International Journal of Analysis and Applications
ISSN 2291-8639
Volume 8, Number 1 (2015), 15-21
http://www.etamaths.com

AN ANALOG OF TITCHMARSH'S THEOREM FOR THE JACOBI-DUNKL TRANSFORM IN THE SPACE $L^2_{\alpha,\beta}(\mathbb{R})$

A. ABOUELAZ, A. BELKHADIR* AND R. DAHER

Abstract. In this paper, using a generalized Jacobi-Dunkl translation operator, we prove an analog of Titchmarsh's theorem for functions satisfying the Jacobi-Dunkl Lipschitz condition in $L^2(\mathbb{R}, A_{\alpha,\beta}(t)dt)$, $\alpha \geq \beta \geq -\frac{1}{2}, \alpha \neq -\frac{1}{2}$.

1. Introduction

Titchmarsh's theorem characterizes the set of functions satisfying the Cauchy-Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namely we have:

Theorem 1.1. [10] *Let* $\alpha \in (0,1)$ *and assume that* $f \in L^2(\mathbb{R})$. *Then the following are equivalents:*

(1) $\|f(t+h) - f(t)\| = O(h^\alpha)$, *as* $\alpha \to 0$;

(2) $\displaystyle\int_{|\lambda|\geq r} |\hat{f}(\lambda)|^2 d\lambda = O(r^{-2\alpha})$, *as* $r \to \infty$.

where $\hat{f}$ *stands for the Fourier transform of* f .

In this paper, we prove an analog of Theorem 1.1 for the Jacobi-Dunkl transform for functions satisfying the Jacobi-Dunkl Lipschitz condition in the space $L^2(\mathbb{R}, A_{\alpha,\beta}(t)dt)$. For this purpose, we use the generalized translation operator. Similar results have been established in the context of noncompact rank one Riemannian symetric spaces [9].

In section 2 below, we recapitulate from [1, 2, 3, 5] some results related to the harmonic analysis associated with Jacobi-Dunkl operator $\Lambda_{\alpha,\beta}$.

Section 3 is devoted to the main result after defining the class $Lip(\delta, 2, \alpha, \beta)$ of functions in $L^2_{\alpha,\beta}(\mathbb{R})$ satisfying the Lipschitz condition correspondent to the generalized Jacobi-Dunkl translation.

2. Notations and preliminaries

The Jacobi-Dunkl function with parameters (α, β) , $\alpha \geq \beta \geq -\frac{1}{2}, \alpha \neq -\frac{1}{2}$, is defined by the formula :

$$(1) \qquad \forall x \in \mathbb{R}, \quad \psi_\lambda^{(\alpha,\beta)}(x) = \begin{cases} \varphi_\mu^{(\alpha,\beta)}(x) - \dfrac{i}{\lambda}\dfrac{d}{dx}\varphi_\mu^{(\alpha,\beta)}(x) & , \text{ if } \lambda \in \mathbb{C}\setminus\{0\}; \\ 1 & , \text{ if } \lambda = 0. \end{cases}$$

2010 *Mathematics Subject Classification.* 33C45.

Key words and phrases. Titchmarsh's theorem; Jacobi-Dunkl transform; generalized Jacobi-Dunkl translation; Jacobi-Dunkl Lipschitz condition.

with $\lambda^2 = \mu^2 + \rho^2$, $\rho = \alpha + \beta + 1$ and $\varphi_\mu^{(\alpha,\beta)}$ is the Jacobi function given by:

$$\varphi_\mu^{(\alpha,\beta)}(x) = F\left(\frac{\rho + i\mu}{2}, \frac{\rho - i\mu}{2}; \alpha + 1, -(sinh(x))^2\right), \tag{2}$$

F is the Gauss hypergeometric function (see [1, 6, 7]).
$\psi_\lambda^{(\alpha,\beta)}$ is the unique C^∞-solution on $\mathbb{R}$ of the differentiel-difference equation

$$\begin{cases} \Lambda_{\alpha,\beta}\mathcal{U} = i\lambda\mathcal{U} & , \ \lambda \in \mathbb{C}; \\ \mathcal{U}(0) = 1. \end{cases} \tag{3}$$

where $\Lambda_{\alpha,\beta}$ is the Jacobi-Dunkl operator given by:

$$\Lambda_{\alpha,\beta}\mathcal{U}(x) = \frac{d\mathcal{U}}{dx}(x) + [(2\alpha + 1)\coth x + (2\beta + 1)\tanh x] \times \frac{\mathcal{U}(x) - \mathcal{U}(-x)}{2}$$

The operator $\Lambda_{\alpha,\beta}$ is a particular case of the operator D given by:

$$D\mathcal{U}(x) = \frac{d\mathcal{U}}{dx}(x) + \frac{A'(x)}{A(x)}\left(\frac{\mathcal{U}(x) - \mathcal{U}(-x)}{2}\right)$$

where $A(x) = |x|^{2\alpha+1}B(x)$, and B a function of class C^∞ on $\mathbb{R}$, even and positive.
The operator $\Lambda_{\alpha,\beta}$ corresponds to the function

$$A(x) = A_{\alpha,\beta}(x) = 2^\rho(\sinh|x|)^{2\alpha+1}(\cosh|x|)^{2\beta+1}.$$

Using the relation

$$\frac{d}{dx}\varphi_\mu^{(\alpha,\beta)}(x) = -\frac{\mu^2 + \rho^2}{4(\alpha + 1)}\sinh(2x)\varphi_\mu^{(\alpha+1,\beta+1)}(x) \ ,$$

the function $\psi_\lambda^{(\alpha,\beta)}$ can be written in the form above (See [2]),

$$\psi_\lambda^{(\alpha,\beta)}(x) = \varphi_\mu^{(\alpha,\beta)}(x) + i\frac{\lambda}{4(\alpha + 1)}\sinh(2x)\varphi_\mu^{(\alpha+1,\beta+1)}(x) \ , \ \forall x \in \mathbb{R} \ , \tag{4}$$

where $\lambda^2 = \mu^2 + \rho^2$, $\quad \rho = \alpha + \beta + 1$.
Denote by $L^2_{\alpha,\beta}(\mathbb{R}) = L^2(\mathbb{R}, A_{\alpha,\beta}(t)dt)$ the space of measurable functions g on $\mathbb{R}$ such that

$$||g||_{L^2_{\alpha,\beta}(\mathbb{R})} = \left(\int_{\mathbb{R}} |g(t)|^2 A_{\alpha,\beta}(t)dt\right)^{1/2} < +\infty$$

.

Using the eigenfunctions $\psi_\lambda^{(\alpha,\beta)}$ of the operator $\Lambda_{\alpha,\beta}$ called the Jacobi-Dunkl kernels, we define the Jacobi-Dunkl transform of a function $f \in L^2_{\alpha,\beta}(\mathbb{R})$ by:

$$\mathcal{F}_{\alpha,\beta}(f)(\lambda) = \int_{\mathbb{R}} f(x)\psi_\lambda^{(\alpha,\beta)}(x)A_{\alpha,\beta}(x)dx, \ \forall\lambda \in \mathbb{R} \ . \tag{5}$$

and the inversion formula

$$f(t) = \int_{\mathbb{R}} \mathcal{F}_{\alpha,\beta}(f)(\lambda)\psi_{-\lambda}^{(\alpha,\beta)}(t)d\sigma(\lambda) \ , \tag{6}$$

where: $\quad d\sigma(\lambda) = \dfrac{|\lambda|}{8\pi\sqrt{\lambda^2 - \rho^2}|C_{\alpha,\beta}(\sqrt{\lambda^2 - \rho^2})|}\mathbb{I}_{\mathbb{R}\setminus]-\rho,\rho[}(\lambda)d\lambda$

Here, $C_{\alpha,\beta}(\mu)=\dfrac{2^{\rho-i\mu}\Gamma(\alpha+1)\Gamma(i\mu)}{\Gamma(\frac{1}{2}(\rho+i\mu))\Gamma(\frac{1}{2}(\alpha-\beta+1+i\mu))}$, $\mu\in\mathbb{C}\setminus(i\mathbb{N})$.

and $\mathbb{I}_{\mathbb{R}\setminus]-\rho,\rho[}$ is the characteristic function of $\mathbb{R}\setminus]-\rho,\rho[$.

Denote $L^2_\sigma(\mathbb{R})=L^2(\mathbb{R},d\sigma(\lambda))$.

The Jacobi-Dunkl transform is a unitary isomorphism from $L^2_{\alpha,\beta}(\mathbb{R})$ onto $L^2_\sigma(\mathbb{R})$, i.e.

$$||f||=||f||_{L^2_{\alpha,\beta}(\mathbb{R})}=||\mathcal{F}_{\alpha,\beta}(f)||_{L^2_\sigma(\mathbb{R})}\ . \tag{7}$$

The operator of Jacobi-Dunkl translation is defined by:

$$T_xf(y)=\int_{\mathbb{R}}f(z)d\nu^{\alpha,\beta}_{x,y}(z)\ ,\ \forall\ x,y\in\mathbb{R}\ . \tag{8}$$

where $\nu^{\alpha,\beta}_{x,y}$, $x,y\in\mathbb{R}$ are the signed measures given by

$$d\nu^{\alpha,\beta}_{x,y}(z)=\begin{cases}K_{\alpha,\beta}(x,y,z)A_{\alpha,\beta}(z)dz & \text{, if } x,y\in\mathbb{R}^\star;\\ \delta_x & \text{, if } y=0;\\ \delta_y & \text{, if } x=0.\end{cases} \tag{9}$$

Here, δ_x is the Dirac measure at x.

And,

$$\begin{aligned}K_{\alpha,\beta}(x,y,z)= \quad & M_{\alpha,\beta}(\sinh(|x|)\sinh(|y|)\sinh(|z|))^{-2\alpha}\mathbb{I}_{I_{x,y}}\times\textstyle\int_0^\pi\rho_\theta(x,y,z)\\ & \times(g_\theta(x,y,z))_+^{\alpha-\beta-1}\sin^{2\beta}\theta d\theta.\end{aligned}$$

$$I_{x,y}=[-|x|-|y|,-||x|-|y||]\cup[||x|+|y||,|x|+|y|]\ ,$$

$$\rho_\theta(x,y,z)=1-\sigma^\theta_{x,y,z}+\sigma^\theta_{z,x,y}+\sigma^\theta_{z,y,x}$$

$$\sigma^\theta_{x,y,z}=\begin{cases}\dfrac{\cosh(x)+\cosh(y)-\cosh(z)\cos(\theta)}{\sinh(x)\sinh(y)} & \text{, if } xy\neq0;\\ 0 & \text{, if } xy=0.\end{cases}\quad ,\ \forall\ x,y,z\in\mathbb{R}\ ,\ \forall\ \theta\in[0,\pi]\ .$$

$$g_\theta(x,y,z)=1-\cosh^2x-\cosh^2y-\cosh^2z+2\cosh x\cosh y\cosh z\cos\theta\ .$$

$$t_+=\begin{cases}t & \text{, if } t>0;\\ 0 & \text{, if } t\leq0.\end{cases}$$

and,

$$M_{\alpha,\beta}=\begin{cases}\dfrac{2^{-2\rho}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})} & \text{, if } \alpha>\beta;\\ 0 & \text{, if } \alpha=\beta.\end{cases}$$

In [2], we have

$$\mathcal{F}_{\alpha,\beta}(T_hf)(\lambda)=\psi^{\alpha,\beta}_\lambda(h).\mathcal{F}_{\alpha,\beta}(f)(\lambda)\quad;\quad h,\lambda\in\mathbb{R}\ . \tag{10}$$

For $\alpha\geq\frac{-1}{2}$, we introduce the bessel normalized function of the first kind defined by

$$j_\alpha(z)=\Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{z}{2})^{2n}}{n!\Gamma(n+\alpha+1)}\quad,\ z\in\mathbb{C}.$$

Moreover, we see that $\lim_{z\to 0}\dfrac{j_\alpha(z)-1}{z^2}\neq 0$, by consequence, there exists $C_1>0$ and $\eta>0$ satisfying

$$|z|\le\eta\Rightarrow\quad |j_\alpha(z)-1|\ge C_1|z|^2\ . \tag{11}$$

Lemma 2.1. *The following inequalities are valids for Jacobi functions* $\varphi_\mu^{\alpha,\beta}(t)$ *:*

(1) $|\varphi_\mu^{(\alpha,\beta)}(t)|\le 1$;
(2) $|1-\varphi_\mu^{(\alpha,\beta)}(t)|\le t^2(\mu^2+\rho^2)$.

Proof. (See [8], Lemma 3.1-3.2) □

Lemma 2.2. *Let* $\alpha\ge\beta\ge\frac{-1}{2}$, $\alpha\neq\frac{-1}{2}$. *Then for* $|\nu|\le\rho$, *there exists a positive constant* C_2 *such that*

$$|1-\varphi_{\mu+i\nu}^{(\alpha,\beta)}(t)|\ge C_2|1-j_\alpha(\mu t)|\ .$$

Proof. (See [4], Lemma 9) □

3. Main result

In this section we introduce and prove an analog of theorem 1.1. Firstly we have to define, for functions in $L^2_{\alpha,\beta}(\mathbb{R})$, the condition of Cauchy-Lipschitz related to the Jacobi-Dunkl translation operator given in (8).

Definition 3.1. *Let* $\delta\in(0,1)$. *A function* $f\in L^2_{\alpha,\beta}(\mathbb{R})$ *is said to be in the Jacobi-Dunkl-Lipschitz class, denoted by* $Lip(\delta,2,\alpha,\beta)$, *if* $||T_hf+T_{-h}f-2f||=O(h^\delta)$, *as* $h\to 0$.

Theorem 3.2. *Let* $f\in L^2_{\alpha,\beta}(\mathbb{R})$. *Then the following are equivalents:*

(1) $f\in Lip(\delta,2,\alpha,\beta)$;
(2) $\displaystyle\int_{|\lambda|\ge r}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda)=O(r^{-2\delta})$, *as* $r\to\infty$.

Proof. 1) ⇒ 2) . Assume that $f\in Lip(\delta,2,\alpha,\beta)$; then we have:

$$||T_hf+T_{-h}f-2f||=O(h^\delta)\quad ,\ as\ h\to 0\,.$$

$$\mathcal{F}_{\alpha,\beta}(T_hf+T_{-h}f-2f)(\lambda)=(\psi_\lambda^{(\alpha,\beta)}(h)+\psi_\lambda^{(\alpha,\beta)}(-h)-2).\mathcal{F}_{\alpha,\beta}(f)(\lambda).$$

Since $\psi_\lambda^{(\alpha,\beta)}(h)=\varphi_\mu^{(\alpha,\beta)}(h)+i\dfrac{\lambda}{4(\alpha+1)}\sinh(2h)\varphi_\mu^{(\alpha+1,\beta+1)}(h)$,

$$\psi_\lambda^{(\alpha,\beta)}(-h)=\varphi_\mu^{(\alpha,\beta)}(-h)-i\frac{\lambda}{4(\alpha+1)}\sinh(2h)\varphi_\mu^{(\alpha+1,\beta+1)}(-h),$$

and $\varphi_\mu^{(\alpha,\beta)}$ is even [See (2)]; then:

$$\mathcal{F}_{\alpha,\beta}(T_hf+T_{-h}f-2f)(\lambda)=2(\varphi_\mu^{(\alpha,\beta)}(h)-1).\mathcal{F}_{\alpha,\beta}(f)(\lambda).$$

From Parseval's identity (7) we write:

$$||T_hf+T_{-h}f-2f||^2=4\int_{\mathbb{R}}|1-\varphi_\mu^{(\alpha,\beta)}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda). \tag{12}$$

By (11) and lemma 2.2, we get:

$$\int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} |1-\varphi_\mu^{(\alpha,\beta)}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda) \geq \int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} C_1^2C_2^2|\mu h|^4|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda)\,,$$

From $\dfrac{\eta}{2h} \leq |\lambda| \leq \dfrac{\eta}{h}$ we have,

$$\left(\frac{\eta}{2h}\right)^2 - \rho^2 \leq \mu^2 \leq \left(\frac{\eta}{h}\right)^2 - \rho^2$$

$$\Rightarrow \mu^2h^2 \geq \frac{\eta 2}{4} - \rho^2h^2$$

Take $h \leq \dfrac{\eta}{3\rho}$, then we have $\mu^2h^2 \geq C_3 = C_3(\eta)$.

So,

$$\int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} |1-\varphi_\mu^{(\alpha,\beta)}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda) \geq C_1^2C_2^2C_3^2 \int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda)\,.$$

There exists then a positive constant C such that:

$$\begin{aligned}\int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda) &\leq C\int_{\mathbb{R}} |1-\varphi_\mu^{(\alpha,\beta)}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda)\\ &\leq Ch^{2\delta}\,,\end{aligned}$$

For all $0 < h \leq \dfrac{\eta}{3\rho}$, (see (12)). Then we have,

$$\int_{r\leq|\lambda|\leq 2r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda) \leq Cr^{-2\delta}\,, \quad as\ r\to\infty.$$

Furthermore, we obtain:

$$\begin{aligned}\int_{|\lambda|\geq r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda) &= \sum_{i=0}^{\infty}\int_{2^ir\leq|\lambda|\leq 2^{i+1}r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda)\\ &\leq C\sum_{i=0}^{\infty}\left(2^ir\right)^{-2\delta}\\ &\leq Cr^{-2\delta}.\end{aligned}$$

This proves that:

$$\int_{|\lambda|\geq r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda) = O(r^{-2\delta}) \quad , \ as\ r\to\infty.$$

2) $\Rightarrow$ 1) . Suppose now that

$$\int_{|\lambda|\geq r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda) = O(r^{-2\delta}) \quad , \ as\ r\to\infty,$$

and write

$$\begin{aligned}\int_{\mathbb{R}} |1-\varphi_\mu^{(\alpha,\beta)}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda) &= \int_{|\lambda|<\frac{1}{h}} |1-\varphi_\mu^{(\alpha,\beta)}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda)\\ &+ \int_{|\lambda|\geq\frac{1}{h}} |1-\varphi_\mu^{(\alpha,\beta)}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda)\end{aligned}$$

— Using the inequality (1) of lemma 2.1, we get:

$$\int_{|\lambda|\geq\frac{1}{h}} |1-\varphi_\mu^{(\alpha,\beta)}(h)|^2 |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \leq 4\int_{|\lambda|\geq\frac{1}{h}} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda)$$

then,

$$\int_{|\lambda|\geq\frac{1}{h}} |1-\varphi_\mu^{(\alpha,\beta)}(h)|^2 |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) = O(h^{2\delta}) \ , \quad as\ h\to 0. \tag{13}$$

— Set $\phi(x) = \displaystyle\int_x^\infty |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda)$.

An integration by parts gives:

$$\begin{aligned}
\int_0^x \lambda^2 |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) &= \int_0^x -\lambda^2\phi'(\lambda)d\lambda \\
&= -x^2\phi(x) + 2\int_0^x \lambda\phi(\lambda)d\lambda \\
&\leq 2\int_0^x O(\lambda^{1-2\delta})d\lambda \\
&= O(x^{2-2\delta}).
\end{aligned}$$

From the second inequality of lemma 2.1, we get

$$\begin{aligned}
\int_{|\lambda|<\frac{1}{h}} |1-\varphi_\mu^{(\alpha,\beta)}(h)|^2 |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) &\leq \int_{|\lambda|<\frac{1}{h}} (\mu^2+\rho^2)h^2 |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \\
&\leq h^2 \int_{|\lambda|<\frac{1}{h}} \lambda^2 |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \\
&= O(h^2.h^{-2+2\delta}).
\end{aligned}$$

Hence,

$$\int_{|\lambda|<\frac{1}{h}} |1-\varphi_\mu^{(\alpha,\beta)}(h)|^2 |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) = O(h^{2\delta}). \tag{14}$$

Finally, we conclude from (13) and (14) that

$$\begin{aligned}
\int_{\mathbb{R}} |1-\varphi_\mu^{(\alpha,\beta)}(h)|^2 |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) &= \int_{|\lambda|<\frac{1}{h}} + \int_{|\lambda|\geq\frac{1}{h}} \\
&= O(h^{2\delta}) + O(h^{2\delta}) \\
&= O(h^{2\delta}).
\end{aligned}$$

And this ends the proof. □

References

[1] Ben Mohamed. H and Mejjaoli. H, *Distributional Jacobi-Dunkl transform and application*, Afr. Diaspora J. Math 1 (2004), 24–46.

[2] Ben Mohamed. H, *The Jacobi-Dunkl transform on* $\mathbb{R}$ *and the convolution product on new spaces of distributions* , Ramanujan J. 21 (2010), 145–175.

[3] Ben Salem. N and Ould Ahmed Salem. A , *Convolution structure associated with the Jacobi-Dunkl operator on* $\mathbb{R}$., Ramanujan J. 12(3) (2006), 359–378.

[4] Bray. W O and Pinsky. M A, *Growth properties of Fourier transforms via moduli of continuity* , Journal of Functional Analysis. 255 (2008), 2256–2285.
[5] Chouchane. F, Mili. M and Trimèche. K, *Positivity of the intertwining opertor and harmonic analysis associated with the Jacobi-Dunkl operator on* $\mathbb{R}$, J. Anal. Appl. 1(4) (2003), 387–412.
[6] Koornwinder. T H , *Jacobi functions and analysis on noncompact semi-simple Lie Groups*, in: Askey. R A , Koornwinder. T H and Schempp. W (eds) Special Functions: Group Theoritical Aspects and Applications. D. Reidel, Dordrecht (1984).
[7] Koornwinder. T H , *A new proof of a Paley-Wiener type theorems for the Jacobi transform*, Ark. Math. 13 (1975), 145–159.
[8] Platonov. S S, *Approximation of functions in* L_2*-metric on noncompact rank 1 symetric spaces*, Algebra Analiz. 11 (1) (1999), 244–270.
[9] Platonov. S S, *The Fourier transform of functions satisfying the Lipschitz condition on rank 1 symetric spaces*, Siberian Math. J. 46(6) (2005), 1108–1118.
[10] Titchmarsh. E C, *Introduction to the Theory of Fourier Integrals* , Claredon, Oxford. 1948, Komkniga, Moscow. 2005.

Department of Mathematics and Informatics, Faculty of Science Ain Chock, University of Hassan II, Casablanca, Morocco,

*Corresponding author