2)=g(E_3,E_3)=\varepsilon$. The (ϕ,ξ,η,g) is (ε)-trans-Sasakian structure for

$$\nabla_X\xi=\varepsilon[-\alpha\phi X+\beta(X-\eta(X)\xi)],$$

with $\alpha=-1$ and $\beta=\frac{1}{z}$.

In (ε,δ)-trans-Sasakian manifold M, we have the following:

$$\begin{aligned} R(X,Y)\xi &= \varepsilon[(Y\alpha)\phi X-(X\alpha)\phi Y]+\delta[(Y\beta)\phi^2X-(X\beta)\phi^2Y]+2\alpha\beta(\delta-\varepsilon)g(\phi X,Y)\xi \\ &+ 2\varepsilon\alpha\beta\delta[\eta(Y)\phi X-\eta(X)\phi Y]+(\alpha^2-\beta^2)[\eta(Y)X-\eta(X)Y], \end{aligned} \tag{2.12}$$
$$\begin{aligned} R(\xi,Y)X &= \varepsilon[(grad\alpha)g(\phi X,Y)+(X\alpha)\phi Y]+\delta[(grad\beta)g(\phi^2X,Y)-(X\beta)\phi^2Y] \\ &+ 2\alpha\beta\varepsilon(\delta-\varepsilon)\eta(Y)\phi X+2\varepsilon\alpha\beta\delta[\varepsilon g(\phi X,Y)\xi+\eta(X)\phi Y] \\ &+ (\alpha^2-\beta^2)[\varepsilon g(X,Y)\xi-\eta(X)Y], \end{aligned} \tag{2.13}$$
$$R(\xi,Y)\xi = [(\alpha^2-\beta^2)-\delta(\xi\beta)][-Y+\eta(Y)\xi]-[\varepsilon(\xi\alpha)+2\varepsilon\alpha\beta\delta](\phi Y), \tag{2.14}$$
$$S(X,\xi) = [(n-1)(\varepsilon\alpha^2-\delta\beta^2)-(\xi\beta)]\eta(X)-\varepsilon((\phi X)\alpha)-(n-2)\varepsilon(X\beta), \tag{2.15}$$
$$\varepsilon(\xi\alpha) + 2\varepsilon\alpha\beta\delta=0. \tag{2.16}$$

Further, in 3-dimensional (ε,δ)-trans-Sasakian manifold, we have

$$\phi(grad\alpha)=grad\beta. \tag{2.17}$$

3. Parallel symmetric second order tensors and Ricci Solitons in (ε,δ)-trans-Sasakian manifolds

Fix h a symmetric tensor field of $(0,2)$-type which we suppose to be parallel with respect to ∇ that is $\nabla h = 0$. Applying the Ricci identity [26]

$$\nabla^2 h(X,Y;Z,W) - \nabla^2 h(X,Y;W,Z) = 0, \tag{3.1}$$

we obtain the relation

$$h(R(X,Y)Z,W) + h(Z,R(X,Y)W) = 0. \tag{3.2}$$

Replacing $Z = W = \xi$ in (3.2) and by using (2.12) and by the symmetry of h, we have

$$\begin{aligned}&2\varepsilon[(Y\alpha)h(\phi X,\xi) - (X\alpha)h(\phi Y,\xi)] + 2\delta[(Y\beta)h(\phi^2 X,\xi) - (X\beta)h(\phi^2 Y,\xi)]\\ &+2(\alpha^2-\beta^2)[\eta(Y)h(X,\xi) - \eta(X)h(Y,\xi)] + 4\varepsilon\alpha\beta\delta[\eta(Y)h(\phi X,\xi) - \eta(X)h(\phi Y,\xi)]\\ &+4\alpha\beta(\delta-\varepsilon)g(\phi X,Y)h(\xi,\xi) = 0.\end{aligned} \tag{3.3}$$

Putting $X = \xi$ in (3.3) and by virtue of (2.1), we obtain

$$-2[\varepsilon(\xi\alpha) + 2\varepsilon\alpha\beta\delta]h(\phi Y,\xi) + 2[(\alpha^2-\beta^2) - \delta(\xi\beta)][\eta(Y)h(\xi,\xi) - h(Y,\xi)] = 0. \tag{3.4}$$

By using (2.16) in (3.4), we have

$$[(\alpha^2-\beta^2) - \delta(\xi\beta)][\eta(Y)h(\xi,\xi) - h(Y,\xi)] = 0. \tag{3.5}$$

Suppose $(\alpha^2-\beta^2) - \delta(\xi\beta) \neq 0$, it results

$$h(Y,\xi) = \eta(Y)h(\xi,\xi). \tag{3.6}$$

Let us call a regular (ε,δ)-trans-Sasakian manifolds with $(\alpha^2-\beta^2) - \delta(\xi\beta) \neq 0$, where regularity means the non-vanishing of the Ricci curvature with respect to the generator of (ε,δ)-trans-Sasakian manifolds.
Differentiating (3.6) covariantly with respect to X, we have

$$\begin{aligned}(\nabla_X h)(Y,\xi) + h(\nabla_X Y,\xi) + h(Y,\nabla_X\xi) &= [\varepsilon g(\nabla_X Y,\xi) + \varepsilon g(Y,\nabla_X\xi)]h(\xi,\xi)\\ &+ \eta(Y)[(\nabla_X h)(Y,\xi) + 2h(\nabla_X\xi,\xi)].\end{aligned} \tag{3.7}$$

By using the parallel condition $\nabla h = 0$, $\eta(\nabla_X\xi) = 0$ and by virtue of (3.6) in (3.7), we get

$$h(Y,\nabla_X\xi) = \varepsilon g(Y,\nabla_X\xi)h(\xi,\xi).$$

By using (2.6) in the above equation, we obtain

$$-\varepsilon\alpha h(Y,\phi X) + \beta\delta h(Y,X) = -\alpha g(Y,\phi X)h(\xi,\xi) + \varepsilon\beta\delta g(Y,X)h(\xi,\xi). \tag{3.8}$$

Replacing $X = \phi X$ in (3.8) and on simplification, we get

$$h(X,Y) = \varepsilon g(X,Y)h(\xi,\xi), \tag{3.9}$$

which together with the standard fact that the parallelism of h implies that $h(\xi,\xi)$ is a constant, via (3.6). Now by considering the above conditions, we can state the following:

Theorem 3.1. *A symmetric parallel second order covariant tensor in a regular (ε,δ)-trans-Sasakian manifold is a constant multiple of the metric tensor.*

Corollary 3.1. *A locally Ricci symmetric $(\nabla S = 0)$ regular (ε,δ)-trans-Sasakian manifold is an Einstein manifold.*

Remark 3.1. *The following statements for indefinite (ε,δ)-trans-Sasakian manifold are equivalent:*

(1) *Einstein,*
(2) *locally Ricci symmetric,*
(3) *Ricci semi-symmetric that is $R\cdot S = 0$.*

The implication (1) $\longrightarrow$ (2) $\longrightarrow$ (3) *is trivial. Now we prove the implication* (3) $\longrightarrow$ (1) *and* $R\cdot S=0$ *means,*

$$(R(X,Y)\cdot S)(U,V) \quad = \quad -S(R(X,Y)U,V)-S(U,R(X,Y)V). \tag{3.10}$$

Considering $R\cdot S=0$ *and putting* $X=\xi$ *in (3.10), we have*

$$S(R(\xi,Y)U,V)+S(U,R(\xi,Y)V)=0. \tag{3.11}$$

By using (2.13) in (3.11), we obtain

$$\begin{aligned}
&\varepsilon[g(\phi U,Y)S(grad\alpha,V)+(U\alpha)S(\phi Y,V)]+\delta[g(\phi^2U,Y)S(grad\beta,V)-(U\beta)S(\phi^2Y,V)]\\
&+2\alpha\beta\varepsilon(\delta-\varepsilon)\eta(Y)S(\phi U,V)+2\varepsilon\alpha\beta\delta[\varepsilon g(\phi U,Y)S(\xi,V)+\eta(U)S(\phi Y,V)]\\
&+(\alpha^2-\beta^2)[\varepsilon g(Y,U)S(\xi,V)-\eta(U)S(Y,V)]+\varepsilon[g(\phi V,Y)S(U,grad\alpha)+(V\alpha)S(U,\phi Y)]\\
&+\delta[g(\phi^2V,Y)S(U,grad\beta)-(V\beta)S(U,\phi^2Y)]+2\alpha\beta\varepsilon(\delta-\varepsilon)\eta(Y)S(U,\phi V)\\
&+2\varepsilon\alpha\beta\delta[\varepsilon g(\phi V,Y)S(U,\xi)+\eta(V)S(U,\phi Y)]+(\alpha^2-\beta^2)[\varepsilon g(Y,V)S(U,\xi)-\eta(V)S(U,Y)]=0.
\end{aligned}$$

Putting $U=\xi$ *in the above equation and by virtue of (2.1), (2.2), (2.15), (2.16) and on simplification, we get*

$$S(Y,V)=(n-1)(\varepsilon\alpha^2-\delta\beta^2)g(Y,V). \tag{3.12}$$

In conclusion, we state the following:

Proposition 3.1. *A Ricci semi-symmetric regular indefinite* (ε,δ)*-trans-Sasakian manifold is an Einstein.*

Corollary 3.2. *Suppose that on a regular* (ε,δ)*-trans-Sasakian manifold the* $(0,2)$*-type field* $\mathcal{L}_Vg+2S$ *is parallel where* V *is a given vector field. Then* (g,V) *yield a Ricci soliton. In particular, if the given regular* (ε,δ)*-trans-Sasakian manifold is Ricci semi-symmetric with* $\mathcal{L}_Vg$ *parallel, we have the same conclusion.*

Proof. Follows from theorem 3.1 and corollary 3.1. ☐

Hence we state the following result:

Corollary 3.3. *A Ricci soliton* (g,ξ,λ) *in an indefinite* (ε,δ)*-trans-Sasakian manifold cannot be steady.*

Proof. From Linear Algebra either the vector field $V\in Span\,\xi$ or $V\perp\xi$. However the second case seems to be complex to analyse in practice. For this reason we investigate for the case $V=\xi$.

A simple computation of $\mathcal{L}_\xi g+2S$ gives

$$(\mathcal{L}_\xi g)(X,Y)=2\beta\delta[g(X,Y)-\varepsilon\eta(X)\eta(Y)]. \tag{3.13}$$

From equation (1.1), we have $h(X,Y)=-2\lambda g(X,Y)$ and then putting $X=Y=\xi$, we get

$$h(\xi,\xi)=-2\lambda\varepsilon, \tag{3.14}$$

where $h(X,Y)=(\mathcal{L}_\xi g)(X,Y)+2S(X,Y)$ and then if we put $X=Y=\xi$ and again by using (3.13) and (2.15), we obtain

$$\begin{aligned}
h(\xi,\xi) \quad &= \quad 2\beta\delta[g(\xi,\xi)-\varepsilon\eta(\xi)\eta(\xi)]+2\{\varepsilon[(n-1)(\varepsilon\alpha^2-\delta\beta^2)-(\xi\beta)]\eta(\xi)\\
&- \quad \varepsilon((\phi\xi)\alpha)-(n-2)\varepsilon(\xi\beta)\}.
\end{aligned}$$

By using (2.1), (2.2) and (2.17) in the above equation, we get

$$h(\xi,\xi) \quad = \quad 2(n-1)\varepsilon(\varepsilon\alpha^2-\delta\beta^2). \tag{3.15}$$

Equating (3.14) and (3.15), we have

$$\lambda=-(n-1)(\varepsilon\alpha^2-\delta\beta^2). \tag{3.16}$$

Since from (3.16), we have $\lambda\neq 0$. Hence the proof. ☐

An indefinite (ε,δ)-trans-Sasakian manifold contains the indefinite Sasakain and Kenmotsu manifolds based on this conditions and from corollary (3.3) we can state the following proposition:

Proposition 3.2.

(1) *A Ricci soliton in a trans-Sasakian manifold is Shrinking if* $\alpha > \beta$.
(2) *A Ricci soliton in an* α*-Sasakian Lorentzian* β*-Kenmotsu manifold is Shrinking if* $\alpha > \beta$.
(3) *A Ricci soliton in an Lorentzian* α*-Sasakian* β*-Kenmotsu manifold is Expanding if* $\alpha > \beta$.
(4) *A Ricci soliton in an Lorentzian trans-Sasakian manifold is Expanding if* $\alpha > \beta$.
(5) *A Ricci soliton in an trans-Sasakian manifold is Expanding if* $\alpha < \beta$.
(6) *A Ricci soliton in an Lorentzian* α*-Sasakian* β*-Kenmotsu manifold is Expanding if* $\alpha < \beta$.
(7) *A Ricci soliton in a Lorentzian trans-Sasakian manifold is Shrinking if* $\alpha < \beta$.
(8) *A Ricci soliton in an* α*-Sasakian Lorentzian* β*-Kenmotsu manifold is Shrinking if* $\alpha < \beta$.

Proof. Proofs of the above condition (1) to (4) follow from equation (3.16) and Remarks (2.1) of (2), (4), (5), (3). Again conditions (5) to (8) follow similarly from equation (3.16) and Remarks (2.1) (2), (5), (3), (4). ☐

4. Ricci solitons in 3-dimensional (ε, δ)-trans-Sasakian manifold

Corollary 4.1. *A Ricci soliton* (g, ξ, λ) *where* $\lambda = -2(\varepsilon\alpha^2 - \delta\beta^2)$ *in an indefinite 3-dimensional* (ε, δ)*-trans-Sasakian manifold with varying scalar curvature cannot be steady.*

Proof. The Riemannian curvature tensor of 3-dimensional (ε, δ)-trans-Sasakian manifold is given by

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &- \frac{r}{2}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{4.1}$$

Putting $Z = \xi$ in (4.1) and by using (2.12) and (2.15) for 3-dimensional (ε, δ)-trans-Sasakian manifold, we get

$$\begin{aligned} &\varepsilon[(Y\alpha)\phi X - (X\alpha)\phi Y] + \delta[(Y\beta)\phi^2 X - (X\beta)\phi^2 Y] + 2\alpha\beta(\delta - \varepsilon)g(\phi X, Y)\xi \\ &+2\varepsilon\alpha\beta\delta[\eta(Y)\phi X - \eta(X)\phi Y] + (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] = \varepsilon\eta(Y)QX - \varepsilon\eta(X)QY \\ &+\varepsilon[(\varepsilon\alpha^2 - \delta\beta^2) - (\xi\beta)](\eta(Y)X - \eta(X)Y) - \varepsilon[((\phi Y)\alpha)X + (Y\beta)X] + \varepsilon[((\phi X)\alpha)Y + (X\beta)Y]. \end{aligned} \tag{4.2}$$

Again, putting $Y = \xi$ in the above equation and by using (2.1) and (2.17), we obtain

$$QX = \left[\frac{r}{2} - 2(\varepsilon\alpha^2 - \delta\beta^2) + \varepsilon(\alpha^2 - \beta^2)\right] X + \left[4(\varepsilon\alpha^2 - \delta\beta^2) - \frac{r}{2} - \varepsilon(\alpha^2 - \beta^2)\right] \eta(X)\xi. \tag{4.3}$$

From (4.3), we have

$$\begin{aligned} S(X,Y) &= \left[\frac{r}{2} - 2(\varepsilon\alpha^2 - \delta\beta^2) + \varepsilon(\alpha^2 - \beta^2)\right] g(X,Y) \\ &+ \left[4(\varepsilon\alpha^2 - \delta\beta^2) - \frac{r}{2} - \varepsilon(\alpha^2 - \beta^2)\right] \varepsilon\eta(X)\eta(Y). \end{aligned} \tag{4.4}$$

Equation (4.4) shows that a 3-dimensional (ε, δ)-trans-Sasakian manifold is η-Einstein.
Now we show that the scalar curvature r is not a constant that is r is varying. Now

$$h(X,Y) = (\mathcal{L}_\xi g)(X,Y) + 2S(X,Y). \tag{4.5}$$

By using (3.13) and (4.4) in (4.5), we have

$$\begin{aligned} h(X,Y) &= [r - 4(\varepsilon\alpha^2 - \delta\beta^2) + 2\varepsilon(\alpha^2 - \beta^2) + 2\beta\delta]g(X,Y) \\ &+ [8(\varepsilon\alpha^2 - \delta\beta^2) - 2\beta\delta - 2\varepsilon(\alpha^2 - \beta^2) - r]\varepsilon\eta(X)\eta(Y). \end{aligned} \tag{4.6}$$

Differentiating (4.6) covariantly with respect to Z, we get

$$\begin{aligned} (\nabla_Z h)(X,Y) &= [\nabla_Z r - 4(2\varepsilon\alpha(Z\alpha) - 2\delta\beta(Z\beta)) + 2\varepsilon(2\alpha(Z\alpha) - 2\beta(Z\beta)) + 2\delta(Z\beta)]g(X,Y) \\ &+ [8(2\varepsilon\alpha(Z\alpha) - 2\delta\beta(Z\beta)) - 2\delta(Z\beta) - 2\varepsilon(2\alpha(Z\alpha) - 2\beta(Z\beta)) - \nabla_Z r]\varepsilon\eta(X)\eta(Y) \\ &+ [8(\varepsilon\alpha^2 - \delta\beta^2) - 2\beta\delta - 2\varepsilon(\alpha^2 - \beta^2) - r][g(X, \nabla_Z\xi)\eta(Y) + g(Y, \nabla_Z\xi)\eta(X)]. \end{aligned} \tag{4.7}$$

Substituting $Z = \xi$, $X = Y \in (Span\,\xi)^\perp$ in (4.7) and by virtue of $\nabla h = 0$ and (2.17), we have

$$\nabla_\xi r - 4\varepsilon\alpha(\xi\alpha) = 0.$$

By using (2.16) in the above equation, we obtain

$$\nabla_\xi r = -8\varepsilon\alpha^2\beta\delta. \tag{4.8}$$

Thus, r is not a constant.

Now we have to check the nature of the soliton that is Ricci soliton (g, ξ, λ) where $\lambda = -2(\varepsilon\alpha^2 - \delta\beta^2)$ in 3-dimensional (ε, δ)-trans-Sasakian manifold:

From (1.1), we have $h(X, Y) = -2\lambda g(X, Y)$ and then putting $X = Y = \xi$, we get

$$h(\xi, \xi) = -2\lambda\varepsilon. \tag{4.9}$$

If $X = Y = \xi$ in (4.6), we obtain

$$h(\xi, \xi) = 4\varepsilon(\varepsilon\alpha^2 - \delta\beta^2). \tag{4.10}$$

Equating (4.9) and (4.10), we have

$$\lambda = -2(\varepsilon\alpha^2 - \delta\beta^2). \tag{4.11}$$

Since from (4.11), we have $\lambda \neq 0$. Therefore Ricci soliton (g, ξ, λ) of 3-dimensional (ε, δ)-trans-Sasakian manifold cannot be steady. □

Now by Corollary (4.1) we can state the following proposition:

Proposition 4.1. *Prove the following:*

(1) *A Ricci soliton in an 3-dimensional trans-Sasakian manifold is Shrinking if $\alpha > \beta$.*
(2) *A Ricci soliton in an 3-dimensional α-Sasakian Lorentzian β-Kenmotsu manifold is Shrinking if $\alpha > \beta$.*
(3) *A Ricci soliton in an 3-dimensional Lorentzian α-Sasakian β-Kenmotsu manifold is Expanding if $\alpha > \beta$.*
(4) *A Ricci soliton in an 3-dimensional Lorentzian trans-Sasakian manifold is Expanding if $\alpha > \beta$.*
(5) *A Ricci soliton in an 3-dimensional trans-Sasakian manifold is Expanding if $\alpha < \beta$.*
(6) *A Ricci soliton in an 3-dimensional Lorentzian α-Sasakian β-Kenmotsu manifold is Expanding if $\alpha < \beta$.*
(7) *A Ricci soliton in a 3-dimensional Lorentzian trans-Sasakian manifold is Shrinking if $\alpha < \beta$.*
(8) *A Ricci soliton in an 3-dimensional α-Sasakian Lorentzian β-Kenmotsu manifold is Shrinking if $\alpha < \beta$.*

Proof. Proofs of the above condition (1) to (4) follow from equation (4.11) and Remarks (2.1) of (2), (4), (5), (3). Again conditions (5) to (8) follow similarly from equation (4.11) and Remarks (2.1) (2), (5), (3), (4). □

5. Example of Ricci solitons for 3-dimensional (ε, δ)-trans-Sasakian manifold

We consider the 3-dimensional manifold $M = \{(x, y, z) : (x, y, z) \in R^3, z \neq 0\}$. Let $\{E_1, E_2, E_3\}$ be linearly independent global frame field on M given by

$$E_1 = z\left(\frac{\partial}{\partial x} + \delta y\frac{\partial}{\partial z}\right), \quad E_2 = \delta z\frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z}. \tag{5.1}$$

Let g be the Riemannian metric defined by $g(E_1, E_2) = g(E_2, E_3) = g(E_1, E_3) = 0$, $g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = \varepsilon$, where g is given by

$$g = \frac{\varepsilon}{z^2}[(1 - y^2z^2)dx \otimes dx + dy \otimes dy + z^2dz \otimes dz].$$

The (ϕ, ξ, η) is given by $\eta = dz - \delta y dx$, $\xi = E_3 = \frac{\partial}{\partial z}$, $\phi E_1 = E_2$, $\phi E_2 = -E_1$, $\phi E_3 = 0$.

Clearly (ϕ, ξ, η, g) structure is an indefinite (ε, δ)-trans-Sasakian structure and satisfy,

$$(\nabla_X\phi)Y = \alpha(g(X, Y)\xi - \varepsilon\eta(Y)X) + \beta(g(\phi X, Y)\xi - \delta\eta(Y)\phi X), \tag{5.2}$$

$$\nabla_X\xi = -\varepsilon\alpha\phi X - \beta\delta\phi^2 X, \tag{5.3}$$

where $\alpha = -\frac{z^2\delta}{2\varepsilon} \neq 0$ and $\beta = -\frac{1}{z\delta} \neq 0$. Hence (ϕ, ξ, η, g) structure defines indefinite (ε, δ)-trans-Sasakian structure. Thus M equipped with indefinite (ε, δ)-trans-Sasakian structure is a (ε, δ)-trans-Sasakian manifold.

Using the above α and β in (4.8), we have

$$\begin{aligned}
\nabla_\xi r &= -8\varepsilon\delta\alpha^2\beta = 8\varepsilon\delta\left(\frac{z^4\delta^2}{4\varepsilon^2}\right)\left(\frac{1}{z\delta}\right)\\
&= 2\varepsilon z^3,\\
\Longrightarrow r &= \frac{z^4\varepsilon}{2}. \qquad (5.4)
\end{aligned}$$

Using the above α and β in (4.11), we have

$$\lambda = \frac{4\delta - \varepsilon Z^6}{2z^2}. \qquad (5.5)$$

Hence Ricci soliton (g, ξ, λ) is given by (5.5) with varying scalar curvature (5.4).

(1) If $\varepsilon = \delta = 1$, in (5.5), then $\lambda = \frac{4-Z^6}{2z^2} = \frac{(2-Z^3)(2+Z^3)}{2z^2}$.
   (a) $\lambda > 0$ in $\{Z : -2^{1/3} < Z < 2^{1/3}\}$:
   Hence by Remark (2.1), Ricci soliton of the given 3-dimensional trans-Sasakian manifold is expanding in the region

$$\{(x,y,z) \in R^3 : -2^{1/3} < Z < 2^{1/3}\}. \qquad (5.6)$$

   (b) Also $\lambda < 0$ in $\{Z : -2^{1/3} > Z > 2^{1/3}\}$:
   Hence by Remark (2.1), Ricci soliton of the given 3-dimensional trans-Sasakian manifold is shrinking in the region

$$\{(x,y,z) \in R^3 : -2^{1/3} > Z > 2^{1/3}\}. \qquad (5.7)$$

   Hence the regions (5.6) and (5.7) are complementary to one another that is,

$$M = \{(x,y,z) \in R^3 : -2^{1/3} < Z < 2^{1/3}\} \cup \{(x,y,z) \in R^3 : -2^{1/3} > Z > 2^{1/3}\}.$$

(2) If $\varepsilon = \delta = -1$, in (5.5), then $\lambda = \frac{Z^6-4}{2z^2} = \frac{(Z^3-2)(Z^3+2)}{2z^2}$.
   (a) $\lambda > 0$ in $\{Z : -2^{1/3} > Z > 2^{1/3}\}$:
   Hence by Remark (2.1), Ricci soliton in Lorentzian trans-Sasakian manifold is expanding in the region

$$\{(x,y,z) \in R^3 : -2^{1/3} > Z > 2^{1/3}\}. \qquad (5.8)$$

   (b) Also $\lambda < 0$ in $\{Z : -2^{1/3} < Z < 2^{1/3}\}$:
   Hence by Remark (2.1), Ricci soliton in Lorentzian trans-Sasakian manifold is shrinking in the region

$$\{(x,y,z) \in R^3 : -2^{1/3} < Z < 2^{1/3}\}. \qquad (5.9)$$

   Hence the regions (5.8) and (5.9) are complementary to one another that is,

$$M = \{(x,y,z) \in R^3 : -2^{1/3} > Z > 2^{1/3}\} \cup \{(x,y,z) \in R^3 : -2^{1/3} < Z < 2^{1/3}\}.$$

   Thus from cases (1) and (2) one can conclude that in a region where the trans-Sasakian manifold is shrinking the Lorentzian trans-Sasakian manifold is expanding and in a region where the trans-Sasakian manifold is expanding the Lorentzian trans-Sasakian manifold is shrinking. Hence in given example trans-Sasakian and Lorentzian trans-Sasakian manifolds are complementary to each other.

(3) If $\varepsilon = -1, \delta = 1$, in (5.5), then $\lambda = \frac{Z^6+4}{2z^2} > 0$. By Remark (2.1), Ricci soliton in Lorentzian α-Sasakian β-Kenmotsu manifold is expanding.

(4) If $\varepsilon = 1, \delta = -1$, in (5.5), then $\lambda = -\frac{(Z^6+4)}{2z^2} < 0$. By Remark (2.1), Ricci soliton in α-Sasakian Lorentzian β-Kenmotsu manifold is shrinking.

6. Conclusion

We know [15, 16] that any compact steady or expanding Ricci soliton is Einstein. In our case we have shown that the Ricci soliton in regular indefinite (ε,δ)-trans-Sasakian manifold is Einstein but it is not steady and it is a manifold of varying scalar curvature because (ε,δ)-trans-Sasakian structure contains both Sasakian as well as Kenmotsu structures for $\varepsilon = \delta = 1$ and Lorentzian condition and the manifold is not compact. Hence it is expanding or shrinking depending upon α and β which characterize the Sasakian and Kenmotsu structure and ε, δ which characterize Lorentz structure or indefinite case.

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