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BEURLING'S THEOREM AND $L^p - L^q$ MORGAN'S THEOREM FOR THE GENERALIZED BESSEL-STRUVE TRANSFORM

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Abstract. The generalized Bessel-Struve transform satisfies some uncertainty principles similar to the Euclidean Fourier transform. A generalization of Beurling's theorem and $L^p - L^q$ Morgan's theorem obtained for the generalized Bessel-Struve transform.

1. Introduction and Preliminaries

There are many theorems known which state that a function and its classical Fourier transform on $\mathbb{R}$ cannot both be sharply localized. That is, it is impossible for a nonzero function and its Fourier transform to be simultaneously small. Here a concept of the smallness had taken different interpretations in different contexts. Morgan [5] and Beurling [3] for example interpreted the smallness as sharp pointwise estimates or integrable decay of functions. In particular Beurling's theorem, which was found by Beurling and his proof was published much later by Hörmander [4], says that

Theorem 1. *If $f \in L^2(\mathbb{R})$ satisfies that*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)||\widehat{f}(y)|e^{|x||y|}dxdy < \infty,$$

then $f = 0$ a.e.

Morgan [5] has established a famous theorem stating that for $\gamma > 2$ and $\eta = \frac{\gamma}{\gamma-1}$, if $(a\gamma)^{\frac{1}{\gamma}}(b\eta)^{\frac{1}{\eta}} > (\sin(\frac{\pi}{2}(\eta-1))^{\frac{1}{\eta}}$, $e^{a|x|^\gamma} f \in L^\infty(\mathbb{R})$ and $e^{b|x|^\eta}\mathcal{F}(f) \in L^\infty(\mathbb{R})$. then f is null almost everywhere. S. Ben Farah and K. Mokni [2] have generalized Morgan's theorem to an $L^p - L^q-$version where $1 \leq p, q \leq +\infty$.

The outline of the content of this paper is as follows. In section 2 we give an analogue of Beurling's theorem for $\mathcal{F}^{B,S}_{\alpha,n}$. Section 3 is devoted to $L^p - L^q$-Morgan's theorem for $\mathcal{F}^{B,S}_{\alpha,n}$.

Let us now be more precise and describe our results. To do so, we need to introduce some notations. Throughout this paper, the letter C indicates a positive constant not necessarily the same in each occurrence. We denote by

- $$a_\alpha = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} \tag{1}$$
  where $\alpha > \frac{-1}{2}$.
- $\mathcal{M}_n$ the map defined by $\mathcal{M}_n(f(x)) = x^{2n}f(x)$.
- $L^p_\alpha(\mathbb{R})$ the class of measurable functions f on $\mathbb{R}$ for which $\|f\|_{p,\alpha} < \infty$, where
  $$\|f\|_{p,\alpha} = \left(\int_{\mathbb{R}} |f(x)|^p|x|^{2\alpha+1}dx\right)^{\frac{1}{p}}, \quad if p < \infty,$$
  and $\|f\|_{\infty,\alpha} = \|f\|_\infty = ess\ sup_{x\geq 0}|f(x)|$.
- $L^p_{\alpha,n}(\mathbb{R})$ the class of measurable functions f on $\mathbb{R}$ for which
  $$\|f\|_{p,\alpha,n} = \|\mathcal{M}_n^{-1}f\|_{p,\alpha+2n} < \infty.$$

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- K_0 the space of functions f infinitely differentiable on $\mathbb{R}^*$ with bounded support verifying for all $n \in \mathbb{N}$,
$$\lim_{y\to 0^-} y^n f^{(n)}(y) \quad and \quad \lim_{y\to 0^+} y^n f^{(n)}(y)$$
exist.
- $\frac{d}{dx^2} = \frac{1}{2x}\frac{d}{dx}$, where $\frac{d}{dx}$ is the first derivative operator.

In this section we recall some facts about harmonic analysis related to the generalized Bessel-Struve operator $\mathcal{F}_{B,S}^{\alpha,n}$. We cite here, as briefly as possible, only some properties. For more details we refer to [1].
For $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$, put

$$\Psi_{\lambda,\alpha,n}(x) = a_{\alpha+2n} x^{2n} \int_0^1 (1-t^2)^{\alpha+2n-\frac{1}{2}} e^{\lambda x t} dt.$$

$\Psi_{\lambda,\alpha,n}$ satisfies

$$\forall \xi \in \mathbb{R},\ \forall \zeta \in \mathbb{R}, \forall x \in \mathbb{R}\ |\Psi_{\lambda,\alpha,n}(i(\xi+i\zeta)x)| \leq x^{2n} e^{|x||\zeta|}. \tag{2}$$

$$\forall n \in \mathbb{N},\ \forall \lambda \in \mathbb{R},\ \forall x \in \mathbb{R},\ \ |\frac{d^n}{dx^n}(x^{-2n}\Psi_{i\lambda,\alpha,n}(x))| \leq |\lambda|^n. \tag{3}$$

Definition 1. *The Generalized Bessel-Struve transform is defined on $L^1_{\alpha,n}(\mathbb{R})$ by*

$$\forall \lambda \in \mathbb{R}, \quad \mathcal{F}_{B,S}^{\alpha,n}(f)(\lambda) = \int_{\mathbb{R}} f(x)\Psi_{-i\lambda,\alpha,n}(x)|x|^{2\alpha+1}dx.$$

Definition 2. *For $f \in L^1_{\alpha,n}(\mathbb{R})$ with bounded support, the integral transform $W_{\alpha,n}$, given by*

$$W_{\alpha,n}(f(x)) = a_{\alpha+2n} \int_{|x|}^{+\infty} (y^2-x^2)^{\alpha+2n-\frac{1}{2}} y^{1-2n} f(sgn(x)y)dy, \quad x \in \mathbb{R}\backslash\{0\}$$

is called the generalized Weyl integral transform associated with Bessel-Struve operator.

Proposition 1. *$W_{\alpha,n}$ is a bounded operator from $L^1_{\alpha,n}(\mathbb{R})$ to $L^1(\mathbb{R})$, where $L^1(\mathbb{R})$ is the space of lebesgue-integrable functions.*

Remark 1. *From Proposition 1 we can find a constant C such that*

$$\int_{\mathbb{R}} |W_{\alpha,n}(f)(x)|dx \quad \leq \quad C\|f\|_{\alpha,n,1}$$

Proposition 2. *If $f \in L^1_{\alpha,n}(\mathbb{R})$ then*

$$\mathcal{F}_{B,S}^{\alpha,n} = \mathcal{F} \circ W_{\alpha,n}, \tag{4}$$

where $\mathcal{F}$ is the classical Fourier transform defined on $L^1(\mathbb{R})$ by

$$\mathcal{F}(g)(\lambda) = \int_{\mathbb{R}} g(x)e^{-i\lambda x}dx.$$

Definition 3. *Let $\alpha = k+\frac{1}{2}$ where $k \in \mathbb{N}$. We define the operator $V_{\alpha,n}$ on K_0 as follows*

$$V_{\alpha,n}f(x) = (-1)^{k+1}\frac{2^{2k+4n+1}(k+2n)!}{(2k+4n+1)!}x^{2n}(\frac{d}{dx^2})^{k+2n+1}(f(x)), \quad x \in \mathbb{R}^*.$$

Theorem 2. *Let $f \in K_0$, $V_{\alpha,n}$ and $W_{\alpha,n}$ are related by the following relation*

$$V_{\alpha,n}(W_{\alpha,n}(f)) = f.$$

2. Beurling's theorem for the Generalized Bessel-Struve transform

In this section we will prove Beurling's theorem for the Generalized Bessel-Struve transform.

Theorem 3. *Let $k \in \mathbb{N}$, $\alpha = k + \frac{1}{2}$ and $f \in L^2_{\alpha,n}(\mathbb{R})$ satisfy*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)||\mathcal{F}^{\alpha,n}_{B,S}(f)(y)| e^{|x||y|} |x|^{2(\alpha+n)+1} dx dy < \infty, \tag{5}$$

then $f = 0$ almost everywhere.

Proof. We start with the following lemma.

Lemma 1. *We suppose that $f \in L^2_{\alpha,n}(\mathbb{R})$ satisfies (5), then $f \in L^1_{\alpha,n}(\mathbb{R})$.*

Proof. We may assume that $f \neq 0$ in $L^2_{\alpha,n}(\mathbb{R})$. (5) and the Fubini theorem for almost every $y \in \mathbb{R}$

$$|\mathcal{F}^{\alpha,n}_{B,S}(f)(y)| \int_{\mathbb{R}} |f(x)| e^{|x||y|} |x|^{2(\alpha+n)+1} dx < \infty,$$

since $\mathcal{F}^{\alpha,n}_{B,S}(f) \neq 0$, there exist $y_0 \in \mathbb{R}, y_0 \neq 0$ such that $\mathcal{F}^{\alpha,n}_{B,S}(f)(y_0) \neq 0$,
therefore

$$\int_{\mathbb{R}} |f(x)| e^{|x||y_0|} |x|^{2(\alpha+n)+1} dx < \infty$$

$$\int_{\mathbb{R}} \frac{|f(x)|}{x^{2n}} e^{|x||y_0|} |x|^{2(\alpha+2n)+1} dx < \infty$$

$$\int_{\mathbb{R}} |\mathcal{M}^{-1}_n f(x)| e^{|x||y_0|} |x|^{2(\alpha+2n)+1} dx < \infty,$$

since $e^{|x||y_0|} \geqslant 1$ for large $|x|$ it follows $\int_{\mathbb{R}} |\mathcal{M}^{-1}_n f(x)||x|^{2(\alpha+2n)+1} dx < \infty$. ■
This lemma and proposition 1 imply that $W_{\alpha,n}(f)$ is well-defined a.e on $\mathbb{R}$.
By Remark 1 we can find a positif constant C such that

$$\begin{aligned}
\int_{\mathbb{R}} |W_{\alpha,n}(f)(x)| dx &\leq C \|f\|_{\alpha,n,1} \\
&\leq C \|\mathcal{M}^{-1}_n f\|_{\alpha+2n,1} \\
&\leq C \int_{\mathbb{R}} |\mathcal{M}^{-1}_n f(x)||x|^{2(\alpha+2n)+1} dx \\
&\leq C \int_{\mathbb{R}} |\frac{f(x)}{x^{2n}}||x|^{2(\alpha+2n)+1} dx \\
&\leq C \int_{\mathbb{R}} |f(x)||x|^{2(\alpha+n)+1} dx.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{\mathbb{R}} \int_{\mathbb{R}} |W_{\alpha,n}(f(x))||\mathcal{F}^{\alpha,n}_{B,S}(f)(y)| e^{|x||y|} dx dy &\leq C \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)||\mathcal{F}^{\alpha,n}_{B,S}(f)(y)| e^{|x||y|} |x|^{2(\alpha+n)+1} dx dy \\
&< \infty.
\end{aligned}$$

It follows from Proposition 2 that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |W_{\alpha,n}(f(x))||\mathcal{F} \circ W_{\alpha,n}(f)(y)| e^{|x||y|} dx dy < \infty.$$

According to Theorem 1, we can deduce that

$$W_{\alpha,n}(f) = 0,$$

applying Lemma 1 we obtain

$$f = V_{\alpha,n} \circ W_{\alpha,n}(f) = 0.$$

■

Corollaire 1. *(Gelfand-Shilov) If $f \in L^2_{\alpha,n}(\mathbb{R})$ such that*

$$\int_{\mathbb{R}} |f(x)| e^{\frac{|x|^p}{p}} |x|^{2(\alpha+n)+1} dx < \infty, \qquad \int_{\mathbb{R}} |\mathcal{F}^{\alpha,n}_{B,S}(f)(y)| e^{\frac{|y|^q}{q}} dy < \infty$$

then $f = 0$.

Proof. Let M and M^* be functions satisfying

$$xy \leq M(x) + M^*(y). \tag{6}$$

If

$$\int_{\mathbb{R}} |f(x)| e^{M(x)} |x|^{2(\alpha+n)+1} dx < \infty, \qquad \int_{\mathbb{R}} |\mathcal{F}^{\alpha}_{B,S}(f)(y)| e^{M^*(y)} dy < \infty$$

then

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)| |\mathcal{F}^{\alpha,n}_{B,S}(f)(y)| e^{|x||y|} |x|^{2(\alpha+n)+1} dx dy \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)| |\mathcal{F}^{\alpha,n}_{B,S}(f)(y)| e^{M(x)+M^*(y)} |x|^{2(\alpha+n)+1} dx dy$$
$$= \int_{\mathbb{R}} |f(x)| e^{M(x)} |x|^{2(\alpha+n)+1} dx \int_{\mathbb{R}} |\mathcal{F}^{\alpha,n}_{B,S}(f)(y)| e^{M^*(y)} dy < \infty.$$

Consequently, Beurling's Theorem implies that $f(y) = 0$. In particular, if $M(x) = \frac{|x|^p}{p}$ and $M^*(y) = \frac{|y|^q}{q}$, where p, q are conjugate exponents $p^{-1} + q^{-1} = 1$, then the pair $(M,\, M^*)$ satisfies the condition (6). Thus, we obtain an analogue of the Gelfand-Shilov uncertainty principle for the Bessel-Struve transform. ■

3. $L^p - L^q$ MORGAN'S THEOREM FOR THE GENERALIZED BESSEL-STRUVE TRANSFORM

In this section, we prove $L^p - L^q$ Morgan's theorem for the Generalized Bessel-Struve transform.

Lemma 2. *We assume that $\rho \in]1, 2[$, $q \in [1, \infty]$, $\sigma > 0$ and $B > \sigma \sin(\frac{\pi}{2}(\rho - 1))$. If g is an entire function on $\mathbb{C}$ verifying*

$$|g(x + iy)| \leq C e^{\sigma |y|^\rho} \in L^p_{\alpha+2n}(\mathbb{R})$$
$$e^{B|x|^\rho} g_{|\mathbb{R}} \in L^q_{\alpha+2n}(\mathbb{R})$$

for all $x,\ y \in \mathbb{R}$, then $g = 0$.

Proof. See [2]. ■

Lemma 3. *Let $p \in [1, \infty]$, $\gamma > 2$ and f a measurable function on $\mathbb{R}$ verifying*

$$\forall a > 0,\ e^{a|x|^\gamma} f \in L^p_{\alpha,n}(\mathbb{R}). \tag{7}$$

Then the function defined on $\mathbb{C}$ by

$$\mathcal{F}^{\alpha,n}_{B,S}(f)(z) = \int_{\mathbb{R}} f(x) \Psi_{-iz,\alpha,n}(x) |x|^{2\alpha+1} dx \tag{8}$$

is well defined and entire on $\mathbb{C}$. Moreover, we have

$$\forall \xi, \zeta \in \mathbb{R},\ |\mathcal{F}^{\alpha,n}_{B,S}(f)(\xi + i\zeta)| \leq \int_{\mathbb{R}} |f(x)| e^{|x||\zeta|} |x|^{2\alpha+1} dx. \tag{9}$$

Proof. Relation (7) assert that $\mathcal{F}^{\alpha,n}_{B,S}(f)$ is well defined. Applying again relation (7), the analytic theorem on (8) and the fact that $\Psi_{-i\lambda,\alpha,n}(x)$ verifies (9), we deduce that $z \to \mathcal{F}^{\alpha,n}_{B,S}(f)(z)$ is an entire on $\mathbb{C}$. The relation (10) is obtained from relation (2). ■

Theorem 4. *Let $p, q \in [1, \infty]$, $a > 0$, $b > 0$, $\gamma > 2$ and $\eta = \frac{\gamma}{\gamma - 1}$. Suppose that f a measurable function on $\mathbb{R}$ such that*

$$e^{a|x|^\gamma} f \in L^p_{\alpha,n}(\mathbb{R}) \text{ and } e^{b|x|^\eta} \mathcal{F}^{\alpha,n}_{B,S}(f) \in L^q_{\alpha+2n}(\mathbb{R}). \tag{10}$$

If $(a\gamma)^{\frac{1}{\gamma}} (b\eta)^{\frac{1}{\eta}} > (\sin(\frac{\pi}{2}(\eta - 1)))^{\frac{1}{\eta}}$, then f is null almost everywhere.

Proof. We notice that $e^{a|x|^\gamma} f \in L^p_{\alpha,n} \Leftrightarrow e^{a|x|^\gamma} \mathcal{M}_n^{-1} f \in L^p_{\alpha+2n}$.
First case: $1 < p < \infty$. Applying Hölder inequality, we get

$$|\mathcal{F}^{\alpha,n}_{B,S}(f)(\xi + i\zeta)| \leq \|f\|_{\alpha,n,p} \left(\int_{\mathbb{R}} e^{-ap'|x|^\gamma} e^{|x||\zeta|p'} |x|^{2\alpha+1} dx \right)^{\frac{1}{p'}}$$

where p' verifies $\frac{1}{p'} + \frac{1}{p} = 1$.
Now, we take $C \in](b\eta)^{-\eta} \sin(\frac{\pi}{2}(\eta - 1))^{\frac{1}{\eta}}, (a\gamma)^{\frac{1}{\gamma}}[$. Using a convexity's inequality, we obtain

$$|x||\zeta| \leq \frac{C^\gamma}{\gamma} |x|^\gamma + \frac{1}{\eta C^\eta} |\zeta|^\eta \tag{11}$$

and the following relation holds

$$\int_{\mathbb{R}} e^{-ap'|x|^\gamma} e^{|x||\zeta|p'} |x|^{2\alpha+1} dx \leq e^{\frac{p'|\zeta|^\eta}{\eta C^\eta}} \int_{\mathbb{R}} e^{-p'(a - \frac{C^\gamma}{\gamma}|x|^\gamma)} |x|^{2\alpha+1} dx.$$

So we get

$$\forall \xi, \zeta \in \mathbb{R}, \ |\mathcal{F}^{\alpha,n}_{B,S}(f)(\xi + i\zeta)| \leq const \ e^{\frac{1}{\eta C^\eta}|\zeta|^\eta}.$$

Second case: $p = 1$ or $p = +\infty$. From relations (2) and (11), we get

$$|\mathcal{F}^{B,S}_{\alpha,n}(f)(\xi + i\zeta)| \leq e^{\frac{1}{\eta C^\eta}|\xi|^\eta} \int_{\mathbb{R}} e^{a|x|^\gamma} |f(x)| e^{-(a - \frac{C^\gamma}{\gamma})} |x|^\gamma |x|^{2\alpha+1} dx.$$

Therefore

$$|\mathcal{F}^{B,S}_{\alpha,n}(f)(\xi + i\zeta)| \leq const \ e^{\frac{1}{\eta C^\eta}|\xi|^\eta}.$$

Hence

$$\forall p \in [1, \infty], \ \forall \xi, \zeta \in \mathbb{R}, \ |\mathcal{F}^{\alpha,n}_{B,S}(f)(\xi + i\zeta)| \leq const \ e^{\frac{1}{\eta C^\eta}|\zeta|^\eta}. \tag{12}$$

By virtue of relations (10), (12) and Lemma 2, we obtain that $\mathcal{F}^{\alpha,n}_{B,S} f = 0$. The injectivity of the generalized Bessel-Struve transform implies that $f = 0$ almost everywhere. ■

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