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HARMONIC ANALYSIS ASSOCIATED WITH THE GENERALIZED q-BESSEL OPERATOR

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Abstract. In this article, we give a new harmonic analysis associated with the generalized q-Bessel operator. We introduce the generalized q-Bessel transform, the generalized q-Bessel translation and the generalized q-Bessel convolution product.

1. Introduction

In this paper we consider a generalized q-Bessel operator $\Delta_{q,\alpha,n}$ defined by

$$\Delta_{q,\alpha,n}f(x) = \frac{1}{x^2}\left[q^{2n}f(q^{-1}x) - (1+q^{2\alpha+4n})f(x) + q^{2\alpha+2n}f(qx)\right] \tag{1}$$

where $n = 0, 1, \ldots$. For $n = 0$, we regain the q-Bessel operator

$$\Delta_{q,\alpha}f(x) = \frac{1}{x^2}\left[f(q^{-1}x) - (1+q^{2\alpha})f(x) + q^{2\alpha}f(qx)\right] \tag{2}$$

Through this paper, we provide a new harmonic analysis corresponding to the generalized q-Bessel operator $\Delta_{d,\alpha,n}$.
The structure of the paper is as follows: In section 2, we set some notations and collect some basic results about q-harmonnic analysis. In section 3, we give some facts about harmonic analysis related to the generalized q-Bessel operator $\Delta_{q,\alpha,n}$, we define the generalized q-Bessel transform and we give some proprieties. In section 4, we define the generalized q-Bessel translation $T^{\alpha}_{q,x,n}$ and the generalized q-Bessel convolution product related to $T^{\alpha}_{q,x,n}$.

2. Element of q-harmonnic analysis

In this section, we recapitulate some facts about harmonic analysis related to the Bessel operator $\Delta_{d,\alpha,n}$. We cite here, as briefly as possible, some properties. For more details we refer to [5, 6, 1, 2].
Throught this paper, we assume that $0 < q < 1$ and $\alpha > -1$.
let $a \in \mathbb{C}$, the q-shifted factorial are defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$$

The q-derivative of a function f is given by

$$D_qf(x) = \frac{f(x)-f(qx)}{(1-q)x} \quad if \quad x \neq 0$$

The q-Jackson integrals from 0 to a and from 0 to ∞ are defined by [3, 4]

$$\int_0^a f(x)d_qx = (1-q)a\sum_0^{\infty} f(aq^n)q^n,$$

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$$\int_0^\infty f(x)d_qx = (1-q)a\sum_{n=-\infty}^{\infty} f(q^n)q^n.$$

We have

$$D_q\int_x^a H(t)d_qt = -H(x).$$

The q-analogue of the integration theorem by a change of variable can be stated as follows

$$\int_a^b H(\frac{\lambda}{r})\lambda^{2\alpha+1}d_q\lambda = r^{2\alpha+2}\int_{\frac{a}{r}}^{\frac{b}{r}} H(t)t^{2\alpha+1}d_qt, \quad \forall r\in\mathbb{R}_q^+$$

The q-integration by parts formula is given by

$$\int_a^b g(x)D_qf(x)d_qx = [f(b)g(b)-f(a)g(a)] - \int_a^b f(qx)D_qg(x)d_q.$$

The third Jackson q-Bessel function J_α (also called Hahn-Exton q-Bessel functions) is defined by the power series [7]

$$J_\alpha(x;q) = \frac{(q^{\alpha+1};q)_\infty}{(q;q)_\infty}x^\alpha\sum_{n=0}^{\infty}(-1)^n\frac{q^{\frac{n(n+1)}{2}}}{(q^{\alpha+1};q)_n(q;q)_n}x^{2n},$$

and has the normalized form

$$j_\alpha(x;q) = \sum_{n=0}^{\infty}(-1)^n\frac{q^{\frac{n(n+1)}{2}}}{(q^{\alpha+1};q)_n(q;q)_n}x^{2n}.$$

It satisfies the following estimate [5]

$$|j_\alpha(q^n,q^2)| \le \frac{(-q^2;q^2)_\infty(-q^{2\alpha+2};q^2)_\infty}{(q^{2\alpha+2};q^2)_\infty}\begin{cases} 1 & \text{if} \quad n\ge 0 \\ q^{n^2-(2\alpha+1)n} & \text{if} \quad n<0\end{cases}$$

If $x\in\mathbb{C}^*\setminus\mathbb{R}$ then we have the following asymptotic expansion as $|x|\to\infty$

$$j_\alpha(x;q^2) \sim \frac{(x^2q^2;q^2)_\infty}{(q^{2\alpha+2};q^2)_\infty}$$

Also the normalized q-Bessel functions satisfies an orthogonality relation

$$c_{q,\alpha}^2\int_0^\infty j_\alpha(xt,q^2)j_\alpha(yt,q^2)t^{2\alpha+1}d_qt = \delta_q(x,y)$$

where

$$\delta_q(x,y) = \begin{cases} 0, & \text{if} \quad y\ne x; \\ \frac{1}{(1-q)x^{2(\alpha+1)}}, & \text{if} \quad x=y.\end{cases}$$

(3) $$c_{q,\alpha} = \frac{1}{1-q}\frac{(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty}.$$

The function $x\mapsto j_\alpha(\lambda x,q^2)$ is a solution of the following q-differential equation

(4) $$\Delta_{q,\alpha}f(x) = -\lambda^2 f(x),$$

where $\Delta_{q,\alpha}$ is the q-Bessel operator given by (2).

For $1\le p<\infty$ we denote by $\mathcal{L}_{q,\alpha}^p$ the set of all real functions on $\mathbb{R}_q^+$ for which

$$\|f\|_{q,p,\alpha} = \left(\int_0^\infty |f(x)|^p x^{2\alpha+1}d_qx\right)^{\frac{1}{P}} < \infty.$$

Proposition 2.1. *Let $f,g\in\mathcal{L}_{q,\alpha}^2$ such that $\Delta_{q,\alpha}f,\Delta_{q,\alpha}g\in\mathcal{L}_{q,\alpha}^2$ then*

(5) $$\int_0^\infty \Delta_{q,\alpha}f(x)g(x)x^{2\alpha+1}d_qx = \int_0^\infty f(x)\Delta_{q,\alpha}g(x)x^{2\alpha+1}d_qx.$$

The q-Bessel Fourier transform $\mathcal{F}_{q,\alpha}$ was introduced and studies in [5, 6]

(6) $$\mathcal{F}_{q,\alpha}f(x) = c_{q,\alpha}\int_0^{\infty} f(t)j_\alpha(xt,q^2)t^{2\alpha+1}d_qt,$$

The q-Bessel translation operator is defined as follows [5, 6]

(7) $$T^{\alpha}_{q,x}f(y) = c_{q,\alpha}\int_0^{\infty}\mathcal{F}_{q,\alpha}(f)(t)j_\alpha(xt,q^2)j_\alpha(yt,q^2)t^{2\alpha+1}d_qt.$$

Proposition 2.2. *We have for all $x, y \in \mathbb{R}_q^+$*

$$T^{\alpha}_{q,x}f(y) = T^{\alpha}_{q,y}f(x)$$

The q-translation operator is positive if

$$T^{\alpha}_{q,x}f \geq 0, \quad \forall f \geq 0, \quad \forall x \in \mathbb{R}_q^+.$$

The domaine of positivity of the q-translation operator is

$$Q_\alpha = \{q \in]0,1[, \quad T^{\alpha}_{q,x} \quad \textit{is positive for all } x \in \mathbb{R}_q^+\}.$$

In [1] it was proved that if $-1 < \alpha < \alpha'$ then $Q_\alpha \subset Q_{\alpha'}$. As a consequence:

- if $0 \leq \alpha$ then $Q_\alpha =]0,1[$.
- if $-\frac{1}{2} \leq \alpha < 0$ then $]0, q_0[\subset Q_{-\frac{1}{2}} \subset Q_\alpha \subsetneq]0,1[, \quad q_0 \simeq 0.43$.
- if $-1 \leq \alpha < -\frac{1}{2}$ then $Q_\alpha \subset Q_{-\frac{1}{2}}$.

In the rest of this section we always assume that the q-translation operator is positive.

The q-convolution product of two functions is given by [5, 6]

(8) $$f *_{q,\alpha} g(x) = c_{q,\alpha}\int_0^{\infty} T^{\alpha}_{q,x}f(y)g(y)y^{2\alpha+1}d_qy.$$

The following Theorem summarize some result about q-Bessel Fourier transform [6]

Theorem 2.3. *The q-Bessel Fourier transform satisfies*

(1) *For all functions $f \in \mathcal{L}^p_{q,\alpha}$,*

$$\mathcal{F}^2_{q,\alpha}f(x) = f(x), \quad \forall x \in \mathbb{R}_q^+.$$

(2) *For all functions $f \in \mathcal{L}^2_{q,\alpha}$,*

(9) $$\| \mathcal{F}^2_{q,\alpha}f \|_{q,\alpha,2} = \| f \|_{q,\alpha,2} .$$

(3) *For all functions $f \in \mathcal{L}^p_{q,\alpha}$, where $p \geq 1$ then $\mathcal{F}_{q,\alpha}f \in \mathcal{L}^{\overline{p}}_{q,\alpha}$. If $1 \leq p \leq 2$ then*

(10) $$\|\mathcal{F}_{q,\alpha}f\|_{q,\alpha,\overline{p}} \leq B_{q,\alpha}^{\frac{2}{p}-1}\|f\|_{q,\alpha,p}.$$

where

(11) $$B_{q,\alpha} = \frac{1}{1-q}\frac{(-q^2;q^2)_\infty(-q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty}$$

(4) *Let $f \in \mathcal{L}^p_{q,\alpha}$ and $g \in \mathcal{L}^r_{q,\alpha}$ the $f *_q g \in \mathcal{L}^s_{q,\alpha}$ and*

(12) $$\mathcal{F}_{q,\alpha}(f *_{q,\alpha} g)(x) = \mathcal{F}_{q,\alpha}f(x) \times \mathcal{F}_{q,\alpha}f(x), \quad \forall x \in \mathbb{R}_q^+.$$

where $1 \leq p, r, s$ such that

$$\frac{1}{p} + \frac{1}{r} - 1 = \frac{1}{s}.$$

Proposition 2.4. [2] *For all $x, y \in \mathbb{R}_q^+$, we have*

(13) $$T^{\alpha}_{q,x}j_\alpha(\lambda y,q^2) = j_\alpha(\lambda x,q^2)j_\alpha(\lambda y,q^2).$$

Proposition 2.5. [2] *For any function $f \in \mathcal{L}^2_{q,\alpha}$ we have*

(14) $$\mathcal{F}_{q,\alpha}(T^{\alpha}_{q,x}f)(\lambda) = j_\alpha(\lambda x,q^2)\mathcal{F}_{q,\alpha}(f)(\lambda), \quad \forall \lambda, x \in \mathbb{R}_q^+.$$

3. Generalized q-Bessel transform

Let

- $\mathcal{M}$ the map defined by

$$\mathcal{M}f(x) = x^{2n} f(x). \tag{15}$$

- $\mathcal{L}^p_{q,\alpha,n}$ the class of measurable functions f on $\mathbb{R}^+_q$ for which

$$\|f\|_{q,\alpha,p,n} = \|\mathcal{M}^{-1} f\|_{q,\alpha+2n,p} < \infty.$$

$\forall x \in \mathbb{R}^+_q$, put

$$\Psi_{\alpha,n}(\lambda x, q^2) = x^{2n} j_{\alpha+2n}(\lambda x, q^2). \tag{16}$$

Proposition 3.1. **(i):** *The map $\mathcal{M}$ is a topological isomorphism from $\mathcal{L}^p_{q,\alpha}$ onto $\mathcal{L}^p_{q,\alpha,n}$*

(ii): *We have*

$$\Delta_{q,\alpha,n} \circ \mathcal{M} = \mathcal{M} \circ \Delta_{q,\alpha+2n}. \tag{17}$$

(iii): *$\Psi_{\alpha,n}(\lambda., q^2)$ satisfies the differential equation*

$$\Delta_{q,\alpha,n} \Psi_{\alpha,n}(\lambda., q^2) = -\lambda^2 \Psi_{\alpha,n}(\lambda., q^2) \tag{18}$$

Proof. Assertion (i) is easily checked.
(ii) easy combination of (1), (2) and (16).
Using (4) and (18), we have

$$\begin{aligned} \Delta_{q,\alpha,n} \Psi_{\alpha,n}(\lambda., q^2) &= \mathcal{M} \circ \Delta_{q,\alpha+2n} \circ \mathcal{M}^{-1} \Psi_{\alpha,n}(\lambda., q^2), \\ &= \mathcal{M} \circ \Delta_{q,\alpha+2n} j_{\alpha+2n}(\lambda., q^2), \\ &= -\lambda^2 \mathcal{M} j_{\alpha+2n}(\lambda., q^2), \\ &= -\lambda^2 \Psi_{\alpha,n}(\lambda., q^2) \end{aligned}$$

which prove (iii). ∎

Definition 3.2. *The generalized q-Bessel transform of a function $f \in \mathcal{L}^1_{q,\alpha,n}$is defined by*

$$\mathcal{F}_{q,\alpha,n}(f)(x) = c_{q,\alpha+2n} \int_0^\infty f(t) \Psi_{\alpha,n}(\lambda t, q^2) t^{2\alpha+1} d_q t \tag{19}$$

where $c_{q,\alpha+2n}$ is given by (3).

Proposition 3.3. **(i):** *For all $f \in \mathcal{L}^1_{q,\alpha,n}$ we have*

$$\mathcal{F}_{q,\alpha,n}(f)(\lambda) = \mathcal{F}_{q,\alpha+2n} \circ \mathcal{M}^{-1} f(\lambda). \tag{20}$$

(ii): *For all $f \in \mathcal{L}^1_{q,\alpha,n}$*

$$\mathcal{F}_{q,\alpha,n}(\Delta_{q,\alpha,n} f)(\lambda) = -\lambda^2 \mathcal{F}_{q,\alpha,n}(f)(\lambda). \tag{21}$$

Proof. Let $f \in \mathcal{L}^1_{q,\alpha,n}$. From (6), (17) and (20), we have

$$\begin{aligned} \mathcal{F}_{q,\alpha,n}(f)(\lambda) &= c_{q,\alpha+2n} \int_0^\infty f(t) \Psi_{\alpha,n}(\lambda t, q^2) t^{2\alpha+1} d_q t, \\ &= c_{q,\alpha+2n} \int_0^\infty f(t) x^{2n} j_{\alpha+2n}(\lambda t, q^2) t^{2\alpha+1} d_q t, \\ &= c_{q,\alpha+2n} \int_0^\infty \mathcal{M}^{-1} f(t) j_{\alpha+2n}(\lambda t, q^2) t^{2\alpha+4n+1} d_q t, \\ &= \mathcal{F}_{q,\alpha+2n} \circ \mathcal{M}^{-1} f(\lambda). \end{aligned}$$

which prove (i).
Let $f \in \mathcal{L}^1_{q,\alpha,n}$. From (5), (19) and (20), we have

$$\begin{aligned}\mathcal{F}_{q,\alpha,n}(\Delta_{q,\alpha,n} f)(\lambda) &= c_{q,\alpha+2n}\int_0^\infty \Delta_{q,\alpha,n} f(t)\Psi_{\alpha,n}(\lambda t, q^2)t^{2\alpha+1}d_qt,\\ &= c_{q,\alpha+2n}\int_0^\infty f(t)\Delta_{q,\alpha,n}\Psi_{\alpha,n}(\lambda t, q^2)t^{2\alpha+1}d_qt,\\ &= c_{q,\alpha+2n}\int_0^\infty (-\lambda^2)f(t)\Psi_{\alpha,n}(\lambda t, q^2)t^{2\alpha+1}d_qt,\\ &= -\lambda^2\mathcal{F}_{q,\alpha,n}(f)(\lambda).\end{aligned}$$

■

Theorem 3.4. (1) *For $f \in \mathcal{L}^p_{q,\alpha,n}$, we have*

$$\|\mathcal{F}_{q,\alpha,n} f\|_{q,\alpha,n,\infty} \le B_{q,\alpha+2n}\|f\|_{q,\alpha,n,1}. \tag{22}$$

where $B_{q,\alpha+2n}$ is given by (11)

(2) *Let $f \in \mathcal{L}^1_{q,\alpha,n}$, then*

$$\|\mathcal{F}_{q,\alpha,n} f\|_{q,\alpha,n,2} = \|f\|_{q,\alpha,n,2}. \tag{23}$$

Proof. Let $f \in \mathcal{L}^1_{q,\alpha,n}$, from (10), (11) and (21) we have

$$\begin{aligned}\|\mathcal{F}_{q,\alpha,n} f\|_{q,\alpha,n,\infty} &= \|\mathcal{F}_{q,\alpha+2n}\circ\mathcal{M}^{-1}f\|_{q,\alpha+2n,\infty},\\ &\le B_{q,\alpha+2n}\|\mathcal{M}^{-1}f\|_{q,\alpha+2n,1},\\ &\le B_{q,\alpha+2n}\|f\|_{q,\alpha,n,1}.\end{aligned}$$

which prove 1).
Let $f \in \mathcal{L}^1_{q,\alpha,n}$. Using (9) and (21), we have

$$\begin{aligned}\|\mathcal{F}_{q,\alpha,n} f\|_{q,\alpha,n,2} &= \|\mathcal{F}_{q,\alpha+2n}\circ\mathcal{M}^{-1}f\|_{q,\alpha+2n,2},\\ &= \|\mathcal{M}^{-1}f\|_{q,\alpha+2n,2},\\ &= \|f\|_{q,\alpha,n,2}.\end{aligned}$$

■

4. Generalized convolution product associated with $\Delta_{q,\alpha,n}$

Definition 4.1. *The generalized q-Bessel translation operators $T^\alpha_{q,x,n}$ associated with $\Delta_{q,\alpha,n}$ are defined by*

$$T^\alpha_{q,x,n} = x^{2n}\mathcal{M}\circ T^{\alpha+2n}_{q,x}\circ\mathcal{M}^{-1} \tag{24}$$

where $T^{\alpha+2n}_{q,x}$ is given by (7).

The generalized q-Bessel translation operator is positive if

$$T^\alpha_{q,x,n} f \ge 0, \quad \forall f \ge 0, \quad \forall x \in \mathbb{R}^+_q.$$

The domaine of positivity of the generalized q-Bessel translation operator is

$$Q_{\alpha,n} = \{q \in]0,1[, \quad T^\alpha_{q,x,n} \quad \textit{is positive for all } x \in \mathbb{R}^+_q\}.$$

In the rest of this paper we always assume that the generalized q-Bessel translation operator is positive.

Proposition 4.2. **(i):** *Let $f \in \mathcal{L}^1_{q,\alpha,n}$, we have*

$$T^\alpha_{q,x,n} f(y) = T^\alpha_{q,y,n} f(x) \tag{25}$$

and

$$T^\alpha_{q,x,n} f(0) = f(x).$$

(ii): $\forall x, y \in \mathbb{R}^+_q$, *we have*

$$T^\alpha_{q,x,n}\Psi_{\alpha,n}(\lambda y, q^2) = \Psi_{\alpha,n}(\lambda x, q^2)\Psi_{\alpha,n}(\lambda y, q^2) \tag{26}$$

(iii): *For any function $f \in \mathcal{L}^2_{q,\alpha,n}$, we have*

$$\mathcal{F}_{q,\alpha,n}(T^{\alpha}_{q,x,n}f)(\lambda) = \Psi_{\alpha,n}(\lambda x, q^2)\mathcal{F}_{q,\alpha,n}(f)(\lambda). \tag{27}$$

Proof. Let $f \in \mathcal{L}^1_{q,\alpha,n}$, from Porosition 2.2 and (25), we have

$$\begin{aligned} T^{\alpha}_{q,x,n}f(y) &= x^{2n}\mathcal{M}\circ T^{\alpha+2n}_{q,x}\circ\mathcal{M}^{-1}f(y), \\ &= x^{2n}y^{2n}T^{\alpha+2n}_{q,y}\circ\mathcal{M}^{-1}f(x), \\ &= y^{2n}\mathcal{M}\circ T^{\alpha+2n}_{q,y}\circ\mathcal{M}^{-1}f(x), \\ &= T^{\alpha}_{q,y,n}f(x). \end{aligned}$$

which prove (i).
Let $x, y \in \mathbb{R}^+_q$. From (13) and (25), we have

$$\begin{aligned} T^{\alpha}_{q,x,n}\Psi_{\alpha,n}(\lambda y, q^2) &= x^{2n}\mathcal{M}\circ T^{\alpha+2n}_{q,x}\circ\mathcal{M}^{-1}(y^{2n}j_{\alpha+2n}(\lambda y, q^2)), \\ &= x^{2n}y^{2n}T^{\alpha+2n}_{q,x}j_{\alpha+2n}(\lambda y, q^2)), \\ &= x^{2n}j_{\alpha+2n}(\lambda x, q^2))y^{2n}j_{\alpha+2n}(\lambda y, q^2)), \\ &= \Psi_{\alpha,n}(\lambda x, q^2)\Psi_{\alpha,n}(\lambda y, q^2). \end{aligned}$$

which prove (ii).
Let $f \in \mathcal{L}^2_{q,\alpha,n}$. From (14), (17), (21) and (25), we have

$$\begin{aligned} \mathcal{F}_{q,\alpha,n}(T^{\alpha}_{q,x,n}f)(\lambda) &= \mathcal{F}_{q,\alpha,n}(x^{2n}\mathcal{M}\circ T^{\alpha+2n}_{q,x}\circ\mathcal{M}^{-1}f)(\lambda), \\ &= x^{2n}\mathcal{F}_{q,\alpha+2n}(T^{\alpha+2n}_{q,x}\circ\mathcal{M}^{-1}f)(\lambda), \\ &= x^{2n}j_{\alpha+2n}(\lambda x, q^2)\mathcal{F}_{q,\alpha+2n}(\mathcal{M}^{-1}f)(\lambda), \\ &= \Psi_{\alpha,n}(\lambda x, q^2)\mathcal{F}_{q,\alpha,n}(f)(\lambda). \end{aligned}$$

which prove (iii). ■

Definition 4.3. *The generalized q-convolution product of both function $f, g \in \mathcal{L}^1_{q,\alpha,n}$ is defined by*

$$f *_{q,\alpha,n} g(x) = c_{q,\alpha+2n}\int_0^{\infty} T^{\alpha}_{q,x,n}f(y)g(y)y^{2\alpha+1}d_qy. \tag{28}$$

where $c_{q,\alpha+2n}$ is given by (3).

Proposition 4.4. *For $f, g \in \mathcal{L}^1_{q,\alpha,n}$*

$$f *_{q,\alpha,n} g = \mathcal{M}\left[(\mathcal{M}^{-1}f) *_{q,\alpha+2n} (\mathcal{M}^{-1}f)\right]. \tag{29}$$

Proof. Let $f, g \in \mathcal{L}^1_{q,\alpha,n}$. From (8), (25) and (29), we have

$$\begin{aligned} f *_{q,\alpha,n} g(x) &= c_{q,\alpha+2n}\int_0^{\infty} T^{\alpha}_{q,x,n}f(y)g(y)y^{2\alpha+1}d_qy, \\ &= c_{q,\alpha+2n}x^{2n}\int_0^{\infty} T^{\alpha+2n}_{q,x}\mathcal{M}^{-1}f(y)g(y)y^{2\alpha+2n+1}d_qy, \\ &= c_{q,\alpha+2n}x^{2n}\int_0^{\infty} T^{\alpha+2n}_{q,x}\mathcal{M}^{-1}f(y)\mathcal{M}^{-1}g(y)y^{2\alpha+4n+1}d_qy, \\ &= x^{2n}\left[(\mathcal{M}^{-1}f) *_{q,\alpha+2n} (\mathcal{M}^{-1}g)\right](x), \\ &= \mathcal{M}\left[(\mathcal{M}^{-1}f) *_{q,\alpha+2n} (\mathcal{M}^{-1}g)\right](x). \end{aligned}$$

■

Proposition 4.5. *For $f, g \in \mathcal{L}^1_{q,\alpha,n}$, then $f *_{q,\alpha,n} g \in \mathcal{L}^1_{q,\alpha,n}$ and*

$$\mathcal{F}_{q,\alpha,n}(f *_{q,n} g)(\lambda) = \mathcal{F}_{q,\alpha,n}(f)(\lambda)\mathcal{F}_{q,\alpha,n}(g)(\lambda). \tag{30}$$

Proof. Let $f, g \in \mathcal{L}^1_{q,\alpha,n}$, we have

$$\begin{aligned}\|f *_{q,n} g\|_{q,\alpha,n,1} &= \|\mathcal{M}^{-1}(f *_{q,n} g)\|_{q,\alpha+2n,1},\\ &\leq \|\mathcal{M}^{-1}f\|_{q,\alpha+2n,1}\|\mathcal{M}^{-1}g\|_{q,\alpha+2n,1},\\ &= \|f\|_{q,\alpha,n,1}\|g\|_{q,\alpha,n,1}.\end{aligned}$$

On the other hand, from (12), (21) and (30), we have

$$\begin{aligned}\mathcal{F}_{q,\alpha,n}(f *_{q,\alpha,n} g)(\lambda) &= \mathcal{F}_{q,\alpha,n}\left(\mathcal{M}\left[(\mathcal{M}^{-1}f) *_{q,\alpha+2n} (\mathcal{M}^{-1}f)\right]\right)(\lambda),\\ &= \mathcal{F}_{q,\alpha+2n} \circ \mathcal{M}^{-1}\left(\mathcal{M}\left[(\mathcal{M}^{-1}f) *_{q,\alpha+2n} (\mathcal{M}^{-1}f)\right]\right)(\lambda),\\ &= \mathcal{F}_{q,\alpha+2n}\left(\mathcal{M}^{-1}f\right)(\lambda) \times \mathcal{F}_{q,\alpha+2n}\left(\mathcal{M}^{-1}g\right)(\lambda),\\ &= \mathcal{F}_{q,\alpha,n}(f)(\lambda)\mathcal{F}_{q,\alpha,n}(g)(\lambda).\end{aligned}$$

■

Proposition 4.6. *Let $f \in \mathcal{L}^1_{q,\alpha,n}$, we have*

$$T^{\alpha}_{q,x,n}f(y) = \int_0^{\infty} f(z)D_{\alpha,n}(x,y,z)z^{2\alpha+1}d_qz, \tag{31}$$

where $D_{\alpha,n}(x,y,z) = c^2_{q,\alpha+2n}\int_0^{\infty}\Psi_{\alpha,n}(xt,q^2)\Psi_{\alpha,n}(yt,q^2)\Psi_{\alpha,n}(zt,q^2)t^{2\alpha+4n+1}d_{qt}$

Proof. Let $f \in \mathcal{L}^1_{q,\alpha,n}$, from (7), (19) and (24) we have

$$\begin{aligned}T^{\alpha}_{q,x,n}f(y) &= x^{2n}\left(\mathcal{M} \circ T^{\alpha+2n}_{q,x} \circ \mathcal{M}^{-1}\right)(f)(y),\\ &= x^{2n}y^{2n}T^{\alpha+2n}_{q,x} \circ \mathcal{M}^{-1}f(y),\\ &= x^{2n}y^{2n}c_{q,\alpha+2n}\int_0^{\infty}\mathcal{F}_{q,\alpha+2n} \circ \mathcal{M}^{-1}f(t)j_{\alpha+2n}(xt,q^2)j_{\alpha+2n}(yt,q^2)t^{2\alpha+4n+1}d_qt,\\ &= c_{q,\alpha+2n}\int_0^{\infty}\mathcal{F}_{q,\alpha,n}(f)(t)\Psi_{\alpha,n}(xt,q^2)\Psi_{\alpha,n}(yt,q^2)t^{2\alpha+4n+1}d_qt,\\ &= \int_0^{\infty} f(z)\left[c^2_{q,\alpha+2n}\int_0^{\infty}\Psi_{\alpha,n}(xt,q^2)\Psi_{\alpha,n}(yt,q^2)\Psi_{\alpha,n}(zt,q^2)t^{2\alpha+4n+1}d_{qt}\right]z^{2\alpha+1}d_qz,\\ &= \int_0^{\infty} f(z)D_{\alpha,n}(x,y,z)z^{2\alpha+1}d_qz.\end{aligned}$$

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