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ON THE GENERALIZED OSTROWSKI TYPE INTEGRAL INEQUALITY FOR DOUBLE INTEGRALS

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Abstract. In this paper, we establish a new generalized Ostrowski type inequality for double integrals involving functions of two independent variables by using fairly elementary analysis.

1. Introduction

In 1938, the classical integral inequality was established by Ostrowski [5] as follows:

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f^{'}:(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty=\sup\limits_{t\in(a,b)}|f'(t)|<\infty$. Then, the inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

In a recent paper [3], Barnett and Dragomir proved the following Ostrowski type inequality for double integrals:

Theorem 1.2. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be continuous on $[a,b]\times[c,d]$, $f''_{x,y}=\frac{\partial^2 f}{\partial x\partial y}$ exists on $(a,b)\times(c,d)$ and is bounded, i.e.,*

$$\left\|f''_{x,y}\right\|_\infty=\sup_{(x,y)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right|<\infty.$$

Then, we have the inequality:

$$\begin{aligned}&\left|\int_a^b\int_c^d f(s,t)dtds-(d-c)(b-a)f(x,y)\right.\\&\qquad\left.-\left[(b-a)\int_c^d f(x,t)dt+(d-c)\int_a^b f(s,y)ds\right]\right| \\ &\leq\left[\frac{1}{4}(b-a)^2+(x-\frac{a+b}{2})^2\right]\left[\frac{1}{4}(d-c)^2+(y-\frac{d+c}{2})^2\right]\left\|f''_{x,y}\right\|_\infty\end{aligned} \tag{1.2}$$

for all $(x,y)\in[a,b]\times[c,d]$.

In [3], the inequality (1.2) is established by the use of integral identity involving Peano kernels. In [7], Pachpatte obtained an inequality in the view (1.2) by using elementary analysis. The interested reader is also refered to ([3], [4], [6]- [13]) for Ostrowski type inequalities in several independent variables and for recent weighted version of these type inequalities see [1], [2], [9] and [11].

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Meanwhile, in [11] Sarikaya and Ogunmez gave the following interesting identity and by using this indentity they establised some interesting integral inequalities:

Lemma 1.1. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be an absolutely continuous function such that the partial derivative of order $\frac{\partial^2 f(t,s)}{\partial t \partial s}$ exists for all $(t,s) \in [a,b] \times [c,d]$ and the weight function $w : [a,b] \to [0,\infty)$ is integrable, nonnegative and*

$$m(a,b) = \int_a^b w(t)dt < \infty. \tag{1.3}$$

Then, we have

$$\begin{aligned} f(x,y) &= \frac{1}{m(a,b)} \int_a^b w(t) f(t,y) dt + \frac{1}{m(c,d)} \int_c^d w(s) f(x,s) ds \\ &\quad - \frac{1}{m(a,b)m(c,d)} \left[\int_a^b \int_c^d w(t)w(s) f(t,s) dsdt - \int_a^b \int_c^d p(x,t) q(y,s) \frac{\partial^2 f(t,s)}{\partial t \partial s} dsdt \right] \end{aligned} \tag{1.4}$$

where

$$p(x,t) = \begin{cases} p_1(a,t) = \int_a^t w(u)du, & a \le t < x \\ \\ p_2(b,t) = \int_b^t w(u)du, & x \le t \le b \end{cases}$$

and

$$q(y,s) = \begin{cases} q_1(c,s) = \int_c^s w(v)dv, & c \le s < y \\ \\ q_2(d,s) = \int_d^s w(v)dv, & y \le s \le d. \end{cases}$$

The main aim of this paper is to establish a new generalized Ostrowski type inequality for double integrals involving functions of two independent variables and their partial derivatives.

2. Main Result

We begin with the following important result:

Lemma 2.1. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be an absolutely continuous function such that the partial derivative of order $\frac{\partial^2 f(t,s)}{\partial t \partial s}$ exists for all $(t,s) \in [a,b] \times [c,d]$, and the function $p : [a,b] \times [c,d] \to [0,\infty)$ is integrable. Then, we have*

$$\begin{aligned} &\left(\int_a^b \int_c^d p(u,v) dvdu \right) f(x,y) - \int_a^b \int_c^d p(t,v) f(t,y) dvdt \\ &- \int_a^b \int_c^d p(u,s) f(x,s) dsdu + \int_a^b \int_c^d p(t,s) f(t,s) dsdt \\ = &\int_a^b \int_c^d P(x,t;y,s) \frac{\partial^2 f(t,s)}{\partial t \partial s} dsdt \end{aligned} \tag{2.1}$$

where

$$P(x,t;y,s)=\begin{cases}\int_a^t\int_c^s p(u,v)dvdu, & a\le t<x,\ c\le s<y\\ \int_a^t\int_d^s p(u,v)dvdu, & a\le t<x,\ y\le s\le d\\ \int_b^t\int_c^s p(u,v)dvdu, & x\le t\le b,\ c\le s<y\\ \int_b^t\int_d^s p(u,v)dvdu, & x\le t\le b,\ \ y\le s\le d.\end{cases}$$

Proof. By definitions of $P(x,t;y,s)$, we have

$$\begin{aligned}&\int_a^b\int_c^d P(x,t;y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt=\int_a^x\int_c^y\left[\int_a^t\int_c^s p(u,v)dvdu\right]\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\\ &+\int_a^x\int_y^d\left[\int_a^t\int_d^s p(u,v)dvdu\right]\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt+\int_x^b\int_c^y\left[\int_b^t\int_c^s p(u,v)dvdu\right]\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\\ &+\int_x^b\int_y^d\left[\int_b^t\int_d^s p(u,v)dvdu\right]\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt.\end{aligned}$$

Integrating by parts, we can state:

$$\begin{aligned}&\int_a^x\int_c^y\left[\int_a^t\int_c^s p(u,v)dvdu\right]\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\\ &=\int_a^x\left[\left(\int_a^t\int_c^y p(u,v)dvdu\right)\frac{\partial f(t,y)}{\partial t}-\int_c^y\left(\int_a^t p(u,s)du\right)\frac{\partial f(t,s)}{\partial t}ds\right]dt\\ &=\left(\int_a^x\int_c^y p(u,v)dvdu\right)f(x,y)-\int_a^x\left(\int_c^y p(t,v)dv\right)f(t,y)dt\\ &-\int_c^y\left(\int_a^x p(u,s)du\right)f(x,s)ds+\int_a^x\int_c^y p(t,s)f(t,s)dsdt,\end{aligned}\tag{2.2}$$

$$\begin{aligned}&\int_a^x\int_y^d\left[\int_a^t\int_d^s p(u,v)dvdu\right]\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\\ &=-\int_a^x\left[\left(\int_a^t\int_d^y p(u,v)dvdu\right)\frac{\partial f(t,y)}{\partial t}+\int_y^d\left(\int_a^t p(u,s)du\right)\frac{\partial f(t,s)}{\partial t}ds\right]dt\\ &=\left(\int_a^x\int_y^d p(u,v)dvdu\right)f(x,y)-\int_a^x\left(\int_y^d p(t,v)dv\right)f(t,y)dt\\ &-\int_y^d\left(\int_a^x p(u,s)du\right)f(x,s)ds+\int_a^x\int_y^d p(t,s)f(t,s)dsdt,\end{aligned}\tag{2.3}$$

$$\begin{aligned}&\int_x^b\int_c^y\left[\int_b^t\int_c^s p(u,v)dvdu\right]\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\\ &=-\int_x^b\left[\left(\int_t^b\int_c^y p(u,v)dvdu\right)\frac{\partial f(t,y)}{\partial t}-\int_c^y\left(\int_t^b p(u,s)du\right)\frac{\partial f(t,s)}{\partial t}ds\right]dt\\ &=\left(\int_x^b\int_c^y p(u,v)dvdu\right)f(x,y)-\int_x^b\left(\int_c^y p(t,v)dv\right)f(t,y)dt\\ &-\int_c^y\left(\int_x^b p(u,s)du\right)f(x,s)ds+\int_x^b\int_c^y p(t,s)f(t,s)dsdt,\end{aligned}\tag{2.4}$$

$$\begin{aligned}
&\int_x^b\int_y^d\left[\int_b^t\int_d^s p(u,v)dvdu\right]\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\\
&=\int_x^b\left[\left(\int_t^b\int_y^d p(u,v)dvdu\right)\frac{\partial f(t,y)}{\partial t}+\int_y^d\left(\int_t^b p(u,s)du\right)\frac{\partial f(t,s)}{\partial t}ds\right]dt\\
&=\left(\int_x^b\int_y^d p(u,v)dvdu\right)f(x,y)-\int_x^b\left(\int_y^d p(t,v)dv\right)f(t,y)dt\\
&-\int_y^d\left(\int_x^b p(u,s)du\right)f(x,s)ds+\int_x^b\int_y^d p(t,s)f(t,s)dsdt.
\end{aligned}\tag{2.5}$$

Adding (2.2)-(2.5) and rewriting, we easily deduce required identity (2.1) which completes the proof. □

Remark 2.1. *If take $p(.,.)\equiv 1$ in Lemma 2.1, we get*

$$\begin{aligned}
&f(x,y)-\frac{1}{(b-a)}\int_a^b f(t,y)dt\\
&-\frac{1}{(d-c)}\int_c^d f(x,s)dsdu+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\\
=&\ \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d P(x,t;y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt
\end{aligned}$$

where

$$P(x,t;y,s)=\begin{cases}(t-a)(s-c), & a\le t<x,\ c\le s<y\\ (t-a)(s-d), & a\le t<x,\ y\le s\le d\\ (t-b)(s-c), & x\le t\le b,\ c\le s<y\\ (t-b)(s-d), & x\le t\le b,\ \ y\le s\le d.\end{cases}$$

which is given by Barnett and Dragomir in [3].

Remark 2.2. *If take $p(u,v)=w(u)w(v)$ in Lemma 2.1, then the Lemma 2.1 reduces to the Lemma 1.1 which is proved by Sarikaya and Ogunmez in [11].*

Theorem 2.1. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be an absolutely continuous function such that the partial derivative of order $\frac{\partial^2 f(t,s)}{\partial t\partial s}$ exists and is bounded, i.e.,*

$$\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty=\sup_{(t,s)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right|<\infty$$

for all $(t,s)\in[a,b]\times[c,d]$, the function $p:[a,b]\times[c,d]\to[0,\infty)$ is integrable. Then, we have

$$\begin{aligned}
&\left|\left(\int_a^b\int_c^d p(u,v)dvdu\right)f(x,y)-\int_a^b\int_c^d p(t,v)f(t,y)dvdt\right.\\
&\left.-\int_a^b\int_c^d p(u,s)f(x,s)dsdu+\int_a^b\int_c^d p(t,s)f(t,s)dsdt\right|\\
\le&\ \left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty\int_a^x(x-u)A(u,y)du+\int_x^b(u-x)A(u,y)du
\end{aligned}\tag{2.6}$$

where

$$A(u,y)=\int_c^y(y-v)\left|p(u,v)\right|dv+\int_y^d(v-y)\left|p(u,v)\right|dv.$$

Proof. From Lemma 2.1 and using the properties of modulus, we observe that

$$\begin{aligned}
&\left|\left(\int_a^b\int_c^d p(u,v)dvdu\right)f(x,y)-\int_a^b\int_c^d p(t,v)f(t,y)dvdt\right.\\
&\left.-\int_a^b\int_c^d p(u,s)f(x,s)dsdu+\int_a^b\int_c^d p(t,s)f(t,s)dsdt\right|\\
\leq\ &\int_a^b\int_c^d |P(x,t;y,s)|\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right|dsdt \qquad (2.7)\\
\leq\ &\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty\int_a^b\int_c^d |P(x,t;y,s)|\,dsdt\\
\leq\ &\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty\left\{\int_a^x\int_c^y\left[\int_a^t\int_c^s |p(u,v)|\,dvdu\right]dsdt\right.\\
&+\int_a^x\int_y^d\left[\int_a^t\int_s^d |p(u,v)|\,dvdu\right]dsdt+\int_x^b\int_c^y\left[\int_t^b\int_c^s |p(u,v)|\,dvdu\right]dsdt\\
&\left.+\int_x^b\int_y^d\left[\int_t^b\int_s^d |p(u,v)|\,dvdu\right]dsdt\right\}\\
\leq\ &\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty\left\{J_1+J_2+J_3+J_4\right\}.
\end{aligned}$$

Now, using the change of order of integration we get

$$\begin{aligned}
J_1 &= \int_a^x\int_c^y\left[\int_a^t\int_c^s |p(u,v)|\,dvdu\right]dsdt\\
&= \int_a^x\int_a^t\left[\int_c^y\int_c^s |p(u,v)|\,dvds\right]dudt\\
&= \int_a^x\int_a^t\left[\int_c^y (y-v)\,|p(u,v)|\,dv\right]dudt\\
&= \int_c^y\left[\int_a^x\int_a^t (y-v)\,|p(u,v)|\,dudt\right]dv \qquad (2.8)\\
&= \int_a^x\int_c^y (x-u)\,(y-v)\,|p(u,v)|\,dvdu
\end{aligned}$$

and similarly,

$$J_2=\int_a^x\int_y^d (x-u)\,(v-y)\,|p(u,v)|\,dvdu, \qquad (2.9)$$

$$J_3=\int_x^b\int_c^y (u-x)\,(y-v)\,|p(u,v)|\,dvdu, \qquad (2.10)$$

$$J_4=\int_x^b\int_y^d (u-x)(v-y)\left|p(u,v)\right|dvdu. \tag{2.11}$$

Thus, using (2.8), (2.9), (2.10) and (2.11) in (2.7), we obtain the inequality (2.6) and the proof is completed. □

Remark 2.3. *If we choose $p(.,.)\equiv 1$ in Theorem 2.1, then the inequality (2.6) reduces the inequality (1.2) which is proved by Barnett and Dragomir in [3].*

Remark 2.4. *If take $p(u,v)=w(u)w(v)$ in Theorem 2.1, then the inequality (2.6) reduces*

$$\begin{aligned}
&\Bigg| f(x,y)-\frac{1}{m(a,b)}\int_a^b w(t)f(t,y)dt \\
&-\frac{1}{m(c,d)}\int_c^d w(s)f(x,s)ds+\frac{1}{m(a,b)m(c,d)}\int_a^b\int_c^d w(s)w(t)f(t,s)dsdt\Bigg| \\
\leq\;& \left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty \int_a^x (x-u)A(u,y)du+\int_x^b (u-x)A(u,y)du
\end{aligned}$$

where

$$A(u,y)=\int_c^y (y-v)w(u)w(v)dv+\int_y^d (v-y)w(u)w(v)dv.$$

which is proved by Sarikaya and Ogunmez in [11].

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