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AN APPROACH TO THE CONCEPT OF SOFT VIETORIS TOPOLOGY

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ABSTRACT. In the present paper, we study the Vietoris topology in the context of soft set. Firstly, we investigate some aspects of first countability in the soft Vietoris topology. Then, we obtain some properties about its second countability.

1. Introduction

In 1999, Molodtsov [22] initiated the concept of a soft set theory as a new approach for coping with uncertainties and also presented the basic results of the new theory. In [22], Molodtsov successfully applied the soft set theory in several directions, such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration and theory of measurement. After presentation of the operations of soft sets [21], the properties and applications of this theory have been studied increasingly ([4], [23], [25]).

Aktaş and Çağman [3] introduced the soft group and also compared soft sets to fuzzy set and rough set. Shabir and Naz [29] initiated the study of soft topological spaces. Rong [28] presented the notions of soft first countable and soft second countable spaces and investigated some their fundamental properties. Recently, many papers concerning the soft set theory have been published ([7], [11], [14], [16], [18], [24], [26]).

Hyperspace theory had its early beginnings in the 1900's, with the work of Hausdorff and Vietoris. This theory plays a fundamental role in mathematics and applied sciences, such as Convex Analysis, Optimization, Economics and Image Processing. Over the years, a lot of research has been performed on this subject ([5], [6], [9], [17], [20]).

As hyperspace of a topological space (X, τ), it means $C(X)$, the set of closed subsets of X, equipped with a topology τ_h such that the function $i : (X, \tau) \to (C(X), \tau_h)$ defined by $i(x) = \{x\}$ is a homeomorphism onto its image. One of the most important and well-studied hyperspace topologies on $C(X)$ is the Vietoris topology. The Vietoris topology is a basic construct due to its usefulness in different areas of mathematics and applications. Therefore, this topology has attracted the attention of many mathematicians in the last few decades ([8], [12], [13], [15], [19], [27], [32]).

Extensions of hypertopologies to the soft sets have been studied by some authors. Akdağ and Erol [1] and Shakir [30] defined independently a hyperspace of soft sets, called soft Vietoris topological space. Later, Akdağ and Erol [2] studied on some hyperspaces of soft sets such as co-quasi H-closed soft topological spaces and D-soft topological spaces.

In this paper, firstly we present a brief synopsis of all necessary definitions and results that will be required. Next, we continue studying the soft Vietoris topology and obtain some results about its first and second countability.

2. Preliminaries

In this section, we recollect some basic notions regarding soft sets. Throughout this work, let X be an initial universe, $P(X)$ be the power set of X and E be a set of parameters for X,

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Definition 2.1 ([22]). *A soft set F on the universe X with the set E of parameters is defined by the set of ordered pairs*

$$F = \{(e, F(e)) : e \in E, F(e) \in P(X)\}$$

where F is a mapping given by $F : E \to P(X)$.

Throughout this paper, the family of all soft sets over X is denoted by $S(X, E)$ [7].

Definition 2.2 ([4], [21], [25]). *Let $F, G \in S(X, E)$. Then,*

(i) The soft set F is called a null soft set, denoted by $\widetilde{\emptyset}$, if $F(e) = \emptyset$ for every $e \in E$.

(ii) The soft set F is called an absolute soft set, denoted by $\widetilde{X}$, if $F(e) = X$ for every $e \in E$.

(iii) F is a soft subset of G if $F(e) \subseteq G(e)$ for every $e \in E$. It is denoted by $F \sqsubseteq G$.

(iv) The complement of F is denoted by F^c, where $F^c : E \to P(X)$ is a mapping defined by $F^c(e) = X - F(e)$ for every $e \in E$. Clearly, $(F^c)^c = F$.

(v) The union of F and G is a soft set H defined by $H(e) = F(e) \cup G(e)$ for every $e \in E$. H is denoted by $F \sqcup G$.

(vi) The intersection of F and G is a soft set H defined by $H(e) = F(e) \cap G(e)$ for every $e \in E$. H is denoted by $F \sqcap G$.

(vii) The difference of F and G is a soft set H defined by $H(e) = F(e) - G(e)$ for every $e \in E$. H is denoted by $F \smile G$.

Definition 2.3 ([10], [19], [24]). *A soft set P over X is said to be a soft point if there exists an $e \in E$ such that $P(e) = \{x\}$ for some $x \in X$ and $P(e') = \emptyset$ for every $e' \in E \backslash \{e\}$. This soft point is denoted as x^e.*

A soft point x^e is said to belongs to a soft set F, denoted by $x^e \widetilde{\in} F$, if $x \in F(e)$.

From now on, let $SP(X)$ be the family of all soft points over X.

Definition 2.4 ([29]). *Let τ be a collection of soft sets over X, then τ is said to be a soft topology on X if*

(st_1) *$\widetilde{\emptyset}, \widetilde{X}$ belong to τ.*

(st_2) *the union of any number of soft sets in τ belongs to τ.*

(st_3) *the intersection of any two soft sets in τ belongs to τ*

(X, τ, E) is called a soft topological space. The members of τ are called soft open sets in X. A soft set F over X is called a soft closed in X if $F^c \in \tau$.

Definition 2.5 ([7], [24]). *Let (X, τ, E) be a soft topological space. A subcollection $\mathcal{B}$ of τ is called a soft base for τ if every member of τ can be expressed as the union of some members of $\mathcal{B}$.*

Definition 2.6. *Let (X, τ, E) be a soft topological space and $F \in S(X, E)$.*

(i) The soft interior of F is the soft set $F^o = \sqcup\{G : G\, is\, soft\, open\, set\, and\, G \sqsubseteq F\}$ [31].

(ii) The soft closure of F is the soft set $\overline{F} = \sqcap\{G : G\, is\, soft\, closed\, set\, and\, F \sqsubseteq G\}$ [29].

Definition 2.7 ([24]). *A soft set F in a soft topological space (X, τ, E) is called a soft neighborhood of the soft point x^e if there exists a soft open set G such that $x^e \widetilde{\in} G \sqsubseteq F$.*

The soft neighborhood system of a soft point x^e, denoted by $\mathcal{N}(x^e)$, is the family of all its soft neighborhoods.

Definition 2.8 ([29]). *Let (X, τ, E) be a soft topological space and Y be a non-empty subset of X. Then, $\tau_Y = \{\widetilde{Y} \sqcap F : F \in \tau\}$ is called the soft relative topology on Y and (Y, τ_Y, E) is called a soft subspace of (X, τ, E).*

Here, $\widetilde{Y}$ is the soft set over X defined by $\widetilde{Y}(e) = Y$ for all $e \in E$.

Definition 2.9 ([28]). *Let (X, τ, E) be a soft topological space and $x^e \widetilde{\in} \widetilde{X}$. A subcollection $\mathcal{B}$ of τ is called a soft local base at a soft point x^e if for every soft open set F containing x^e, there exists a $G \in \mathcal{B}$ such that $x^e \widetilde{\in} G \sqsubseteq F$.*

Definition 2.10 ([11]). *Let (X, τ, E) be a soft topological space, $\{x_n^{e_n} : n \in \mathbb{N}\}$ be a sequence of soft points in (X, τ, E) and $x^e \in SP(X)$. The sequence $\{x_n^{e_n} : n \in \mathbb{N}\}$ is said to converge to x^e, and we write $x_n^{e_n} \to x^e$, if for every $F \in \mathcal{N}(x^e)$, there exists an $n_0 \in \mathbb{N}$ such that $x_n^{e_n} \widetilde{\in} F$ for all $n \geq n_0$.*

Definition 2.11 ([24]). *Let (X, τ, E) be a soft topological space. A soft point $x^e \in SP(X)$ is called a limiting soft point of a soft set F over X if every soft open set containing x^e contains at least one soft point of F other than x^e, i.e., if $F \sqcap (G \smile x^e) \neq \widetilde{\emptyset}$ for every $G \in \tau$ containing x^e.*

The union of all limiting soft points of F is called the derived soft set of F and is denoted by F'.

Definition 2.12 ([28]). *Let (X, τ, E) be a soft topological space.*

(i) If each soft point in X has a countable soft local base, then it is called a soft first countable space.

(ii) If there exists a countable soft base for τ, then it is a soft second countable space.

Definition 2.13 ([28]). *Let (X, τ, E) be a soft topological space. If there exists a family $\{x_n^{e_n} : n \in \mathbb{N}\}$ of countable many soft points in X such that $\overline{\bigsqcup_{n \in \mathbb{N}} x_n^{e_n}} = \widetilde{X}$, then (X, τ, E) is called a soft separable space.*

Definition 2.14. *Let (X, τ, E) be a soft topological space. Then,*

(i) It is called a soft T_1-space if every soft point in X is a soft closed set [18].

(ii) It is called a soft Hausdorff space or a soft T_2-space if for any two distinct soft points $x_1^{e_1}$, $x_2^{e_2} \in SP(X)$ there exist soft open sets F and G such that $x_1^{e_1} \widetilde{\in} F, x_2^{e_2} \widetilde{\in} G$ and $F \sqcap G = \widetilde{\emptyset}$ [11].

(iii) It is called a soft regular space if for every $x^e \widetilde{\in} \widetilde{X}$ and every soft closed set F such that $x^e \widetilde{\notin} F$, there exist soft open sets F_1 and F_2 such that $x^e \widetilde{\in} F_1$, $F \sqsubseteq F_2$ and $F_1 \sqcap F_2 = \widetilde{\emptyset}$ [16].

Theorem 2.15 ([16]). *Let (X, τ, E) be a soft topological space. Then, the following statements are equivalent:*

(1) *(X, τ, E) is a soft regular space.*

(2) *For any soft open set F in (X, τ, E) and $x^e \widetilde{\in} F$, there exists a soft open set G containing x^e such that $x^e \widetilde{\in} \overline{G} \sqsubseteq F$.*

(3) *For any soft closed set H in (X, τ, E) and $x^e \widetilde{\notin} H$, there exists a soft open set G containing x^e such that $\overline{G} \sqcap H = \widetilde{\emptyset}$.*

Definition 2.16 ([7], [31]). *Let (X, τ, E) be a soft topological space and $F \in S(X, E)$.*

(i) A family $\mathcal{C} = \{F_i : i \in I\}$ of soft sets over X is called a cover of F if it satisfies $F \sqsubseteq \bigsqcup_{i \in I} F_i$. It is called a soft open cover if each member of $\mathcal{C}$ is a soft open set. A subfamily of $\mathcal{C}$ is called a subcover of $\mathcal{C}$ if it is also a cover of F.

(ii) (X, τ, E) is called a soft compact space if every soft open cover of $\widetilde{X}$ has a finite subcover.

Theorem 2.17 ([26]). *Let (Y, τ_Y, E) be a soft subspace of a soft topological space (X, τ, E). Then, (Y, τ_Y, E) is a soft compact space if and only if every cover of $\widetilde{Y}$ by soft open sets over X contains a finite subcover.*

Theorem 2.18 ([26]). *Every soft compact subspace of a soft Hausdorff space is soft closed.*

Definition 2.19 ([1])**.** *Let (X, τ, E) be a soft topological space and $F \in \tau$. Then, the families of soft sets F^+ and F^- are defined as follows:*

$$F^+ = \{G \in SC(X) : G \sqsubseteq F\} \qquad \textit{and} \qquad F^- = \{G \in SC(X) : F \sqcap G \neq \widetilde{\emptyset}\},$$

where $SC(X)$ is the family of non-null soft closed sets over X.

Proposition 2.20 ([1])**.** *Let (X, τ, E) be a soft topological space. For non-null soft sets F and G, the following statements are true:*

(i) $F^+ \cap G^+ = (F \sqcap G)^+$.
(ii) $F^+ \cup G^+ \subseteq (F \sqcup G)^+$.
(iii) $(F \sqcap G)^- \subseteq F^- \cap G^-$.
(iv) $F^- \cup G^- = (F \sqcup G)^-$.
(v) $F \sqsubseteq G$ if and only if $F^+ \subseteq G^+$.
(vi) $F \sqsubseteq G$ if and only if $F^- \subseteq G^-$.

Proposition 2.21 ([1])**.** *Let (X, τ, E) be a soft topological space. Then, the families*

$$\delta^+_{SV} = \{F^+ : F \in \tau\} \qquad \textit{and} \qquad \delta^-_{SV} = \{F^- : F \in \tau\}$$

are subbases for the topological spaces τ^+_{SV} and τ^-_{SV} on $SC(X)$, respectively.

Definition 2.22 ([1])**.** *The topological spaces τ^+_{SV} and τ^-_{SV} on $SC(X)$ which mentioned in above proposition are called a soft upper Vietoris topological space and a soft lower Vietoris topological space, respectively.*

Definition 2.23 ([1])**.** *A topological space on $SC(X)$ with $\delta_{SV} = \delta^+_{SV} \cup \delta^-_{SV}$ as subbase is called a soft Vietoris topological space, denoted by τ_{SV}.*

3. First and Second Countability of the Soft Vietoris Topology

The aim of this section is to present some properties related to countability of soft Vietoris topology. Firstly, we focus on its first countability.

Theorem 3.1. *Let (X, τ, E) be a soft T_1-space. Then, the following statements are equivalent:*
(1) *$(SC(X), \tau_{SV})$ is a first countable space.*
(2) *$(SC(X), \tau^+_{SV})$ and $(SC(X), \tau^-_{SV})$ are first countable spaces.*

Proof. (2) $\Rightarrow$ (1) is obvious from Definition 2.22 and 2.23.
To prove (1) $\Rightarrow$ (2), let $(SC(X), \tau_{SV})$ be a first countable space and take $H \in SC(X)$. Let

$$\mathcal{L} = \left\{ \ell = \bigcap_{i \in I} F_i^+ \cap \bigcap_{j \in J} G_j^- : I, J \textit{ is finite} \right\}$$

be a countable local base at H for τ_{SV}. Then, the family

$$\mathcal{L}^+ = \left\{ \bigcap_{i \in I} F_i^+ : \bigcap_{i \in I} F_i^+ \textit{ occurs in some } \ell \in \mathcal{L} \right\} \cup \{SC(X)\}$$

forms a countable local base at H for τ^+_{SV}. Indeed, if there is no $\bigcap_{i \in I} F_i^+ \in \ell$ with $H \in \bigcap_{i \in I} F_i^+$, then $SC(X)$ is the only open set in τ^+_{SV} containing H. If $H \in F^+$ and $F^+ \in \tau^+_{SV}$, where $F \neq \widetilde{X}$, then there exists an $\ell \in \mathcal{L}$ such that $H \in \ell \subseteq F^+$. Therefore, ℓ must be of the form $\bigcap_{i \in I} F_i^+ \cap \bigcap_{j \in J} G_j^-$ where I is nonempty. Suppose that $I = \emptyset$. Then, since $H \in \bigcap_{j \in J} G_j^-$, we obtain $(H \sqcup F^c) \in \bigcap_{j \in J} G_j^-$. Also, we know that $(H \sqcup F^c) \notin F^+$. But this contradicts the fact that $\bigcap_{j \in J} G_j^- \subseteq F^+$. Next, observe that we have $F^c \sqsubseteq \bigsqcup_{i \in I} F_i^c$, since otherwise, there would be an $x^e \widetilde{\in} \widetilde{X}$ such that $x^e \widetilde{\in} F^c$, $x^e \widetilde{\notin} \bigsqcup_{i \in I} F_i^c$ and $(H \sqcup x^e) \in \ell - F^+$. Thus, by Proposition 2.20 (i) and (v), we obtain $\bigcap_{i \in I} F_i^+ \subseteq F^+$.

Now, we shall show that there exists a countable local base at H for τ^-_{SV}. Let $H \in G^-$ and

$G^- \in \tau_{SV}^-$, where $G \neq \widetilde{X}$. Since $G^- \in \tau_{SV}$, there exists an $\ell \in \mathcal{L}$ such that $H \in \ell \subseteq G^-$. Without loss of generality we can suppose that in the expression of every element from $\mathcal{L}$ the family J is nonempty $\big($for example, $\bigcap_{i\in I} F_i^+ = \bigcap_{i\in I} F_i^+ \cap \widetilde{X}^-\big)$. So, we have $\ell = \bigcap_{i\in I} F_i^+ \cap \bigcap_{j\in J} G_j^-$. For each $j \in J$, let us take a soft set $K_j = G_j \smile \bigsqcup_{i\in I} F_i^c$ and define $K_\ell = \bigcap_{j\in J} K_j^-$. It easy to see that $H \in K_\ell$. Then, $\mathcal{L}^- = \big\{K_\ell : \ell \in \mathcal{L}\big\}$ forms a countable local base at H for τ_{SV}^-. Indeed, since there exists a $j \in J$ with $K_j \sqsubseteq G$, we obtain $K_\ell \subseteq G^-$. Suppose that for each $j \in J$ there exists an $x_j^{e_j} \widetilde{\in} K_j \smile G$. Therefore, we get $\bigsqcup_{j\in J} x_j^{e_j} \in \ell - G^-$, which yields a contradiction.

Theorem 3.2. *Let (X,τ,E) be a soft T_1-space. Then,*

$(SC(X),\tau_{SV}^-)$ is a first countable space if and only if (X,τ,E) is a soft first countable space and each soft closed set over X is a soft separable.

Proof. Let $(SC(X),\tau_{SV}^-)$ be a first countable space. From the fact that each soft point in X is a soft closed set it follows that (X,τ,E) is a soft first countable space. Let $H \in SC(X)$ and $\{F_n : n \in \mathbb{N}\}$ be a countable family of nonempty soft open sets which determines a countable local base at H for τ_{SV}^-. Because $H \sqcap F_n \neq \widetilde{\emptyset}$ for each $n \in \mathbb{N}$, we may choose a soft point $x_n^{e_n} \widetilde{\in} H \sqcap F_n$. Now, we shall show that $\overline{\bigsqcup_{n\in\mathbb{N}} x_n^{e_n}} = H$. It is easy to see that $\overline{\bigsqcup_{n\in\mathbb{N}} x_n^{e_n}} \sqsubseteq H$. Let $x^e \widetilde{\in} H$. For each $G \in \tau$ containing x^e, we have $G \sqcap H \neq \widetilde{\emptyset}$. Then, there exist $n_1, ..., n_k \in \mathbb{N}$ such that $F_{n_1}^- \cap ... \cap F_{n_k}^- \subseteq G^-$. Therefore, we obtain $F_{n_j} \sqsubseteq G$ for some $j \in \{1,...,k\}$. Thus, since $x_j^{e_j} \widetilde{\in} F_{n_j} \sqsubseteq G$, we get $G \sqcap \big(\bigsqcup_{n\in\mathbb{N}} x_n^{e_n}\big) \neq \widetilde{\emptyset}$ and so that $x^e \widetilde{\in} \overline{\bigsqcup_{n\in\mathbb{N}} x_n^{e_n}}$.

On the other hand, let $H \in SC(X)$. Then, by hypothesis, there exists a family $\{x_n^{e_n} : n \in \mathbb{N}\}$ of countable many soft points in H such that $\overline{\bigsqcup_{n\in\mathbb{N}} x_n^{e_n}} = H$. Now, let $\mathcal{B}(x_n^{e_n})$ be a countable soft local base at $x_n^{e_n}$ for each $n \in \mathbb{N}$. Thus, one can readily verify that $\mathcal{B}(H) = \big\{\bigcap_{j\in J} G_j^- : G_j \in \mathcal{B}(x_j^{e_j}), J\, is\, finite\big\}$ is a countable local base at H for τ_{SV}^-.

For each $H \in SC(X)$, we define $H^* = \{F \in SC(X) : F \sqcap H = \widetilde{\emptyset}\}$. Then, we get the following theorem.

Theorem 3.3. *Let (X,τ,E) be a soft T_1-space. Then,*

$(SC(X),\tau_{SV}^+)$ is a first countable space if and only if for each $H \in SC(X)$, there exists a countable family $\mathcal{L}_H \subseteq H^*$ such that for each $F \in H^*$ there exist $F_1, F_2, ..., F_n \in \mathcal{L}_H$ with $F \sqsubseteq F_1 \sqcup F_2 \sqcup ... \sqcup F_n$.

Proof. The sufficiency is clear.

To prove necessity, let $(SC(X),\tau_{SV}^+)$ be a first countable space and let $H \in SC(X)$. Then, the family

$$\mathcal{H} = \big\{\bigcap_{i\in I} F_i^+ : F_i^c \in H^*, I\, is\, finite\big\} \cup \{SC(X)\}$$

is a countable local base at H for τ_{SV}^+. Now, put

$$\mathcal{L}_H = \big\{G \in H^* : G\, occurs\, in\, some\, element\, of\, \mathcal{H}\big\}.$$

It is easy to see that $\mathcal{L}_H$ is a countable family of H^*. Let $F \in H^*$. Then, we obtain $H \in (F^c)^+$ and $(F^c)^+ \in \tau_{SV}^+$. Therefore, there exists a member $\bigcap_{i=1}^n F_i^+$ of $\mathcal{H}$ such that $H \in \bigcap_{i=1}^n F_i^+ \subseteq (F^c)^+$. By Proposition 2.20 (i) and (v), we have $F \sqsubseteq F_1^c \sqcup ... \sqcup F_n^c$. Thus, it follows from $F_i^c \in \mathcal{L}_H$ for each $i \in \{1,...,n\}$ that the family $\mathcal{L}_H$ has the required properties.

Definition 3.4. *Let (X,τ,E) be a soft topological space, $F \in S(X,E)$ and let $\mathcal{F} = \big\{H : H \sqsubseteq F\big\}$ be a family of non-null soft closed sets. Then, F is called a hemi-SC(X) if there exists a countable cofinal subfamily of $\mathcal{F}$ with respect to inclusion relation.*

Example 3.5. *Let $X = \{x, y, z\}$, $E = \{e_1, e_2\}$ and let F be a soft set over X, where $F = \{(e_1, \{x, z\}), (e_2, \{y\})\}$. Consider a discrete soft topology τ on X and a family*

$$\mathcal{F} = \big\{x^{e_1}, z^{e_1}, y^{e_2}, \{(e_1, \{x\}), (e_2, \{y\})\}, \{(e_1, \{z\}), (e_2, \{y\})\}, \{(e_1, \{x, z\})\}, F\big\}$$

of non-null soft closed sets contained in F. Then, a subfamily $\mathcal{G} = \{F\} \subset \mathcal{F}$ is cofinal in $\mathcal{F}$ since there exists an $F \in \mathcal{G}$ such that $H \sqsubseteq F$ for every $H \in \mathcal{F}$. Thus, F is a hemi-$SC(X)$.

Definition 3.6. *The character of a soft set $F \in S(X, E)$ in a soft topological space (X, τ, E) is defined as the smallest cardinal number of the form $|\mathcal{B}(F)|$, where $\mathcal{B}(F)$ is a soft local base at F for τ. This cardinal number is denoted by $\chi(F)$.*

Example 3.7. *Let $X = \{x, y, z\}$, $E = \{e_1, e_2\}$ and let F be a soft set over X, where $F = \{(e_1, \{z\}), (e_2, \{x, y\})\}$. Let us define a soft topology on X as the following:*

$$\tau = \{G : x^{e_1} \widetilde{\in} G\} \cup \{\widetilde{\emptyset}\}.$$

Then, $\mathcal{B}(F) = \{H\}$ is a soft local base at F for τ, where $H = \{(e_1, \{x, z\}), (e_2, \{x, y\})\}$. Thus, the character of F is $\chi(F) = 1$.

Using the above definitions and theorems, we can easily prove the following corollaries.

Corollary 3.8. *Let (X, τ, E) be a soft T_1-space. Then, the following statements are satisfied:*

(1) $(SC(X), \tau^+_{SV})$ is a first countable space if and only if each soft open set F, where $F \neq \widetilde{X}$, is a hemi-$SC(X)$.

(2) $(SC(X), \tau^+_{SV})$ is a first countable space if and only if $\chi(F) \leq |\mathbb{N}|$ for each $F \in SC(X)$, where $|\mathbb{N}|$ denotes the cardinal number of $\mathbb{N}$.

Corollary 3.9. *Let (X, τ, E) be a soft T_1-space. Then, $(SC(X), \tau_{SV})$ is a first countable space if and only if the following three conditions hold:*

(i) (X, τ, E) is a soft first countable space.
(ii) Each soft closed set over X is a separable.
(iii) Each soft open set over X is a hemi-$SC(X)$.

Lemma 3.10. *Let (X, τ, E) be a soft topological space and let $\{x_n^{e_n} : n \in \mathbb{N}\}$ be a sequence of soft points in X. If $x_n^{e_n} \to x^e \in SP(X)$, then $F = \bigsqcup_{n\in\mathbb{N}} x_n^{e_n} \sqcup x^e$ is a soft compact set.*

Proof. Let $\mathcal{C} = \{F_i : i \in I\}$ be a cover of F by soft open sets over X. Then, there exists an $i_0 \in I$ such that $x^e \widetilde{\in} F_{i_0}$. Since $x_n^{e_n} \to x^e$, there exists an $n_0 \in \mathbb{N}$ such that $x_n^{e_n} \widetilde{\in} F_{i_0}$ for all $n \geq n_0$. Now, let us take an $F_n \in \mathcal{C}$ satisfying $x_n^{e_n} \widetilde{\in} F_n$ for all $n < n_0$. Therefore, we have $x_n^{e_n} \widetilde{\in} \bigsqcup_{i=1}^{n_0-1} F_i$ for all $n < n_0$ and so that $F \sqsubseteq F_{i_0} \sqcup \bigsqcup_{i=1}^{n_0-1} F_i$. Hence, the family $\{F_{i_0}\} \cup \{F_i : i = 1, ..., n_0 - 1\}$ is a finite subcover of $\mathcal{C}$. Thus, from Theorem 2.17 it follows that F is a compact set.

Theorem 3.11. *Let (X, τ, E) be a soft Hausdorff space. If $(SC(X), \tau^+_{SV})$ is a first countable space, then (X, τ, E) is a soft regular space.*

Proof. Let F be a soft open set and $x^e \widetilde{\in} F$. By Theorem 2.15, we shall show that there exists a soft open set G such that $x^e \widetilde{\in} G \sqsubseteq \overline{G} \sqsubseteq F$. Without loss of generality we can suppose that $F \neq \widetilde{X}$. The first countability of $(SC(X), \tau^+_{SV})$ implies that F is a hemi-$SC(X)$. Let $\{F_1, F_2, ..., F_n, ...\}$ be a countable cofinal subfamily in the family $\{H \in SC(X) : H \sqsubseteq F\}$. Also, one can easily verify that (X, τ, E) is a soft first countable space. Let us denote by $\{U_n : n \in \mathbb{N}\}$ a countable soft local base at x^e with $U_n \sqsubseteq F$ for each $n \in \mathbb{N}$. Now, we claim that there exists an $n \in \mathbb{N}$ such that $x^e \widetilde{\in} F_n^o$. Indeed, suppose that $x^e \widetilde{\notin} F_n^o$ for each $n \in \mathbb{N}$. Then, there exists a soft point $x_n^{e_n} \widetilde{\in} U_n \smile F_n$ for each $n \in \mathbb{N}$. Therefore, we see that the sequence $\{x_n^{e_n} : n \in \mathbb{N}\}$ converges to x^e. From Lemma 3.10 it follows that $L = \bigsqcup_{n\in\mathbb{N}} x_n^{e_n} \sqcup x^e$ is a soft compact set. Also, we have $L \sqsubseteq F$ and by Theorem 2.18, we obtain

$L \in SC(X)$. Hence, using the cofinality condition, we get $L \sqsubseteq F_n$ for some $n \in \mathbb{N}$, which yields a contradiction. Thus, there exists an $n \in \mathbb{N}$ such that $x^e \,\widetilde{\in}\, F_n^o \sqsubseteq \overline{F_n^o} \sqsubseteq F$.

Definition 3.12. *Let (X, τ, E) be a soft topological space and $\{x_i^{e_i} : i \in I\}$ a family of infinitely soft points in X. (X, τ, E) is called a countably soft compact space if the soft set $\bigsqcup_{i\in I} x_i^{e_i}$ has a limiting soft point.*

Theorem 3.13. *Let (X, τ, E) be a soft Hausdorff space. If $(SC(X), \tau_{SV}^+)$ is a first countable space, then the derived soft set $\widetilde{X}'$ of $\widetilde{X}$ is a countably soft compact.*

Proof. Suppose that $\widetilde{X}'$ is not countably soft compact. Then, there exists a family $\{x_n^{e_n} : n \in \mathbb{N}\}$ of countable many soft points in $\widetilde{X}'$ such that $F = \bigsqcup_{n\in N} x_n^{e_n}$ does not have a limiting soft point. By Theorem 3.11, (X, τ, E) is a soft regular space and therefore there exists a pairwise disjoint countable family of soft open sets $\{U_n : n \in \mathbb{N}\}$ such that $x_n^{e_n} \,\widetilde{\in}\, U_n$ for every $n \in \mathbb{N}$. Also, from Corollary 3.8(2) it follows that there exists a countable soft local base $\{G_n : n \in \mathbb{N}\}$ of soft open sets at F.

Now, for every $n \in \mathbb{N}$, we can choose a $y_n^{k_n} \in SP(X)$ with $y_n^{k_n} \,\widetilde{\in}\, (U_n \sqcap G_n) \smile F$ because $x_n^{e_n}$ is a limiting soft point of $\widetilde{X}$. Let $G = \bigsqcup_{n\in\mathbb{N}} y_n^{k_n}$. Then, we have $F \sqcap \overline{G} = \widetilde{\emptyset}$. Indeed, suppose that there exists a soft point $x_{n_0}^{e_{n_0}}$ such that $x_{n_0}^{e_{n_0}} \,\widetilde{\in}\, F$ and $x_{n_0}^{e_{n_0}} \,\widetilde{\in}\, \overline{G}$. Take a soft set $H = U_{n_0} \smile y_{n_0}^{k_{n_0}}$. Since the family $\{U_n : n \in \mathbb{N}\}$ is pairwise disjoint, H is a soft open neighborhood of $x_{n_0}^{e_{n_0}}$ such that $H \sqcap G = \widetilde{\emptyset}$. This is a contradiction since $x_{n_0}^{e_{n_0}} \,\widetilde{\in}\, \overline{G}$.

Hence, since $F \sqcap \overline{G} = \widetilde{\emptyset}$, there exists a G_n such that $F \sqsubseteq G_n \sqsubseteq \overline{G}^c$. But, this is a contradiction to the fact that $y_n^{k_n} \,\widetilde{\in}\, \overline{G}$. Thus, $\widetilde{X}'$ is a countably soft compact.

We now consider the second countability of the soft Vietoris topology.

Theorem 3.14. *Let (X, τ, E) be a soft topological space. Then, the following statements are equivalent:*

(1) *$(SC(X), \tau_{SV}^-)$ is a second countable space.*
(2) *(X, τ, E) is a soft second countable space.*

Proof. It is clear from Proposition 2.20 (iii)-(iv) and (vi).

Theorem 3.15. *Let (X, τ, E) be a soft T_1-space. Then, the following statements are equivalent:*

(1) *$(SC(X), \tau_{SV})$ is a second countable space.*
(2) *$(SC(X), \tau_{SV}^+)$ and $(SC(X), \tau_{SV}^-)$ are second countable spaces.*

Proof. (2) $\Rightarrow$ (1) follows immediately from Definition 2.22 and 2.23.

To prove (1) $\Rightarrow$ (2), let $(SC(X), \tau_{SV})$ be a second countable space. From the fact that every soft point in X is a soft closed set it follows that (X, τ, E) is a soft second countable space. Hence, by Theorem 3.14, $(SC(X), \tau_{SV}^-)$ is a second countable space.

Let

$$\mathcal{L} = \left\{ \ell = \bigcap_{i\in I} F_i^+ \cap \bigcap_{j\in J} G_j^- : I, J \textit{ is finite} \right\}$$

be a countable base for τ_{SV}. Then, the family

$$\mathcal{L}^+ = \left\{ \bigcap_{i\in I} F_i^+ : \bigcap_{i\in I} F_i^+ \textit{ occurs in some } \ell \in \mathcal{L} \right\} \cup \{SC(X)\}$$

forms a countable base for τ_{SV}^+. Indeed, if there is no $\bigcap_{i\in I} F_i^+ \in \ell$ with $H \in \bigcap_{i\in I} F_i^+$, then $SC(X)$ is the only open set in τ_{SV}^+ containing H. Let $H \in F^+$ and $F^+ \in \tau_{SV}^+$, where $F \neq \widetilde{X}$. Since $F^+ \in \tau_{SV}$, there exists an $\ell \in \mathcal{L}$ such that $H \in \ell \subseteq F^+$. Therefore, ℓ must be of the form $\bigcap_{i\in I} F_i^+ \cap \bigcap_{j\in J} G_j^-$ where I is nonempty (see the proof of Theorem 3.1). Thus, as is shown in the proof of Theorem 3.1, we get $\bigcap_{i\in I} F_i^+ \subseteq F^+$, which completes the proof.

Theorem 3.16. *Let (X,τ,E) be a soft T_1-space. Then,*

$(SC(X),\tau_{SV}^+)$ is a second countable space if and only if there exists a countable family $\Delta \subseteq SC(X)$ such that for each $F\in SC(X)$ and for each $G\in\tau$ satisfying $F\sqsubseteq G$ there exist $F_1,F_2,...,F_n\in\Delta$ with $F\sqsubseteq F_1\sqcup F_2\sqcup ...\sqcup F_n\sqsubseteq G$.

Proof. Let $\mathcal{L}$ be a countable base for τ_{SV}^+. We know that every element in $\mathcal{L}$ can be written as $\bigcap\left\{H_j^+ : j\in J\right\}$, where J is a finite set. Take

$$\Delta = \left\{F\in SC(X) : F\ occurs\ in\ the\ presentation\ of\ some\ element\ from\ \mathcal{L}\right\}.$$

One can readily verify that Δ is a countable family of $SC(X)$. Let $F\in SC(X)$ and $G\in\tau$ such that $F\sqsubseteq G$. If $F=\widetilde{X}$, then we are done. So, suppose that $F\neq\widetilde{X}$ and also $G\neq\widetilde{X}$. Now, let us take a soft set $K=G^c$. Then, we have $K\in SC(X)$ and $K\in (F^c)^+$. Therefore, there exists a member $\bigcap_{j\in J} H_j^+$ of $\mathcal{L}$ such that $K\in\bigcap_{j\in J} H_j^+\subseteq (F^c)^+$. Hence, from Proposition 2.20 (i) and (v) it follows that

$$F\sqsubseteq \bigsqcup_{j\in J} H_j^c \sqsubseteq K^c = G.$$

Thus, since $H_j^c\in\Delta$ for each $j\in J$, the family $\Delta\subseteq SC(X)$ has the required properties.

For the converse, the family of sets of the form $\bigcap\left\{(F_j^c)^+ : j\in J\right\}$, where $F_j\in\Delta$ and J is a finite set, together with $SC(X)$ is a countable base for τ_{SV}^+.

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