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A NEW MONOTONE ITERATION PRINCIPLE IN THE THEORY OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper the author proves the algorithms for the existence as well as approximations of the solutions for the initial value problems of nonlinear fractional differential equations using the operator theoretic techniques in a partially ordered metric space. The main results rely on the Dhage iteration principle embodied in the recent hybrid fixed point theorems of Dhage (2014) in a partially ordered normed linear space and the existence and approximations of the solutions of the considered nonlinear fractional differential equations are obtained under weak mixed partial continuity and partial Lipschitz conditions. Our hypotheses and existence and approximation results are also well illustrated by some numerical examples.

1. Introduction

The Dhage iteration principle or method (in short DIP or DIM) developed in [3, 4, 5, 6, 7] is relatively new to the literature on nonlinear analysis, particularly in the theory of nonlinear differential and integral equations. But it is is a powerful tool in the subject of nonlinear analysis because of its utility of applications to nonlinear equations for different qualitative aspects of the solutions. See Dhage and Dhage [8], Dhage *et al.* [13], Pathak and Lopez [16] and the references therein. Very recently, the above method has been applied in Dhage [3, 4, 6, 7] and Dhage and Dhage [8, 9, 10, 11] to nonlinear ordinary differential and integral equations for proving the existence and algorithms of the solutions. Similarly, DIP has also some interesting applications to the theory of nonlinear fractional differential and integral equations and in the present paper we prove the existence as well as approximations of the solutions for the initial value problems of fractional differential equations.

Before stating the main differential problems of the paper, we recall the following basic definitions of fractional calculus [15, 17] which are useful in what follows.

Definition 1.1. *Let $J = [t_0, t_0 + a]$ be a closed and bounded of the real line $\mathbb{R}$ for some $t_0, a \in \mathbb{R}$ with $t_0 \geq 0$ and $a > 0$. If $x \in AC^n(J, \mathbb{R})$, then the Caputo derivative ${}^C D^q x$ of x of fractional order q is defined as*

$$ {}^C D^q x(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} x^{(n)}(s)ds, \ n-1 < q < n, n = [q]+1, $$

where $[q]$ denotes the integer part of the real number q, and Γ is the Euler's gamma function. Here $AC^n(J, \mathbb{R})$ denote the space of real valued functions $x(t)$ which have continuous derivatives up to order $n-1$ on J such that the n^{th} derivative $x^{(n)} \in AC(J, \mathbb{R})$.

Definition 1.2. *If $J_\infty = [t_0, \infty)$ is a closed interval of the real line $\mathbb{R}$ for some $t_0 \in \mathbb{R}$ with $t_0 \geq 0$, then for any $x \in C(J, \mathbb{R})$, the Riemann-Liouville fractional integral of order $q > 0$ is defined as*

$$ I^q x(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} \frac{x(s)}{(t-s)^{1-q}} ds, \ t \in J_\infty, $$

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provided the right hand side is defined pointwise on (t_0, ∞), *where* Γ *is the Euler's gamma function.*

Given a closed and bounded interval $J = [t_0, t_0 + a]$ of the real line $\mathbb{R}$ for some $t_0, a \in \mathbb{R}$ with $t_0 \geq 0$ and $a > 0$, consider the initial value problem (in short IVP) of nonlinear fractional differential equation (FDE),

$$\left.\begin{aligned} {}^CD^q x(t) &= f(t, x(t)),\ t \in J, \\ x(t_0) &= \alpha_0, \end{aligned}\right\} \tag{1.1}$$

where ${}^CD^q$ is the Caputo derivative of fractional order q, $0 < q < 1$ and $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

By a *solution* of the FDE (1.1) we mean a function $x \in C^1(J, \mathbb{R})$ that satisfies equation (1.1), where $C^1(J, \mathbb{R})$ is the space of continuously differentiable real-valued functions defined on J.

The nonlinear FDE (1.1) is well-known and it has been discussed at length for the existence and the uniqueness of the solutions under compactness and Lipschitz conditions which are considered to be very strong in the theory of nonlinear differential and integral equations. In the present paper we prove the existence and uniqueness of the solutions of FDE (1.1) under weaker partially compactness and partially Lipschitz type conditions via Dhage iteration principle and also indicate some realizations.

The rest of the paper will be organized as follows. In Section 2 we give some preliminaries and key fixed point theorem that will be used in subsequent part of the paper. In Section 3 we discuss the existence result for initial value problems and in Section 4 we discuss the existence result for initial value problems of hybrid differential differential equations with linear perturbation of first type.

2. Auxiliary Results

Unless otherwise mentioned, throughout this paper that follows, let E denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\|\cdot\|$ in which the addition and the scalar multiplication by positive real numbers are preserved by $\preceq$. A few details of such partially ordered spaces appear in Dhage [2] and the references therein.

Two elements x and y in E are said to be **comparable** if either the relation $x \preceq$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all elements of C are comparable. It is known that E is **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Heikkilä and Lakshmikantham [14] and the references therein.

We need the following definitions in the sequel.

Definition 2.1. *A mapping* $\mathcal{T} : E \to E$ *is called* **isotone** *or* **monotone nondecreasing** *if it preserves the order relation* $\preceq$, *that is, if* $x \preceq y$ *implies* $\mathcal{T}x \preceq \mathcal{T}y$ *for all* $x, y \in E$. *Similarly,* $\mathcal{T}$ *is called* **monotone nonincreasing** *if* $x \preceq y$ *implies* $\mathcal{T}x \succeq \mathcal{T}y$ *for all* $x, y \in E$. *Finally,* $\mathcal{T}$ *is called* **monotonic** *or simply* **monotone** *if it is either monotone nondecreasing or monotone nonincreasing on* E.

The following terminologies may be found in any book on nonlinear analysis and applications.

Definition 2.2. *An operator* $\mathcal{T}$ *on a normed linear space* E *into itself is called* **compact** *if* $\mathcal{T}(E)$ *is a relatively compact subset of* E. $\mathcal{T}$ *is called* **totally bounded** *if for any bounded subset* S *of* E, $\mathcal{T}(S)$ *is a relatively compact subset of* E. *If* $\mathcal{T}$ *is continuous and totally bounded, then it is called a* **completely continuous** *on* E.

Definition 2.3 (Dhage [3])**.** *A mapping* $\mathcal{T} : E \to E$ *is called* **partially continuous** *at a point* $a \in E$ *if for* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that* $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ *whenever* x *is comparable to* a *and* $\|x - a\| < \delta$. $\mathcal{T}$ *is called a partially continuous on* E *if it is partially continuous at every point of it. It is clear that if* $\mathcal{T}$ *is a partially continuous on* E, *then it is continuous on every chain* C *contained in* E.

Definition 2.4 (Dhage [3, 4])**.** *A non-empty subset S of the partially ordered Banach space E is called partially bounded if every chain C in S is bounded. An operator $\mathcal{T}$ on a partially normed linear space E into itself is called* **partially bounded** *if $\mathcal{T}(C)$ is bounded for every chain C in E. $\mathcal{T}$ is called* **uniformly partially bounded** *if all chains $\mathcal{T}(C)$ in E are bounded by a unique constant.*

Definition 2.5 (Dhage [3, 4])**.** *A non-empty subset S of the partially ordered Banach space E is called partially compact if every chain C in S is compact. The operator $\mathcal{T}$ is called* **partially compact** *if $\mathcal{T}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. $\mathcal{T}$ is called* **uniformly partially compact** *if $\mathcal{T}$ is a uniformly partially bounded and partially compact operator on E. $\mathcal{T}$ is called* **partially totally bounded** *if for any totally ordered and bounded subset C of E, $\mathcal{T}(C)$ is a relatively compact subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called a* **partially completely continuous** *operator on E.*

Remark 2.6. Note that every compact mapping on a partially normed linear space is partially compact and every partially compact mapping is partially totally bounded, however the reverse implications do not hold. Again, every completely continuous mapping is partially completely continuous and every partially completely continuous mapping is partially continuous and partially totally bounded, but the converse may not be true.

Definition 2.7 (Dhage [3])**.** *The order relation $\preceq$ and the metric d on a non-empty set E are said to be* ***compatible*** *if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible.*

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function has this property. Similarly, every finite dimensional euclidean space $\mathbb{R}^n$ possesses compatibility property with respect to the usual component-wise order relation $\leq$ and the standard norm $\|\cdot\|$ in $\mathbb{R}^n$.

The **Dhage iteration principle** or **method** developed in Dhage [3, 4, 6] may be described as "***the monotonic convergence of the sequence of successive approximations to the solution of a nonlinear equation beginning with a lower or an upper solution of the equation as its initial or first approximation***" which forms a useful tool in the subject of existence theory of nonlinear analysis. It is clear that Dhage iteration method is different from usual Picard's successive iterations and embodied in some of the following applicable hybrid fixed point theorems of Dhage [4] which are the key tools for our work contained in the present paper. A few other hybrid fixed point theorems containing the Dhage iteration method along with their applications also appear in Dhage [3, 4].

Theorem 2.8 (Dhage [4])**.** *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear space such that the order relation $\preceq$ and the norm $\|\cdot\|$ in E are compatible in every compact chain C of E. Let $\mathcal{T} : E \to E$ be a partially continuous, nondecreasing and partially compact operator. If there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{T}x_0$ or $\mathcal{T}x_0 \preceq x_0$, then the operator equation $\mathcal{T}x = x$ has a solution x^* in E and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Remark 2.9. The regularity of E in above Theorem 2.8 may be replaced with a stronger continuity condition of the operator $\mathcal{T}$ on E which is a result proved in Dhage [4].

The following hybrid fixed point theorems are employed for proving the existence and uniqueness of the solutions for the FDE considered in the subsequent section of the paper. Before stating these results, we consider the following definition in what follows.

Definition 2.10. *An upper semi-continuous and nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function provided $\psi(0) = 0$. An operator $\mathcal{T} : E \to E$ is called partially nonlinear $\mathcal{D}$-contraction if* *there exists a $\mathcal{D}$-function ψ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \le \psi\big(\|x - y\|\big) \tag{2.1}$$

for all comparable elements $x, y \in E$, where $0 < \psi(r) < r$ for $r > 0$. In particular if $\psi(r) = k\,r$, $\mathcal{T}$ is a partially linear contraction on E with a contraction constant k.

Theorem 2.11 (Dhage [3])**.** *Let $(E, \preceq, \|\cdot\|)$ be a partially ordered Banach space and let $\mathcal{T} : E \to E$ be a nondecreasing and partially nonlinear $\mathcal{D}$-contraction. Suppose that there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$. If $\mathcal{T}$ is continuous or E is regular, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*. Moreover, the fixed point x^* is unique if every pair of elements in E has a lower and an upper bound.*

Theorem 2.12 (Dhage [4])**.** *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear space such that the order relation $\preceq$ and the norm $\|\cdot\|$ in E are compatible in every compact chain C of E. Let $\mathcal{A}, \mathcal{B} : E \to E$ be two nondecreasing operators such that*

(a) *$\mathcal{A}$ is partially bounded and partially nonlinear $\mathcal{D}$-contraction,*
(b) *$\mathcal{B}$ is partially continuous and partially compact, and*
(c) *there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{A}x_0 + \mathcal{B}x_0$ or $x_0 \succeq \mathcal{A}x_0 + \mathcal{B}x_0$.*

Then the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n + \mathcal{B}x_n$, n=0,1,..., converges monotonically to x^*.

Remark 2.13. The compatibility of the order relation $\preceq$ and the norm $\|\cdot\|$ in every compact chain of E holds if every partially compact subset of E possesses the compatibility property with respect to $\preceq$ and $\|\cdot\|$. This simple fact has been utilized in proving the main existence and approximations results for the considered the FDE (3.1) on J.

Note that the Dhage iteration method presented in the above hybrid fixed point theorems have been employed in Dhage and Dhage [9, 10, 11] for approximating the solutions of initial value problems of nonlinear first order ordinary differential equation under some natural hybrid conditions. In the following section we approximate the solutions of certain IVPs of nonlinear integro-differential equations via successive approximations beginning with the loser or upper solution.

3. Existence and Uniqueness Theorems

The equivalent integral form of the FDE (1.1) is considered in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\le$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{3.1}$$

and

$$x \le y \iff x(t) \le y(t) \quad \text{for all } t \in J. \tag{3.2}$$

Clearly, $C(J, \mathbb{R})$ is a Banach space with respect to above supremum norm and also partially ordered w.r.t. the above partially order relation $\le$. It is known that the partially ordered Banach space $C(J, \mathbb{R})$ is regular and a lattice so that every pair of elements of E has a lower and an upper bound in it. It is known that the partially ordered Banach space $C(J, \mathbb{R})$ has some nice properties w.r.t. the above order relation in it. The following crucial lemma concerning the compatibility of the order relation and the norm in $C(J, \mathbb{R})$ follows by an application of Arzellá-Ascoli theorem.

Lemma 3.1. *Let $\big(C(J, \mathbb{R}), \le, \|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\le$ defined by (3.1) and (3.2) respectively. Then $\|\cdot\|$ and $\le$ are compatible in every partially compact subset of $C(J, \mathbb{R})$.*

Proof. Let S be a partially compact subset of $C(J,\mathbb{R})$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a monotone nondecreasing sequence of points in S. Then we have

$$x_1(t) \le x_2(t) \le \cdots \le x_n(t) \le \cdots\,, \tag{ND}$$

for each $t \in J$.

Suppose that a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges to a point x in S. Then the subsequence $\{x_{n_k}(t)\}_{k\in\mathbb{N}}$ of the monotone real sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is convergent. By monotone characterization, the whole sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is convergent and converges to a point $x(t)$ in S for each $t \in J$. This shows that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x point-wise on J. To show the convergence is uniform, it is enough to show that the sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is equicontinuous. Since S is partially compact, every chain or totally ordered set and consequently $\{x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence by Arzelá-Ascoli theorem. Hence $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges uniformly to x. As a result, $\|\cdot\|$ and $\le$ are compatible in S. This completes the proof. □

We need the following definition in what follows.

Definition 3.2. *A function $u \in C^1(J,\mathbb{R})$ is said to be a lower solution of the FDE (1.1) if it satisfies*

$$\left.\begin{aligned} {}^{C}D^q u(t) &\le f(t,u(t)),\ t \in J,\\ u(t_0) &\le \alpha_0, \end{aligned}\right\} \tag{$*$}$$

Similarly, an upper solution $v \in C^1(J,\mathbb{R})$ to the FDE (3.1) *is defined on J, by the above inequalities with reverse sign.*

3.1. **Existence theorem.** We consider the following set of assumptions in what follows:

(H$_1$) There exists a constant $M_f > 0$ such that $|f(t,x)| \le M_f$ for all $t \in J$ and $x \in \mathbb{R}$.
(H$_2$) The function $f(t,x)$ is monotone nondecreasing in x for each $t \in J$.
(H$_3$) The FDE (3.1) has a lower solution $u \in C^1(J,\mathbb{R})$.

The following lemma is useful in what follows and may be found in Kilbas *et.al.* [15] and Podlubny [17].

Lemma 3.3. *For a given continuous function $h : J \to \mathbb{R}$, a function $x \in C^1(J,\mathbb{R})$ is a solution of the FDE*

$$\left.\begin{aligned} {}^{C}D^q x(t) &= h(t),\ t \in J,\\ x(t_0) &= \alpha_0, \end{aligned}\right\} \tag{3.3}$$

if and only if it is a solution of the nonlinear integral equation,

$$x(t) = \alpha_0 + \frac{1}{\Gamma q}\int_{t_0}^{t} (t-s)^{q-1}\, h(s)\, ds,\ t \in J. \tag{3.4}$$

Theorem 3.4. *Assume that the hypotheses (H$_1$) through (H$_3$) hold. Then the FDE (3.1) has a solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by*

$$x_0 = u, \quad x_{n+1}(t) = \alpha_0 + \frac{1}{\Gamma q}\int_{t_0}^{t} (t-s)^{q-1} f((s,x_n(s))\, ds, \tag{3.5}$$

for all $t \in J$, converges monotonically to x^*.

Proof. By Lemma 3.3, the FDE (1.1) is equivalent to the nonlinear integral equation

$$x(t) = \alpha_0 + \frac{1}{\Gamma q}\int_{t_0}^{t} (t-s)^{q-1} f(s,x(s))\, ds,\ t \in J. \tag{3.6}$$

Set $E = C(J,\mathbb{R})$. Then, from Lemma 3.1 it follows that every compact chain in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\le$ in E.

Define the operator $\mathcal{T}$ on E by

$$\mathcal{T}x(t) = \alpha_0 + \frac{1}{\Gamma q}\int_{t_0}^{t} (t-s)^{q-1} f(s, x(s))\, ds,\ \ t \in J. \tag{3.7}$$

From the continuity of the integral, it follows that $\mathcal{T}$ defines the map $\mathcal{T} : E \to E$. Then, the FDE (3.1) is equivalent to the operator equation

$$\mathcal{T}x(t) = x(t),\ \ t \in J. \tag{3.8}$$

We shall show that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.8. This is achieved in the series of following steps.

Step I: *$\mathcal{T}$ is nondecreasing operator on E.*

Let $x, y \in E$ be such that $x \leq y$. Then by hypothesis (H_2), we obtain

$$\begin{aligned}
\mathcal{T}x(t) &= \alpha_0 + \frac{1}{\Gamma q}\int_{t_0}^{t} (t-s)^{q-1} f(s, x(s))\, ds \\
&\leq \alpha_0 + \frac{1}{\Gamma q}\int_{t_0}^{t} (t-s)^{q-1} f(s, y(s))\, ds \\
&= \mathcal{T}y(t),
\end{aligned}$$

for all $t \in J$. This shows that $\mathcal{T}$ is nondecreasing operator on E into E.

Step II: *$\mathcal{T}$ is a partially continuous operator on E.*

Let $\{x_n\}$ be a sequence of points of a chain C in E such that $x_n \to x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{T}x_n(t) &= \lim_{n\to\infty} \left[\alpha_0 + \frac{1}{\Gamma q}\int_{t_0}^{t} (t-s)^{q-1} f(s, x_n(s))\, ds\right] \\
&= \alpha_0 + \frac{1}{\Gamma q}\int_{t_0}^{t} (t-s)^{q-1} \left[\lim_{n\to\infty} f(s, x_n(s))\right] ds \\
&= \alpha_0 + \frac{1}{\Gamma q}\int_{t_0}^{t} (t-s)^{q-1} f(s, x(s))\, ds \\
&= \mathcal{T}x(t),
\end{aligned}$$

for all $t \in J$. This shows that $\{\mathcal{T}x_n\}$ converges to $\mathcal{T}x$ pointwise on J.

Next, we will show that $\{\mathcal{T}x_n\}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$\begin{aligned}
&|\mathcal{T}x_n(t_2) - \mathcal{T}x_n(t_1)| \\
&\quad \leq \left|\frac{1}{\Gamma q}\int_{t_0}^{t_2} (t_2-s)^{q-1} f(s, x_n(s))\, ds - \frac{1}{\Gamma q}\int_{t_0}^{t_2} (t_1-s)^{q-1} f(s, x_n(s))\, ds\right| \\
&\quad\quad + \left|\frac{1}{\Gamma q}\int_{t_0}^{t_2} (t_1-s)^{q-1} f(s, x_n(s))\, ds - \frac{1}{\Gamma q}\int_{t_0}^{t_1} (t_1-s)^{q-1} f(s, x_n(s))\, ds\right| \\
&\quad \leq \frac{1}{\Gamma q}\left|\int_{t_0}^{t_2} |(t_2-s)^{q-1} - (t_1-s)^{q-1}|\, |f(s, x_n(s))|\, ds\right| \\
&\quad\quad + \frac{1}{\Gamma q}\left|\int_{t_1}^{t_2} (t_1-s)^{q-1} |f(s, x_n(s))|\, ds\right|
\end{aligned}$$

$$\leq \frac{M_f}{\Gamma q}\left|\int_{t_0}^{t_2} |(t_2-s)^{q-1}-(t_1-s)^{q-1}|\,ds\right| + \frac{M_f}{\Gamma q}\left|\int_{t_1}^{t_2}(t_1-s)^{q-1}\,ds\right|$$

$$\leq \frac{M_f}{\Gamma q}\left|\int_{t_0}^{t_0+a} |(t_2-s)^{q-1}-(t_1-s)^{q-1}|\,ds\right| + \frac{M_f}{\Gamma q}\,|p(t_1)-p(t_2)|$$

where, $p(t) = \dfrac{M_f}{\Gamma q}\displaystyle\int_{t_0}^{t}(t_0+a-s)^{q-1}\,ds$.

Since the functions $t \mapsto (t-s)^{q-1}$ and $t \mapsto p(t)$ are uniformly continuous on compact $J = [t_0, t_0+a]$, we have that

$$|\mathcal{T}x_n(t_2) - \mathcal{T}x_n(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{T}x_n \to \mathcal{T}x$ is uniformly and hence $\mathcal{T}$ is a partially continuous on E.

Step III: *$\mathcal{T}$ is a partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{T}(C)$ is a uniformly bounded and equicontinuous set in E. First we show that $\mathcal{T}(C)$ is uniformly bounded. Let $x \in C$ be arbitrary. Then,

$$\begin{aligned}
|\mathcal{T}x(t)| &\leq |\alpha_0| + \frac{1}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}|f(s,x(s))|\,ds \\
&\leq |\alpha_0| + \frac{1}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}|f(s,x(s))|\,ds \\
&\leq |\alpha_0| + \frac{a^q\,M_f}{\Gamma(q+1)} \\
&= r,
\end{aligned}$$

for all $t \in J$. Taking the supremum over t, we obtain $\|\mathcal{T}x\| \leq r$ for all $x \in C$. Hence $\mathcal{T}(C)$ is a uniformly bounded subset of E. Next, we will show that $\mathcal{T}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$\begin{aligned}
&|\mathcal{T}x(t_2) - \mathcal{T}x(t_1)| \\
&\quad\leq \left|\frac{1}{\Gamma q}\int_{t_0}^{t_2}(t_2-s)^{q-1}f(s,x(s))\,ds - \frac{1}{\Gamma q}\int_{t_0}^{t_2}(t_1-s)^{q-1}f(s,x(s))\,ds\right| \\
&\qquad + \left|\frac{1}{\Gamma q}\int_{t_0}^{t_2}(t_1-s)^{q-1}f(s,x(s))\,ds - \frac{1}{\Gamma q}\int_{t_0}^{t_1}(t_1-s)^{q-1}f(s,x(s))\,ds\right| \\
&\quad\leq \frac{1}{\Gamma q}\left|\int_{t_0}^{t_2} |(t_2-s)^{q-1}-(t_1-s)^{q-1}|\,|f(s,x(s))|\,ds\right| \\
&\qquad + \frac{1}{\Gamma q}\left|\int_{t_1}^{t_2}(t_1-s)^{q-1}|f(s,x(s))|\,ds\right| \\
&\quad\leq \frac{M_f}{\Gamma q}\left|\int_{t_0}^{t_2} |(t_2-s)^{q-1}-(t_1-s)^{q-1}|\,ds\right| + \frac{M_f}{\Gamma q}\left|\int_{t_1}^{t_2}(t_1-s)^{q-1}\,ds\right| \\
&\quad\leq \frac{M_f}{\Gamma q}\left|\int_{t_0}^{t_0+a} |(t_2-s)^{q-1}-(t_1-s)^{q-1}|\,ds\right| + \frac{M_f}{\Gamma q}\,|p(t_1)-p(t_2)|\,.
\end{aligned}$$

Since the functions $t \mapsto (t-s)^{q-1}$ and $t \mapsto p(t)$ are uniformly continuous on compact $J = [t_0, t_0+a]$, we have that

$$|\mathcal{T}x(t_2) - \mathcal{T}x(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1$$

uniformly for all $x \in C$. This shows that $\mathcal{T}(C)$ is an equicontinuous set in E. Hence $\mathcal{T}(C)$ is compact subset of E and consequently $\mathcal{T}$ is a partially compact operator on E into itself.

Step IV: *u satisfies the operator inequality $u \le \mathcal{T}u$.*

Since the hypothesis (H_3) holds, u is a lower solution of (3.1) defined on J. Then,

$$ {}^{C}D^{q}u(t) \le f(t,u(t)), \tag{3.9}$$

satisfying,

$$ u(t_0) \le \alpha_0, \tag{3.10}$$

for all $t \in J$.

Integrating (3.9) from t_0 to t, we obtain

$$ u(t) \le \alpha_0 + \frac{1}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}f(s,u(s))\,ds, \tag{3.11}$$

for all $t \in J$. This show that u is a lower solution of the operator equation $x = \mathcal{T}x$.

Thus $\mathcal{T}$ satisfies all the conditions of Theorem 2.8 in view of Remark 2.9 and we apply to conclude that the operator equation $\mathcal{T}x = x$ has a solution. Consequently the fractional integral equation (3.6) and the FDE (1.1) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}$ of successive approximations defined by (3.5) converges monotonically to x^*. This completes the proof. □

Remark 3.5. The conclusion of Theorem 3.4 also remains true if we replace the hypothesis (H_3) with the following one:

(H_3') The FDE (3.1) has an upper solution $v \in C^1(J,\mathbb{R})$.

Example 3.6. *Given a closed and bounded interval $J = [0,1]$, consider the FDE,*

$$\left.\begin{aligned} {}^{C}D^{1/2}x(t) &= \pi + \tanh x(t),\ t \in J,\\ x(0) &= 1. \end{aligned}\right\} \tag{3.12}$$

Here, $f(t,x) = \pi + \tanh x$. Clearly, the function f is continuous on $J \times \mathbb{R}$. The function f satisfies the hypothesis (H_1) with $M_f = \pi + 2$. Moreover, the function $f(t,x) = \pi + \tanh x$ is nondecreasing in x for each $t \in J$ and so the hypothesis (H_2) is satisfied.

Finally, the FDE (3.12) has a lower solution u defined by

$$ u(t) = 1 - \frac{1}{\Gamma(1/2)}\int_0^t (t-s)^{-1/2}\,ds = 1 + \frac{2\,t^{1/2}}{\sqrt{\pi}}$$

defined on J. Thus all the hypotheses of Theorem 3.4 are satisfied. Hence we conclude that the FDE (3.12) has a solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$ x_0 = u, \quad x_{n+1}(t) = 1 + \frac{1}{\sqrt{\pi}}\int_0^t (t-s)^{-1/2}\,\tanh x_n(s)\,ds,$$

for all $t \in J$, converges monotonically to x^*.

3.2. **Uniqueness theorem.** Next, we prove the uniqueness theorem for the FDE (3.1) under weak Lipschitz condition. We need the following hypothesis in what follows.

(H_4) There exists a $\mathcal{D}$-function ϕ such that

$$0 \le f(t,x) - f(t,y) \le \phi(x-y)$$

for all $x, y \in \mathbb{R}$, $x \ge y$. Moreover, $0 < \dfrac{a^q}{\Gamma(q+1)}\,\phi(r) < r$ for each $r > 0$.

Theorem 3.7. *Assume that hypotheses (H_2) through (H_4) hold. Then the FDE (3.1) has a unique solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by* (3.5) *converges monotonically to x^*.*

Proof. Set $E = C(J,\mathbb{R})$. Clearly, E is a lattice w.r.t. the order relation $\leq$ and so the lower and the upper bound for every pair of elements in E exist. Define the operator $\mathcal{T}$ by (3.7). Then, the FDE (1.1) is equivalent to the operator equation (3.8). We shall show that $\mathcal{T}$ satisfies all the conditions of Theorem 2.11 in E.

Clearly, $\mathcal{T}$ is a nondecreasing operator on E into itself. We shall simply show that the operator $\mathcal{T}$ is a nonlinear $\mathcal{D}$-contraction on E. Let $x, y \in E$ be any two elements such that $x \geq y$. Then, by hypothesis (H_4),

$$\begin{aligned}
|\mathcal{T}x(t) - \mathcal{T}y(t)| &\leq \left| \frac{1}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s))\, ds - \frac{1}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} f(s, y(s))\, ds \right| \\
&\leq \frac{1}{\Gamma q} \left| \int_{t_0}^{t} (t-s)^{q-1} |f(s, x(s)) - f(s, y(s))|\, ds \right| \\
&\leq \frac{1}{\Gamma q} \left| \int_{t_0}^{t} (t-s)^{q-1} \phi(|x(s) - y(s)|)\, ds \right| \\
&\leq \frac{1}{\Gamma q} \left| \int_{t_0}^{t} (t-s)^{q-1}\, ds \right| \phi(\|x - y\|) \\
&\leq \frac{a^q}{\Gamma(q+1)} \phi(\|x - y\|) \\
&= \psi(\|x - y\|) \qquad (3.13)
\end{aligned}$$

for all $t \in J$, where $\psi(r) = \dfrac{a^q}{\Gamma(q+1)}\, \phi(r) < r$, $r > 0$.

Taking the supremum over t, we obtain

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi(\|x - y\|)$$

for all $x, y \in E$, $x \geq y$. As a result $\mathcal{T}$ is a partially nonlinear $\mathcal{D}$-contraction on E. Furthermore, it can be shown as in the proof of Theorem 3.4 that the function u given in hypothesis (H_3) satisfies the operator inequality $u \leq \mathcal{T}u$ on J. Now a direct application of Theorem 2.11 yields that the FDE (1.1) has a unique solution x^* and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.5) converges monotonically to x^*. □

Remark 3.8. The conclusion of Theorem 3.7 also remains true if we replace the hypothesis (H_3) with the following one:

(H_3') The FDE (3.1) has an upper solution $v \in C^1(J,\mathbb{R})$.

Example 3.9. *Given a closed and bounded interval $J = [0,1]$, consider the FDE,*

$$\left.\begin{aligned}
{}^{C}D^{1/2}x(t) &= \pi + \frac{1}{2}\arctan x(t),\ t \in J, \\
x(0) &= 1.
\end{aligned}\right\} \qquad (3.14)$$

Here, $f(t,x) = \pi + \frac{1}{2}\arctan x$. Clearly, the function f is continuous on $J \times \mathbb{R}$. The function f satisfies the hypothesis (H_1) with $M_f = \pi + \dfrac{\pi}{4}$. Moreover, the function $f(t,x)$ is nondecreasing in x for each $t \in J$ and so the hypothesis (H_2) is satisfied. We show that f satisfies the hypothesis (H_4) on $J \times \mathbb{R}$. Let $x, y \in \mathbb{R}$ be such that $x \geq y$. Then, we have

$$0 \leq f(t,x) - f(t,y) \leq \frac{1}{2}\big[\arctan x - \arctan y\big] \leq \phi(x - y)$$

for all $t \in J$, where ψ is $\mathcal{D}$-function defined by $\phi(r) = \dfrac{1}{2} \cdot \dfrac{r}{1+\xi^2}$ and $x > \xi > y$. Furthermore, $\dfrac{a^q}{\Gamma(q+1)}\,\phi(r) = \dfrac{1}{\sqrt{\pi}} \cdot \dfrac{r}{1+\xi^2} < r$ for each $0 < \xi < r$. Finally, the FDE (3.1) has a lower solution

$$u(t) = 1 - \frac{\pi}{2} \cdot \frac{1}{\Gamma(1/2)} \cdot \int_0^t (t-s)^{-1/2}\, ds = 1 + \sqrt{\pi}\, t^{1/2}$$

defined on J. Thus all the hypotheses of Theorem 3.7 are satisfied and hence we conclude that the FDE (3.14) has a unique solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_0 = u, \quad x_{n+1}(t) = 1 + \frac{1}{2\sqrt{\pi}} \int_0^t (t-s)^{-1/2} \arctan x_n(s)\, ds,$$

for all $t \in J$, converges monotonically to x^*.

4. Linear Perturbations of First Type

It is possible sometimes that the nonlinearity f involved in the FDE (1.1) neither satisfies the hypothesis of Theorem 3.4 nor satisfies the hypothesis of Theorem 3.7, however the splitting functions f_1 and f_2 of f in the form $f = f_1 + f_2$ satisfy the conditions of Theorems 3.4 and 3.7 respectively. In the terminology of Dhage [2] it is called a hybrid fractional differential equation with a linear perturbation of the FDE (1.1) of first type. Then in this case it of interest to obtain the conclusion of Theorem 3.4 with stated algorithm for the solutions which is a problem of this section.

Given the notations of previous section, we consider the following nonlinear hybrid fractional differential equation (in short HFDE),

$$\left.\begin{aligned} {}^C D^q x(t) &= f(t,x(t)) + g(t,x(t)),\ t \in J,\\ x(t_0) &= \alpha_0, \end{aligned}\right\} \tag{4.1}$$

where $f, g : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

By a *solution* of the HFDE (4.1) we mean a function $x \in C^1(J,\mathbb{R})$ that satisfies equation (4.1) on J.

The HFDE (4.1) is a hybrid fractional differential equation with a linear perturbation of first type. See Dhage [1, 2] and the references therein. The HFDE (4.1) is well-known in the literature and discussed at length for existence as well as other aspects of the solutions. In the present discussion, it is proved that the existence of the solutions may be proved under mixed partially Lipschitz and partially compactness type conditions.

We need the following definition in what follows.

Definition 4.1. *A function $u \in C^1(J,\mathbb{R})$ is said to be a lower solution of the HFDE (4.1) if it satisfies*

$$\left.\begin{aligned} {}^C D^q u(t) &\le f(t,u(t)) + g(t,u(t)),\ t \in J,\\ u(t_0) &\le \alpha_0. \end{aligned}\right\} \tag{$**$}$$

Similarly, an upper solution $v \in C^1(J,\mathbb{R})$ to the HFDE (4.1) *is defined on J by the above inequalities with reverse sign.*

We consider the following set of assumptions in what follows:

(H$_5$) There exists a constant $M_g > 0$ such that $|g(t,x)| \le M_g$ for all $t \in J$ and $x \in \mathbb{R}$.
(H$_6$) The mapping $g(t,x)$ is monotone nondecreasing in x for each $t \in J$.
(H$_7$) The HFDE (4.1) has a lower solution $u \in C^1(J,\mathbb{R})$.

Theorem 4.2. *Assume that the hypotheses (H$_1$) and (H$_4$) through (H$_7$) hold. Then the HFDE (4.1) has a solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by*

$$
\begin{aligned}
x_0 = u, \quad x_{n+1}(t) &= \alpha_0 + \frac{1}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1} f(s, x_n(s))\, ds \\
&\quad + \frac{1}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1} g(s, x_n(s)),
\end{aligned} \tag{4.2}
$$

for all $t \in J$, converges monotonically to x^*.

Proof. Set $E = C(J, \mathbb{R})$. Then, from Lemma 3.1 it follows that every compact chain C in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in E.

By Lemma 3.3, the HFDE (4.1) is equivalent to the nonlinear integral equation

$$
\begin{aligned}
x(t) &= \alpha_0 + \frac{1}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1} f(s, x(s))\, ds \\
&\quad + \frac{1}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1} g(s, x(s))\, ds, \ t \in J.
\end{aligned} \tag{4.3}
$$

Set $E = C(J, \mathbb{R})$ and define the operators $\mathcal{A}$ and $\mathcal{B}$ on E by

$$
\mathcal{A}x(t) = \frac{1}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1} f(s, x(s))\, ds, \ t \in J, \tag{4.4}
$$

and

$$
\mathcal{B}x(t) = \alpha_0 + \frac{1}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1} g(s, x(s))\, ds, \ t \in J. \tag{4.5}
$$

From the continuity of the integrals, it follows that $\mathcal{A}$ and $\mathcal{B}$ define the operators $\mathcal{A}, \mathcal{B} : E \to E$. Now, the FDE (4.1) is equivalent to the operator equation

$$
\mathcal{A}x(t) + \mathcal{B}x(t) = x(t), \ \ t \in J. \tag{4.6}
$$

Then following the arguments similar to those given in Theorems 3.4 and 3.7, it can be shown that the operator $\mathcal{A}$ is partially bounded and nonlinear $\mathcal{D}$-contraction and $\mathcal{B}$ is partially continuous and partially compact operator on E into itself. Finally, by hypothesis (H$_7$), we have an element $u \in C^1(J, \mathbb{R})$ such that $u \leq \mathcal{A}u + \mathcal{B}u$. Now by a direct application of Theorem 2.12 we conclude that the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution x^*. Consequently the FDE (4.1) has a solution x^* and the sequence $\{x_n\}_{n=1}^{\infty}$ defined by (4.2) converges monotonically to x^*. This completes the proof. □

The conclusion of Theorems 4.2 also remains true if we replace the hypothesis (H$_7$) with the following one:

(H$_7'$) The HFDE (4.1) has an upper solution $v \in C^1(J, \mathbb{R})$.

Example 4.3. *Given a closed and bounded interval $J = [0, 1]$, consider the HFDE,*

$$
\left.
\begin{aligned}
{}^{C}D^{1/2}x(t) = \pi + \frac{1}{2}\arctan x(t) + \tanh x(t), \ \ t \in J,& \\
x(0) = 1.&
\end{aligned}
\right\} \tag{4.7}
$$

Here, $f(t, x) = \pi + \frac{1}{2}\arctan x$ and $g(t, x) = \tanh y$. Then the function f satisfies the hypothesis (H$_1$) with $M_f = \pi + \dfrac{\pi}{4}$. Again, f satisfies (H$_4$) with $\psi(r) = \dfrac{1}{2} \cdot \dfrac{r}{1+\xi^2}$, $0 < \xi < r$. Next, g satisfies

(H_5) with $M_g = 1$. Similarly, g satisfies the hypothesis (H_6). Finally, the function

$$\begin{aligned} u(t) &= 1 - \frac{1}{\Gamma(1/2)} \int_0^t (t-s)^{-1/2}\, ds - \frac{1}{\Gamma(1/2)} \cdot \int_0^t (t-s)^{-1/2}\, ds \\ &= 1 + \frac{4}{\sqrt{\pi}} t^{1/2}, \end{aligned}$$

for all $t \in J$ is a lower solution of the HFDE (4.7) defined on J. As a result, we apply Theorem 4.2 and conclude that the FDE (4.7) has a solution x^* on J and the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} x_0 = u, \quad x_{n+1}(t) = 1 &+ \frac{1}{2\sqrt{\pi}} \int_0^t (t-s)^{-1/2} \arctan x(s) \\ &+ \frac{1}{\sqrt{\pi}} \int_0^t (t-s)^{-1/2} \tanh x_n(s)\, ds \end{aligned}$$

for each $t \in J$, converges monotonically x^*.

Remark 4.4. In this paper we have proved only the existence and uniqueness results for the FDE (1.1) and its linear perturbations of first type (4.1). However, other aspects of the solutions of these FDEs such as the existence of minimal and maximal solutions and comparison theorems could also be proved using the same Dhage iteration method with appropriate modifications. Furthermore, if the FDE (1.1) has a lower solution u and an upper solution v such that $u \leq v$, then the corresponding solutions x_* and x^* of the FDE (1.1) satisfy $x_* \leq x^*$ and are the minimal and maximal solutions in the vector segment $[u, v]$ of the Banach space $E = C(J, \mathbb{R})$. Because the order relation $\leq$ defined by (3.1) is equivalent to the order relation defined by the order cone

$$\mathcal{K} = \big\{ x \in C(J, \mathbb{R}) \mid x(t) \geq 0 \quad \text{for all} \quad t \in J \big\}, \tag{4.8}$$

which is a closed set in $C(J, \mathbb{R})$.

5. Dhage iteration and Other principles

We remark that the existence theorem for the FDE (1.1) can also be proved via classical Schauder fixed point principle under the compactness type arguments. The procedure involves the construction of a closed convex and bounded subset S of the Banach space $C(J, \mathbb{R})$ and a continuous and compact operator $\mathcal{T}$ which maps S into itself. This is somewhat a troublesome job and moreover, it does not yield any computational scheme for the solutions. In the present approach of the Dhage iteration principle, we do not need any convexity argument and along with the existence we obtain an algorithm for the numerical solutions via construction of a sequence of successive approximations. Again, the existence and uniqueness theorem for the FDE (1.1) is generally proved via classical Banach contraction mapping principle which involves a strong Lipschitz condition on the nonlinearity f, but here in the present approach, the Lipschitz condition is replaced by a weaker partial nonlinear Lipschitz condition and obtained the same conclusion with some additional information about the characterizations of convergence of the sequence of successive approximations. Similarly, the existence theorem for the HFDE (4.1) may be proved via classical Krasnoselskii type fixed point principle under the usual mixed Lipschitz and compactness type conditions which again does not yield any numerical or approximate solution. See Dhage [1] and the references therein. But in the case of Dhage iteration principle for such nonlinear hybrid equations, we need the weaker partial Lipschitz and partial compactness type conditions and we obtain an algorithm in terms of a sequence of successive approximations for the numerical solutions. Thus in all cases, the Dhage iteration principle has some advantages over the classical existing methods involving the basic Schauder, Banach and Krasnoselskii fixed point principles. The limitation of the Dhage iteration principle is that one needs an extra monotone feature of the nonlinearity in the unknown variable, but as a result we obtain an additional information about the monotonic characterization of the convergence of the sequence of successive approximations to the solutions of the problems in question.

Another approach of classic upper and lower solution method in the existence theory for the FDEs (1.1), (4.1) and similar nonlinear problems is well-known in the literature in which one needs both the upper and lower solutions to exist in order to prove the existence of the solutions of the related differential equation. Moreover, the lower solution u and an upper solution v of the related problem should satisfy the inequality $u \leq v$. The work along this line may be found in Heikkilä and Lakshmikatham [14] and the references therein. The novelty of our present approach with respect to the existing ones lies in the fact that we do not assume both lower and upper solutions in our existence theorems even if they exist, but assume only one of them and prove the existence of solutions in a constructive way. Furthermore, even if the lower and an upper solution exist, they may not satisfy the inequality mentioned above. Therefore, in such a case the usual classic upper and lower solution method does not work. This is the main advantage of the present approach over the previous classic upper and lower solution method for the nonlinear differential and other different types of equations.

6. Conclusion

From the foregoing discussion it is clear that the Dhage iteration principle forms an interesting powerful tool for discussing the existence results of certain nonlinear hybrid fractional differential equations. However, it has some limitations that unlike Picard's method, the new method does not give any idea about the rate of convergence of the sequence of successive approximations. But in a way we have been able to prove the numerical solutions of the considered nonlinear fractional differential equation (1.1) and its perturbation (4.1). Finally, while concluding this paper we mention that the fractional differential equation considered here is of very simple nature for which we have illustrated the Dhage iteration principle to obtain the algorithms for the solutions under weaker partially Lipschitz and compactness conditions. However, an analogous study could also be made for other complex fractional differential equations using similar method with appropriate modifications. Some of the results along this line will be reported elsewhere.

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