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ITERATIVE SOLUTIONS OF NONLINEAR INTEGRAL EQUATIONS OF HAMMERSTEIN TYPE

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Abstract. Let H be a real Hilbert space. Let $F, K : H \to H$ be Lipschitz monotone mappings with Lipschtiz constants L_1 and L_2, respectively. Suppose that the Hammerstein type equation $u + KFu = 0$ has a solution in H. It is our purpose in this paper to construct a new explicit iterative sequence and prove strong convergence of the sequence to a solution of the generalized Hammerstein type equation. The results obtained in this paper improve and extend known results in the literature.

1. Introduction

Let H be a real Hilbert space. A mapping $A : D(A) \subset H \to H$ is said to be $L-$*Lipschitz* if there exists $L \geq 0$ such that

$$||Ax - Ay|| \leq L||x - y||, \text{ for all } x, y \in D(A). \tag{1.1}$$

A is called *nonexpansive* mapping if $L = 1$ and it is called *contraction* mapping if $L < 1$. It is easy to observe that the class of Lipschitz mappings includes the class of nonexpansive and hence the class of contraction mappings.

A mapping $A : D(A) \subset H \to H$ is said to be $\gamma-$ *inverse strongly monotone* if there exists a positive real number γ such that

$$\langle x - y, Ax - Ay\rangle \geq \gamma||Ax - Ay||^2, \text{ for all } x, y \in D(A). \tag{1.2}$$

If A is $\gamma-$inverse strongly monotone, then it is *Lipschitz continuous* with Lipschitz constant $\frac{1}{\gamma}$. A is said to be α-*strongly monotone* if for each $x, y \in D(A)$ there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha||x - y||^2. \tag{1.3}$$

A mapping $A : D(A) \subset H \to H$ is called *monotone* if for each $x, y \in D(A)$, the following inequality holds:

$$\langle x - y, Ax - Ay\rangle \geq 0. \tag{1.4}$$

Evidently the set of γ-inverse strongly monotone and the set of α-strongly monotone mappings are included in the set of monotone mappings.

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A monotone mapping $A : H \to H$ is said to be *maximal monotone* if $R(I + \lambda A)$, the range of $(I + \lambda A)$, is H for every $\lambda > 0$, where I is the identity mapping on H. This is equivalent to saying that, a monotone mapping A is said to be maximal monotone if it is not properly contained in any other monotone mapping.

For a maximal monotone mapping A and $r > 0$, a mapping $J_r : R(I+rA) \to D(A)$ given by $J_r = (I + rA)^{-1}$ is called the *resolvent of* A. It is well known that the resolvent operator, J_r, is single valued and nonexpansive mapping.

The class of monotone mappings is one of the most important classes of mappings among nonlinear mappings. Interests in monotone mappings stems mainly from the fact that many physically significant problems (see e.g [20]) can be modelled by initial value problems of the form:

$$x'(t) + Ax(t) = 0, x(0) = x_0, \tag{1.5}$$

where A is a monotone mapping in an Hilbert space H. Such evolution equation can be found in the heat, wave and Schrödinger equations. If $x(t)$ is independent of t, the equation (1.5) reduces to

$$Au = 0, \tag{1.6}$$

whose solutions correspond to the equilibrium points of the system (1.5). A variety of problems, for example, convex optimization, linear programming, and elliptic differential equations can be formulated as finding a zero of maximal monotone mappings. Consequently, many research efforts (see, e.g., Zarantonello [16], Minty [11], Kacurovskii [9] and Vainberg and Kacurovskii [14]) have been devoted to methods of finding appropriate solutions, if it exists, of equation (1.6) and then

$$u + Au = 0. \tag{1.7}$$

One important generalization of equation (1.7) is the so-called *equation of Hammerstein type* (see e.g., [8]), where a nonlinear integral equation of Hammerstein type is one of the form:

$$u(x) + \int_{\Omega} k(x,y)f(y,u(y))dy = h(x), \tag{1.8}$$

where dy is a σ-finite measure on the measure space Ω, the real kernel k is defined on $\Omega \times \Omega$, f is a real-valued function defined on $\Omega \times \mathbb{R}$ and is, in general, nonlinear and h is a given function on Ω. If we now define a mapping K by

$$Kv(x) := \int_{\Omega} k(x,y)v(y)dy; \; x \in \Omega,$$

and the so-called *superposition* or *Nemytskii* mapping by $Fu(y) := f(y,u(y))$ then, the integral equation (1.8) can be put in operator theoretic form as follows:

$$u + KFu = 0, \tag{1.9}$$

where, without loss of generality, we have taken $h \equiv 0$. Given h in the function space H, the integral equation then asks for some u in H such that $(I+KF)(u) = h$. We note that if K and F are monotone, then $A := I + KF$ need not be necessarily monotone.

Equations of Hammerstein type play a crucial role in the theory that arise in differential equations, for instance, elliptic boundary value problems whose linear parts possess Greens functions can, as a rule, be transformed into the form (1.9) (see e.g., [12], Chapter IV).

Several existence and uniqueness theorems have been proved for equations of Hammerstein type (see e.g., [1, 2, 3, 4, 7]). In general, equations of Hammerstein type (1.9) are nonlinear and there is no known standard method to find solutions for them. Consequently, methods of approximating solutions of such equations are of interest.

In 2004, Chidume and Zegeye [6] used an auxiliary operator (in their proof), defined in a real Hilbert space in terms of K and F that is monotone whenever K and F are, and constructed an iterative procedure that converges strongly to the solution of Equation (1.9). In fact, they proved the following theorem.

Theorem 1.1. ([6]) *Let H be a real Hilbert space. Let $F : D(F) \subset H \to H, K : D(K) \subset H \to H$ be bounded monotone mappings with $R(F) \subseteq D(K)$ where $D(F)$ and $D(K)$ are closed convex subsets of H satisfying certain condition. Suppose the equation $0 = u+KFu$ has a solution in $D(F)$. Let $\{\lambda_n\}$ and $\{\theta_n\}$ be real sequences in $(0, 1]$ satisfying the following conditions:*

(i) $\lim\limits_{n\to\infty} \theta_n = 0$, *(ii)* $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty, \lim\limits_{n\to\infty} \dfrac{\lambda_n}{\theta_n} = 0$, *(iii)* $\lim\limits_{n\to\infty} \dfrac{\left(\frac{\theta_{n-1}}{\theta_n} - 1\right)}{\lambda_n\theta_n} = 0.$

Let sequences $\{u_n\} \subseteq D(F)$ and $\{v_n\} \subseteq D(K)$ be generated from $u_0 \in D(F)$ and $v_0 \in D(K)$, respectively by

$$\begin{cases} u_{n+1} = P_{D(F)}\big(u_n - \lambda_n(Fu_n - v_n + \theta_n(u_n - w_1))\big), \\ v_{n+1} = P_{D(K)}\big(v_n - \lambda_n(Kv_n + u_n + \theta_n(v_n - w_2))\big) \end{cases} \tag{1.10}$$

where $w_1 \in D(F)$ and $w_2 \in D(K)$ are arbitrary but fixed. Then, there exists $d > 0$ such that if $\lambda_n \le d$ and $\frac{\lambda_n}{\theta_n} \le d^2$ for all $n \ge 0$, the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u^* and v^*, respectively in H, where u^* is a solution of the equation $0 = u + KFu$ and $v^* = Fu^*$.

In 2012, Chidume and Djitte [5] introduced an iterative scheme and proved the following Theorem.

Theorem 1.2. ([5]) *Let H be a real Hilbert space. Let $F, K : H \to H$ be a bounded, monotone mapping and satisfy the range condition. Let $\{u_n\}$ and $\{v_n\}$ be sequences in H defined iteratively from arbitrary $u_1, v_1 \in H$ by*

$$\begin{cases} u_{n+1} = u_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(u_n - u_1), n \ge 1 \\ v_{n+1} = v_n - \lambda_n(Kv_n + u_n) - \lambda_n\theta_n(v_n - v_1), n \ge 1 \end{cases} \tag{1.11}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions.

(i) $\lim\limits_{n\to\infty} \theta_n = 0$, *(ii)* $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty$, $\lambda_n = o(\theta_n)$, *(iii)* $\lim\limits_{n\to\infty} \dfrac{\left(\frac{\theta_{n-1}}{\theta_n} - 1\right)}{\lambda_n\theta_n} = 0.$

Suppose that $u+KFu = 0$ has a solution in H. Then, there exists a constant $d_0 > 0$ such that if $\lambda_n \le d_0\theta_n$, for all $n \ge n_0$ for some $n_0 \ge 1$, then the sequence $\{u_n\}$ converges to u^*, a solution of $u + KFu = 0$.

More recently, Zegeye and Malonza [17] introduced a method which contains an auxiliary operator, defined in an Hilbert space in terms of K and F which, under certain conditions, is *monotone* whenever K and F are, and whose zeros are solutions of equation (1.9). They proved the following Theorem.

Theorem 1.3. ([17]) *Let H be a real Hilbert space. Let $F : H \to H$ and $K : H \to H$ be continuous and bounded monotone operators. Let $E := H \times H$ with norm $||z||_E^2 = ||u||_H^2 + ||v||_H^2$, for $z = (u, v) \in E$ and let a map $T : E \to E$ defined by $Tz = T(u, v) := (Fu - v, u + Kv)$ be $\gamma-$inverse strongly monotone. Let a sequence $\{x_n\}$ be generated by:*

$$\begin{cases} x_0 = w \in E \text{ chosen arbitrarily,} \\ w_n = x_n - \gamma_n T x_n, \\ x_{n+1} = \alpha_n w + \beta_n x_n + \lambda_n w_n, \end{cases} \tag{1.12}$$

where $\alpha_n, \beta_n, \gamma_n, \lambda_n \in (0, 1)$ satisfy $\alpha_n + \beta_n + \lambda_n = 1$ and $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$; $0 < \beta \leq \beta_n, \lambda_n$, for all $n \geq 0$ and $0 < a_0 \leq \gamma_n \leq \gamma$, for some $a_0, \beta \in \mathbb{R}$. Then the sequence $\{x_n\}$ converges strongly to $x^* = [u^*, v^*] \in E$, where u^* is a solution of the equation $0 = u + KFu$ and $v^* = Fu^*$.

We observe that in Theorem 1.1 and Theorem 1.2, the convergence of the Schemes to the solution of the equation $u+KFu = 0$ is granted by the existence of a constant which is not clear how it is calculated. In Theorem 1.3, the auxiliary operator T is used in the iteration scheme and the condition imposed on T, which is $\gamma-$inverse strongly monotone, is strong. These lead us to the following question.

Question: *Is it possible to construct an iterative scheme which converges strongly to a solution of Hammerstein type equation (1.9) which does not require the existence of a constant and does not involve an auxiliary operator?*

It is our purpose in this paper to construct a new explicit iterative sequence and prove strong convergence of the sequence to a solution of the generalized Hammerstein type equation (1.9). Our theorems provide an affirmative answer to the above question in Hilbert spaces. The results obtained in this paper improve and extend the results in this direction.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. It is well known that for every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, i.e,

$$||x - P_C x|| \leq ||x - y|| \text{ for all } y \in C. \tag{2.1}$$

The mapping P_C is called the metric projection of H onto C and characterized by the following property (see, *e.g.*, [13]):

$$P_C x \in C \text{ and } \langle x - P_C x, P_C x - y\rangle \geq 0, \text{ for all } x \in H, y \in C. \tag{2.2}$$

In the sequel we shall make use of the following lemmas.

Lemma 2.1. [13] *Let H be a real Hilbert space and $A : H \to H$ be a monotone mapping. Then, A is maximal monotone if and only if $R(I + rA) = H$ for some $r > 0$.*

Lemma 2.2. [20] *Let H be a real Hilbert space. If $A : H \to H$ is monotone and continuous, then A is maximal monotone.*

Lemma 2.3. [18] *Let H be a real Hilbert space. Then for all $x_i \in H$ and $\alpha_i \in [0, 1]$ for $i = 0, 1, 2, 3, ..., n$ such that $\alpha_0 + \alpha_1 + \alpha_2 + ... + \alpha_n = 1$ the following equality holds:*

$$||\alpha_0 x_0 + \alpha_1 x_1 + ... + \alpha_n x_n||^2 = \sum_{i=1}^{n} \alpha_i ||x_i||^2 - \sum_{1 \le i,j \le n} \alpha_i \alpha_j ||x_i - x_j||^2.$$

Lemma 2.4. *Let H be a real Hilbert space. Then, for any given $x, y \in H$, the following inequality holds:*

$$||x + y||^2 \le ||x||^2 + 2\langle y, x + y\rangle.$$

Lemma 2.5. [15] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n \delta_n, n \ge n_0,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying the following conditions:
$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\limsup_{n\to\infty} \delta_n \le 0$. *Then,* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6. [19] *Let H be a real Hilbert space and let $A : H \to H$ be a continuous monotone mapping. Then, $N(A) = \{x \in H : Ax = 0\}$ is closed and convex.*

Lemma 2.7. [10] *Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$, for all $i \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \le a_{m_k+1} \text{ and } a_k \le a_{m_k+1}.$$

In fact, $m_k = \max\{j \le k : a_j < a_{j+1}\}$.

Lemma 2.8. [6] *Let H be a real Hilbert space. Let $E = H \times H$ with norm $||z||_E^2 = ||u||_H^2 + ||v||_H^2$ for $z = (u, v) \in E$. Then, E is a real Hilbert space and for $w_1 = (u_1, v_1), w_2 = (u_2, v_2) \in E$, we have that $\langle w_1, w_2\rangle = \langle u_1, u_2\rangle + \langle v_1, v_2\rangle$.*

Lemma 2.9. [6] *Let C and D be nonempty subsets of a real Hilbert space H. Let $F : C \to H$, $K : D \to H$ be monotone mappings. Let $E = H \times H$ with norm $||z||_E^2 = ||u||_H^2 + ||v||_H^2$ for $z = (u, v) \in E$. Define a mapping $T : C \times D \to E$ by $Tz = T(u, v) := (Fu - v, Kv + u)$. Then, T is monotone mapping.*

3. Main Result

We first prove the following lemma which will be used in the sequel.

Lemma 3.1. *Let C and D be nonempty subsets of a real Hilbert space H. Let $F : C \to H$, $K : D \to H$ be monotone mappings. Let $E = H \times H$ with norm $||z||_E^2 = ||u||_H^2 + ||v||_H^2$ for $z = (u, v) \in E$. Define a mapping $T : C \times D \to E$ by $Tz = T(u, v) := (Fu - v, Kv + u)$. Then we have the following.*

(a) *If F and K are Lipschitz, then T is Lipschitz.*
(b) *If F and K are maximal monotone, then T is maximal monotone.*

Proof. Since F and K are monotone, by Lemma 2.9, T is monotone mapping.

(a) Let $z_1 = (u_1, v_1), z_2 = (u_2, v_2) \in C \times D$ and let L_1 and L_2 be Lipschitz constants of F and K, respectively. Then, we have

$$\begin{aligned}
||Tz_1 - Tz_2||^2 &= ||(Fu_1 - v_1, Kv_1 + u_1) - (Fu_2 - v_2, Kv_2 + u_2)||^2 \\
&= ||Fu_1 - Fu_2 - (v_1 - v_2)||^2 + ||Kv_1 - Kv_2 + (u_1 - u_2)||^2 \\
&\leq ||Fu_1 - Fu_2||^2 + 2||Fu_1 - Fu_2||||v_1 - v_2|| + ||v_1 - v_2||^2 \\
&\quad + ||Kv_1 - Kv_2||^2 + 2||Kv_1 - Kv_2||||u_1 - u_2|| + ||u_1 - u_2||^2 \\
&\leq ||Fu_1 - Fu_2||^2 + ||Fu_1 - Fu_2||^2 + ||v_1 - v_2||^2 + ||v_1 - v_2||^2 \\
&\quad + ||Kv_1 - Kv_2||^2 + ||Kv_1 - Kv_2||^2 + ||u_1 - u_2||^2 + ||u_1 - u_2||^2 \\
&\leq 2||Fu_1 - Fu_2||^2 + 2||v_1 - v_2||^2 + 2||Kv_1 - Kv_2||^2 + 2||u_1 - u_2||^2 \\
&\leq 2L_1^2||u_1 - u_2||^2 + 2||v_1 - v_2||^2 + 2L_2^2||v_1 - v_2||^2 + 2||u_1 - u_2||^2 \\
&\leq 2(L_1^2 + 1)||u_1 - u_2||^2 + 2(L_2^2 + 1)||v_1 - v_2||^2 \\
&\leq L^2(||u_1 - u_2||^2 + ||v_1 - v_2||^2),
\end{aligned}$$

where $L = \sqrt{2}\max\{\sqrt{L_1^2 + 1}, \sqrt{L_2^2 + 1}\}$. Thus $||Tz_1 - Tz_2|| \leq L||z_1 - z_2||$ and hence T is Lipschitz mapping.

(b) Let $0 < r < 1$. Then, since F and K are maximal monotone we have that $R(I + rF) = H$ and $R(I + rK) = H$. Moreover, the resolvent $J_r^F = (I + rF)^{-1}$ of F and $J_r^K = (I + rK)^{-1}$ of K are nonexpansive. Now, let $h = (h_1, h_2) \in E$. Define $G := E \to E$ by $Gw = (J_r^F(h_1 + rv), J_r^K(h_2 - ru))$ for all $w = (u, v) \in E$. By the nonexpansiveness of J_r^F and J_r^K, we have $||Gw_1 - Gw_2|| \leq r||w_1 - w_2||$, for all $w_1, w_2 \in E$. Thus, G is a contraction mapping. Then, by the Banach contraction principle, G has a unique fixed point say $w^* = (u^*, v^*) \in E$. That is, $Gw^* = w^*$, where $u^* = J_r^F(h_1 + rv^*)$ and $v^* = J_r^K(h_2 - ru^*)$. Thus, for every $h = (h_1, h_2) \in E$, there exists $w^* = (u^*, v^*) \in E$ such that $(I + rT)(w^*) = h$. Hence, $R(I + rT) = E$. Therefore, by Lemma 2.1, T is maximal monotone.

□

Now, consider the sequences $\{u_n\}$, $\{v_n\} \subset H$ and let $u'_n = F(u_n - \gamma_n(Fu_n - v_n)), v'_n = K(v_n - \gamma_n(Kv_n + u_n))$. Then through out the rest of the paper, we use the following notations.

i) $t_n = u_n - \gamma_n\big[u'_n - v_n + \gamma_n(Kv_n + u_n)\big]$,
ii) $s_n = v_n - \gamma_n\big[v'_n + u_n - \gamma_n(Fu_n - v_n)\big]$.

We now prove the following theorem.

Theorem 3.2. *Let H be a real Hilbert space. Let $F, K : H \to H$ be Lipschitz monotone mappings with Lipschtiz constants L_1 and L_2, respectively. Suppose that the equation $0 = u + KFu$ has a solution in H. Let $\bar{u}, \bar{v} \in H$ and the sequences $\{u_n\}$, $\{v_n\} \subset H$ be generated from arbitrary u_0, $v_0 \in H$ by*

$$\begin{cases} u_{n+1} = \alpha_n\bar{u} + (1 - \alpha_n)(a_nu_n + (1 - a_n)t_n), \\ v_{n+1} = \alpha_n\bar{v} + (1 - \alpha_n)(a_nv_n + (1 - a_n)s_n), \end{cases} \tag{3.1}$$

where $\gamma_n \subset [a,b] \subset (0, \frac{1}{L})$*, for* $L := \sqrt{2}\max\{\sqrt{L_1^2+1}, \sqrt{L_2^2+1}\}$*,* $\{a_n\} \subset (0,r] \subset (0,1)$ *and* $\{\alpha_n\} \subset (0,c] \subset (0,1)$ *for all* $n \geq 0$ *satisfies* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum \alpha_n = \infty$*. Then, the sequences* $\{u_n\}$ *and* $\{v_n\}$ *converge strongly to* u^* *and* v^* *respectively, in* H*, where* u^* *is the solution of* $0 = u + KFu$ *and* $v^* = Fu^*$*.*

Proof. F and K are maximal monotone by Lemma 2.2. Now, let $E := H \times H$ be endowed with the norm $||z||_E^2 = ||u||_H^2 + ||v||_H^2$, for $z = (u,v) \in E$. Define $T : E \to E$ by $T(z) = T(u,v) := (Fu - v, Kv + u)$. Then, by Lemma 3.1, T is Lipschtiz and maximal monotone mapping. we also observe that u^* is the solution of $0 = u + KFu$ if and only if $z^* = (u^*, v^*)$ is a solution of $0 = Tz$ for $v^* = Fu^*$. Thus, $N(T) = \{z \in E : Tz = 0\} \neq \emptyset$. Now, for initial point $z_0 = (u_0, v_0) \in E$, define the sequence $\{z_n\}$ by

$$\begin{cases} x_n = z_n - \gamma_n T z_n, \\ z_{n+1} = \alpha_n w + (1-\alpha_n)[a_n z_n + (1-a_n)(z_n - \gamma_n T x_n)], \end{cases} \tag{3.2}$$

where $w = (\bar{u}, \bar{v})$. Observe that we have $z_n = [u_n, v_n]$, where $\{u_n\}$ and $\{v_n\}$ are sequences in (3.1). Let $y_n = z_n - \gamma_n T x_n$ and $p \in N(T)$. Then, by the monotonicity of T, we have

$$\begin{aligned} ||y_n - p||^2 &= ||z_n - \gamma_n T x_n - p||^2 - ||z_n - \gamma_n T x_n - y_n||^2 \\ &= ||z_n - p||^2 - ||z_n - y_n||^2 + 2\gamma_n \langle T x_n, p - y_n \rangle \\ &= ||z_n - p||^2 - ||z_n - y_n||^2 + 2\gamma_n \big(\langle T x_n - Tp, p - x_n \rangle \\ &\quad + \langle Tp, p - x_n \rangle + \langle T x_n, x_n - y_n \rangle \big) \\ &\leq ||z_n - p||^2 - ||z_n - y_n||^2 + 2\gamma_n \langle T x_n, x_n - y_n \rangle \\ &= ||z_n - p||^2 - ||z_n - x_n||^2 - 2\langle z_n - x_n, x_n - y_n \rangle \\ &\quad - ||x_n - y_n||^2 + 2\gamma_n \langle T x_n, x_n - y_n \rangle \\ &= ||z_n - p||^2 - ||z_n - x_n||^2 - ||x_n - y_n||^2 \\ &\quad + 2\langle z_n - \gamma_n T x_n - x_n, y_n - x_n \rangle. \end{aligned} \tag{3.3}$$

But since $x_n = z_n - \gamma_n T z_n$ and T is Lipschitzian we obtain

$$\begin{aligned} &\langle z_n - \gamma_n T x_n - x_n, y_n - x_n \rangle \\ &= \langle z_n - \gamma_n T z_n - x_n, y_n - x_n \rangle + \langle \gamma_n T z_n - \gamma_n T x_n, y_n - x_n \rangle \\ &\leq \langle \gamma_n T z_n - \gamma_n T x_n, y_n - x_n \rangle \leq \gamma_n L ||z_n - x_n|| ||y_n - x_n||. \end{aligned} \tag{3.4}$$

Thus, from (3.3) and (3.4) we have that

$$\begin{aligned} ||y_n - p||^2 &\leq ||z_n - p||^2 - ||z_n - x_n||^2 - ||x_n - y_n||^2 + 2L\gamma_n ||z_n - x_n|| ||y_n - x_n|| \\ &\leq ||z_n - p||^2 - ||z_n - x_n||^2 - ||x_n - y_n||^2 \\ &\quad + \gamma_n L (||z_n - x_n||^2 + ||x_n - y_n||^2) \\ &\leq ||z_n - p||^2 + (\gamma_n L - 1)||z_n - x_n||^2 + (\gamma_n L - 1)||x_n - y_n||^2. \end{aligned} \tag{3.5}$$

Thus, from (3.2), Lemma 2.3, and (3.5) we have the following:

$$
\begin{aligned}
||z_{n+1}-p||^2 &= ||\alpha_n w+(1-\alpha_n)[a_n z_n+(1-a_n)y_n]-p||^2\\
&\le \alpha_n||w-p||^2+(1-\alpha_n)||a_n(z_n-p)+(1-a_n)(y_n-p)||^2\\
&\le \alpha_n||w-p||^2+(1-\alpha_n)\big[a_n||z_n-p||^2+(1-a_n)||y_n-p||^2\big]\\
&\le \alpha_n||w-p||^2+(1-\alpha_n)a_n||z_n-p||^2+(1-\alpha_n)(1-a_n)\big[||z_n-p||^2\\
&\quad +(\gamma_n L-1)||z_n-x_n||^2+(\gamma_n L-1)||x_n-y_n||^2\big].\\
&= \alpha_n||w-p||^2+(1-\alpha_n)||z_n-p||^2+(1-\alpha_n)(1-a_n)\\
&\quad \times(\gamma_n L-1)\big[||z_n-x_n||^2+||x_n-y_n||^2\big] \qquad (3.6)
\end{aligned}
$$

Now, since from the hypotheses, we have $\gamma_n < \frac{1}{L}$ for all $n \ge 1$, the inequality (3.6) implies that

$$
||z_{n+1}-p||^2 \le \alpha_n||w-p||^2+(1-\alpha_n)||z_n-p||^2. \qquad (3.7)
$$

Therefore, by induction we get that

$$
||z_{n+1}-p||^2 \le \max\{||z_0-p||^2, ||w-p||^2\}, \forall n \ge 0,
$$

which implies that $\{z_n\}$, $\{x_n\}$, and $\{y_n\}$ are bounded.

Let $z^* = P_{N(T)}w$. Then, using (3.2), Lemma 2.4, Lemma 2.3, (3.5), (3.6) and the fact that $\gamma_n < \frac{1}{L}$, we obtain the following:

$$
\begin{aligned}
||z_{n+1}-z^*||^2 &= ||\alpha_n(w-z^*)+(1-\alpha_n)\big[a_n z_n+(1-a_n)y_n-z^*\big]||^2\\
&\le (1-\alpha_n)||a_n z_n+(1-a_n)y_n-z^*||^2\\
&\quad +2\alpha_n\langle w-z^*, z_{n+1}-z^*\rangle\\
&\le (1-\alpha_n)a_n||z_n-z^*||^2+(1-\alpha_n)(1-a_n)||y_n-z^*||^2\\
&\quad +2\alpha_n\langle w-z^*, z_{n+1}-z^*\rangle\\
&\le (1-\alpha_n)\Big(a_n||z_n-z^*||^2+(1-a_n)[||z_n-z^*||^2+(\gamma_n L-1)(||z_n-x_n||^2\\
&\quad +||x_n-y_n||^2)]\Big)+2\alpha_n\langle w-z^*, z_{n+1}-z^*\rangle\\
&= (1-\alpha_n)||z_n-z^*||^2+(1-\alpha_n)(1-a_n)(\gamma_n L-1)(||z_n-x_n||^2\\
&\quad +||x_n-y_n||^2)+2\alpha_n\langle w-z^*, z_{n+1}-z^*\rangle \qquad (3.8)\\
&\le (1-\alpha_n)||z_n-z^*||^2+2\alpha_n\langle w-z^*, z_{n+1}-z^*\rangle\\
&\le (1-\alpha_n)||z_n-z^*||^2+2\alpha_n\langle w-z^*, z_n-z^*\rangle+2\alpha_n||z_{n+1}-z_n||||w-z^*||. \qquad (3.9)
\end{aligned}
$$

Now, we consider two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{||z_n-z^*||\}$ is decreasing for all $n \ge n_0$. Then, we get that, $\{||z_n-z^*||)\}$ is convergent. Thus, from (3.8), the fact that $\gamma_n < b < \frac{1}{L}$ for all $n \ge 0$ and $\alpha_n \to 0$ as $n \to \infty$, we have that

$$
y_n-x_n \to 0, z_n-x_n \to 0 \text{ as } n \to \infty. \qquad (3.10)
$$

Moreover, from (3.2), (3.10) and letting $n \to \infty$, we get

$$
z_{n+1}-z_n = \alpha_n(w-z_n)+(1-\alpha_n)(1-a_n)(y_n-z_n) \to 0. \qquad (3.11)
$$

Furthermore, since $\{z_n\}$ is bounded subset of H, which is reflexive, we can choose a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that $z_{n_j} \rightharpoonup \hat{z}$ and $\limsup\limits_{n\to\infty}\langle w-z^*, z_n-z^*\rangle =$

$\lim_{j\to\infty}\langle w-z^*, z_{n_j}-z^*\rangle$. This together with (3.10) implies that $y_{n_j} \rightharpoonup \hat{z}$ and $x_{n_j} \rightharpoonup \hat{z}$.

Now, we show that $\hat{z} \in N(T)$. But, since T is Lipschitz continuous, we have

$$||Ty_{n_j} - Tx_{n_j}|| \to 0 \text{ as } j \to \infty.$$

Let $(s,t) \in G(T)$. Then, we have $t-Ts = 0$ and hence we get $\langle s-z, t-Ts\rangle = 0$, for all $z \in E$. On the other hand, since $y_{n_j} = z_{n_j} - \gamma_{n_j} Tx_{n_j}$, we have $\langle z_{n_j} - \gamma_{n_j} Tx_{n_j} - y_{n_j}, y_{n_j} - s\rangle = 0$, and hence, $\langle s - y_{n_j}, (y_{n_j} - z_{n_j})/\gamma_{n_j} + Tx_{n_j}\rangle = 0$. Thus, we get

$$\begin{aligned}\langle s - y_{n_j}, t\rangle = \langle s - y_{n_j}, Ts\rangle &= \langle s - y_{n_j}, Ts\rangle - \langle s - y_{n_j}, (y_{n_j} - z_{n_j})/\gamma_{n_j} + Tx_{n_j}\rangle \\ &= \langle s - y_{n_j}, Ts - Ty_{n_j}\rangle + \langle s - y_{n_j}, Ty_{n_j} - Tx_{n_j}\rangle \\ &\quad -\langle s - y_{n_j}, (y_{n_j} - z_{n_j})/\gamma_{n_j}\rangle \\ &\geq \langle s - y_{n_j}, Ty_{n_j} - Tx_{n_j}\rangle - \langle s - y_{n_j}, (y_{n_j} - z_{n_j})/\gamma_{n_j}\rangle.\end{aligned}$$

This implies that $\langle s - \hat{z}, t\rangle \geq 0$, as $j \to \infty$. Then, maximality of T gives that $\hat{z} \in N(T)$. Thus, from (2.2), we immediately obtain that

$$\begin{aligned}\limsup_{n\to\infty}\langle w - z^*, z_n - z^*\rangle &= \lim_{j\to\infty}\langle w - z^*, z_{n_j} - z^*\rangle \\ &= \langle w - z^*, \hat{z} - z^*\rangle \leq 0. \qquad (3.12)\end{aligned}$$

Hence, it follows from (3.9), (3.11), (3.12) and Lemma 2.5 that $||z_n - z^*|| \to 0$ as $n \to \infty$. Consequently, $z_n \to z^* = (u^*, v^*) = P_{N(T)}w$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$||z_{n_i} - z^*|| < ||z_{n_i+1} - z^*||,$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.7, there exist a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, and

$$||z_{m_k} - z^*|| \leq ||z_{m_k+1} - z^*|| \text{ and } ||z_k - z^*|| \leq ||z_{m_k+1} - z^*||, \qquad (3.13)$$

for all $k \in \mathbb{N}$. Now, from (3.8), the fact that $\gamma_n < \frac{1}{L}$ for all $n \geq 0$ and $\alpha_n \to 0$ as $n \to \infty$, we get that $y_{m_k} - x_{m_k} \to 0, z_{m_k} - x_{m_k} \to 0$ as $k \to \infty$. Thus, following the method in Case 1, we obtain

$$\limsup_{k\to\infty}\langle w - z^*, z_{m_k} - z^*\rangle \leq 0. \qquad (3.14)$$

Now, replacing z_n by z_{m_k} in (3.9), we have that

$$\begin{aligned}||z_{m_k+1} - z^*||^2 &\leq (1 - \alpha_{m_k}||z_{m_k} - z^*||^2 + 2\alpha_{m_k}\langle w - z^*, z_{m_k} - z^*\rangle, \\ &\quad +2\alpha_{m_k}||z_{m_k+1} - z_{m_k}||.||w - z^*||, \qquad (3.15)\end{aligned}$$

and hence (3.13) and (3.15) imply that

$$\alpha_{m_k}||z_{m_k} - x^*||^2 \leq 2\alpha_{m_k}\langle w - z^*, z_{m_k} - z^*\rangle + 2\alpha_{m_k}||z_{m_k+1} - z_{m_k}||.||w - z^*||.$$

But the fact that $\alpha_{m_k} > 0$ implies that

$$||z_{m_k} - z^*||^2 \leq 2\langle w - z^*, z_{m_k} - z^*\rangle + 2||z_{m_k+1} - z_{m_k}||.||w - z^*||.$$

Thus, using (3.14) and (3.11) we get that $||z_{m_k} - z^*|| \to 0$ as $k \to \infty$. This together with (3.15) implies that $||z_{m_k+1} - z^*|| \to 0$ as $k \to \infty$. But $||z_k - z^*|| \leq ||z_{m_k+1} - z^*||$ for all $k \in \mathbb{N}$ gives that $x_k \to z^*$. Therefore, from the above two cases, we can conclude that $\{z_n\}$ converges strongly to a point $z^* = (u^*, v^*) = P_{N(T)}w$, where u^* is the solution of $0 = u + KFu$ and $v^* = Fu^*$. The proof is complete.

□

If, in Theorem 3.2, we assume that F is γ_1-inverse strongly monotone and K is γ_2-inverse strongly monotone, then both F and K are Lipschitz with Lipschitz constant $L' = \max\{\frac{1}{\gamma_1}, \frac{1}{\gamma_2}\}$ and hence we get the following corollary.

Corollary 3.3. *Let H be a real Hilbert space. Let $F : H \to H$ be γ_1-inverse strongly monotone and $K : H \to H$ be γ_2-inverse strongly monotone mappings. Suppose that the equation $0 = u + KFu$ has a solution in H. Let $\bar{u}, \bar{v} \in H$ and the sequences $\{u_n\}$,$\{v_n\} \subset H$ be generated from arbitrary u_0 and v_0 in H by*

$$\begin{cases} u_{n+1} = \alpha_n \bar{u} + (1-\alpha_n)(a_n u_n + (1-a_n)t_n), \\ v_{n+1} = \alpha_n \bar{v} + (1-\alpha_n)[a_n v_n + (1-a_n)s_n), \end{cases} \tag{3.16}$$

where $\gamma_n \subset [a,b] \subset (0, \frac{1}{L})$, for $L := \sqrt{2((L')^2+1)}$, $\{a_n\} \subset (0,r] \subset (0,1)$ and $\{\alpha_n\} \subset (0,c] \subset (0,1)$ for all $n \geq 0$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then, the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u^* and v^* respectively, where u^* is the solution of the equation $0 = u + KFu$ and $v^* = Fu^*$.

If, in Theorem 3.2, we assume that F is Lipschitz α_1-strongly monotone with Lipschitz constant L_1 and K is Lipschitz α_2-strongly monotone with Lipschitz constant L_2, then one can show that F is $\frac{\alpha_1}{L_1^2}$-inverse strongly monotone and K is $\frac{\alpha_2}{L_2^2}$-inverse strongly monotone and hence we get the following corollary.

Corollary 3.4. *Let H be a real Hilbert space. Let $F : H \to H$ be Lipschitz α_1-strongly monotone and $K : H \to H$ be Lipschitz α_2-strongly monotone mappings. Suppose that the equation $0 = u + KFu$ has a solution in H. Let $\bar{u}, \bar{v} \in H$ and the sequences $\{u_n\}$,$\{v_n\} \subset H$ be generated from arbitrary u_0 and v_0 in H by*

$$\begin{cases} u_{n+1} = \alpha_n \bar{u} + (1-\alpha_n)(a_n u_n + (1-a_n)t_n), \\ v_{n+1} = \alpha_n \bar{v} + (1-\alpha_n)[a_n v_n + (1-a_n)s_n), \end{cases} \tag{3.17}$$

where $\gamma_n \subset [a,b] \subset (0, \frac{1}{L})$, for $L := \sqrt{2((L'')^2+1)}$ and $L'' = \max\{\frac{L_1^2}{\alpha_1}, \frac{L_2^2}{\alpha_2}\}$, $\{a_n\} \subset (0,r] \subset (0,1)$ and $\{\alpha_n\} \subset (0,c] \subset (0,1)$ for all $n \geq 0$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then, the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u^* and v^* respectively, where u^* is the solution of the equation $0 = u + KFu$ and $v^* = Fu^*$.

If, in Theorem 3.2, we assume that $F = I$, an identity mapping on H, then F is Lipschitz monotone with Lipschitz constant $L_1 = 1$ and the sequences $\{t_n\}$ and $\{s_n\}$ reduce to:

i) $t'_n = (1-\gamma_n)u_n + (1-\gamma_n)\gamma_n v_n - \gamma_n^2 K v_n$,
ii) $s'_n = (1-\gamma_n^2)v_n + (\gamma_n - 1)\gamma_n u_n - \gamma_n v'_n$,

where $v'_n = K(v_n - \gamma_n(Kv_n + u_n))$ and hence we get the following corollary.

Corollary 3.5. *Let H be a real Hilbert space. Let $K : H \to H$ be Lipschitz monotone mapping with Lipschtiz constant L_2. Suppose that the equation $0 = u + Ku$ has a solution in H. Let $\bar{u}, \bar{v} \in H$ and the sequences $\{u_n\}$,$\{v_n\} \subset H$ be generated from arbitrary u_0 and v_0 in H by*

$$\begin{cases} u_{n+1} = \alpha_n \bar{u} + (1-\alpha_n)(a_n u_n + (1-a_n)t'_n), \\ v_{n+1} = \alpha_n \bar{v} + (1-\alpha_n)[a_n v_n + (1-a_n)s'_n), \end{cases} \tag{3.18}$$

where $\gamma_n \subset [a,b] \subset (0, \frac{1}{L})$, *for* $L := \sqrt{2}\max\{\sqrt{2}, \sqrt{L_2^2+1}\}$, $\{a_n\} \subset (0,r] \subset (0,1)$ *and* $\{\alpha_n\} \subset (0,c] \subset (0,1)$ *for all* $n \geq 0$ *satisfies* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum \alpha_n = \infty$. *Then, the sequences* $\{u_n\}$ *and* $\{v_n\}$ *both converge strongly to* u^*, *where* u^* *is the solution of the equation* $0 = u + Ku$.

If, in Theorem 3.2, we assume that $K = I$, an identity mapping on H, then K is Lipschitz monotone with Lipschitz constant $L_2 = 1$ and the sequences $\{t_n\}$ and $\{s_n\}$ reduce to:

i) $t_n'' = (1-\gamma_n^2)u_n + (1-\gamma_n)\gamma_n v_n - \gamma_n u_n'$,

ii) $s_n'' = (1-\gamma_n)v_n + (\gamma_n - 1)\gamma_n u_n + \gamma_n^2 F u_n$,

where $u_n' = F(u_n - \gamma_n(Fu_n - v_n))$ and hence we get the following corollary.

Corollary 3.6. *Let* H *be a real Hilbert space. Let* $F : H \to H$ *be Lipschitz monotone mapping with Lipschtiz constant* L_1. *Suppose that the equation* $0 = u+Fu$ *has a solution in* H. *Let* $\bar{u}, \bar{v} \in H$ *and the sequences* $\{u_n\}$, $\{v_n\} \subset H$ *be generated from arbitrary* u_0 *and* v_0 *in* H *by*

$$\left\{ \begin{array}{l} u_{n+1} = \alpha_n \bar{u} + (1-\alpha_n)(a_n u_n + (1-a_n)t_n''), \\ v_{n+1} = \alpha_n \bar{v} + (1-\alpha_n)[a_n v_n + (1-a_n)s_n''), \end{array} \right. \tag{3.19}$$

where $\gamma_n \subset [a,b] \subset (0, \frac{1}{L})$, *for* $L := \sqrt{2}\max\{\sqrt{2}, \sqrt{L_1^2+1}\}$, $\{a_n\} \subset (0,r] \subset (0,1)$ *and* $\{\alpha_n\} \subset (0,c] \subset (0,1)$ *for all* $n \geq 0$ *satisfies* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum \alpha_n = \infty$. *Then, the sequences* $\{u_n\}$ *and* $\{v_n\}$ *converge strongly to* u^* *and* $-u^*$, *respectively, where* u^* *is the solution of the equation* $0 = u + Ku$.

We note that the method of proof of Theorem 3.2 provides the following theorem for approximating the unique minimum norm point of solution of the Hammerstein type equation.

Theorem 3.7. *Let* H *be a real Hilbert space. Let* $F, K : H \to H$ *be Lipschitz monotone mappings with Lipschtiz constants* L_1 *and* L_2, *respectively. Suppose that the equation* $0 = u + KFu$ *has a solution in* H. *Let the sequences* $\{u_n\}$,$\{v_n\} \subset H$ *be generated from arbitrary* u_0 *and* v_0 *in* H *by*

$$\left\{ \begin{array}{l} u_{n+1} = (1-\alpha_n)(a_n u_n + (1-a_n)t_n), \\ v_{n+1} = (1-\alpha_n)(a_n v_n + (1-a_n)s_n), \end{array} \right. \tag{3.20}$$

where $\gamma_n \subset [a,b] \subset (0, \frac{1}{L})$, *for* $L := \sqrt{2}\max\{\sqrt{L_1^2+1}, \sqrt{L_2^2+1}\}$, $\{a_n\} \subset (0,r] \subset (0,1)$ *and* $\{\alpha_n\} \subset (0,c] \subset (0,1)$ *for all* $n \geq 0$ *satisfies* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum \alpha_n = \infty$. *Then, the sequence* $\{z_n\} = \{(u_n, v_n)\}$ *converges strongly to the unique minimum norm point* $z^* = (u^*, v^*)$ *in* $H \times H$, *where* u^* *is a solution of* $0 = u + KFu$ *and* $v^* = Fu^*$.

Remark 3.8. *Theorem 3.2 improves Theorem 3.4 of Chidume and Zegeye* [6] *and Theorem 3.1 of Chidume and Djitte* [5] *in the sense that the convergence of our scheme does not require the existence of a constant number.*

Remark 3.9. *Theorem 3.2 extends Theorem 3.4 of Zegeye and Malonza* [17] *in the sense that our scheme, which does not involve the auxiliary mapping, provides strong convergence to a solution of Hammerstein type equation for a more general class of monotone mappings. Our theorems provide an affirmative answer to the above question in Hilbert spaces.*

4. Numerical example

Now, we give an example of Lipschitz monotone mappings satisfying conditions of Theorem 3.2 and some numerical experiment result to explain the conclusion of the theorem.

Let $H = \mathbb{R}$ with absolute value norm. Let $F, K : \mathbb{R} \to \mathbb{R}$ be defined by

$$Fx = 3x \quad and \qquad Kx = 2x - 14. \tag{4.1}$$

Clearly, F and K are Lipschitz maximal monotone mappings with constants 3 and 2, respectively. Furthermore, we observe that $u^* = 2$ is the solution of $u + KFu = 0$.

Now if we take, $\alpha_n = \frac{1}{n+100}$, $\gamma_n = \frac{1}{n+200} + 0.01$, $a_n = \frac{1}{n+100} + 0.01$, and $w = (\bar{u}, \bar{v}) = (1, 0)$, we observe that the conditions of Theorem 3.2 are satisfied and Scheme (3.1) reduces to

$$\begin{cases} u_{n+1} = \alpha_n \bar{u} + (1-\alpha_n)(a_n u_n + (1-a_n)t_n), \\ v_{n+1} = \alpha_n \bar{v} + (1-\alpha_n)(a_n v_n + (1-a_n)s_n), \end{cases} \tag{4.2}$$

where $t_n = (1 - 3\gamma_n + 8\gamma_n^2)u_n + (1 - 5\gamma_n)\gamma_n v_n + 14\gamma_n^2$,
and $s_n = (3\gamma_n^2 - 2\gamma_n + 1)v_n + (5\gamma_n - 1)\gamma_n u_n - 28\gamma_n^2 + 14\gamma_n$.

Thus, for $(u_0, v_0) = (1, 3)$, (u_n, v_n) converges strongly to $(u^*, v^*) = (2, 6) = P_{N(T)}(w)$, where 2 is the solution of $u + KFu = 0$ and $6 = F(2)$. See the following Table and Figure.

n	1	101	1001	2001	3001	4001	5001	6001	7001	7901
u_n	1.0000	1.5552	1.9025	1.9470	1.9636	1.9723	1.9776	1.9812	1.9838	1.9856
v_n	3.0000	5.0504	5.7888	5.8853	5.9213	5.9400	5.9516	5.9594	5.9650	5.9689

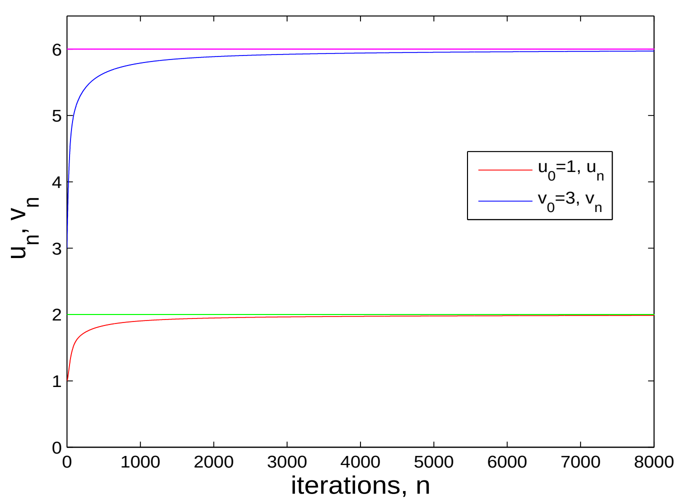

Figure 1

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