International Journal of Analysis and Applications
ISSN 2291-8639
Volume 6, Number 2 (2014), 195-204
http://www.etamaths.com

FIXED POINT THEOREM OF MODIFIED S-ITERATION PROCESS FOR CIRIC QUASI CONTRACTIVE OPERATOR IN CAT(0) SPACES

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Abstract. The aim of this paper is to study the strong convergence of modified S-iteration process for Ciric quasi contractive operator in the framework of $CAT(0)$ spaces. Also we give an application of our result with supporting example. Our result improves and extends some corresponding previous result from the existing literature (see, e.g., [3, 29] and many others).

1. Introduction and Preliminaries

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces (see [6]), $\mathbb{R}$-trees (see [22]), Euclidean buildings (see [7]), the complex Hilbert ball with a hyperbolic metric (see [15]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [6].

Fixed point theory in a CAT(0) space was first studied by Kirk (see [23, 24]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since, then the fixed point theory for single-valued and multi-valued mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared (see, e.g., [2], [9], [12]-[14], [16], [20]-[21], [25]-[26], [28], [30]-[31] and references therein). It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(k) space with $k \leq 0$ since any CAT(k) space is a CAT(k') space for every $k' \geq k$ (see,e.g., [6]).

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry, and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y. We say X is (i) a *geodesic space* if any two points of X are joined by a geodesic and (ii) a *uniquely geodesic* if there is exactly

2010 *Mathematics Subject Classification.* 54H25, 54E40.
Key words and phrases. Ciric quasi contractive operator; modified S-iteration process; fixed point; strong convergence; CAT(0) space.

one geodesic joining x and y for each $x, y \in X$, which we will denoted by $[x, y]$, called the segment joining x to y.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [6]).

CAT(0) space

A geodesic metric space is said to be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y}). \tag{1.1}$$

Complete $CAT(0)$ spaces are often called *Hadamard spaces* (see [19]). If x, y_1, y_2 are points of a $CAT(0)$ space and y_0 is the midpoint of the segment $[y_1, y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the $CAT(0)$ inequality implies

$$d^2\Big(x, \frac{y_1 \oplus y_2}{2}\Big) \leq \frac{1}{2} d^2(x, y_1) + \frac{1}{2} d^2(x, y_2) - \frac{1}{4} d^2(y_1, y_2). \tag{1.2}$$

The inequality (1.2) is the (CN) inequality of Bruhat and Tits [8]. The above inequality was extended in [13] as

$$\begin{aligned} d^2(z, \alpha x \oplus (1-\alpha) y) &\leq \alpha d^2(z, x) + (1-\alpha) d^2(z, y) \\ &\quad -\alpha(1-\alpha) d^2(x, y) \end{aligned} \tag{1.3}$$

for any $\alpha \in [0, 1]$ and $x, y, z \in X$.

Let us recall that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (see [[6], p.163]). Moreover, if X is a $CAT(0)$ metric space and $x, y \in X$, then for any $\alpha \in [0, 1]$, there exists a unique point $\alpha x \oplus (1-\alpha) y \in [x, y]$ such that

$$d(z, \alpha x \oplus (1-\alpha) y) \leq \alpha d(z, x) + (1-\alpha) d(z, y), \tag{1.4}$$

for any $z \in X$ and $[x, y] = \{\alpha x \oplus (1-\alpha) y : \alpha \in [0, 1]\}$.

A subset C of a $CAT(0)$ space X is convex if for any $x, y \in C$, we have $[x, y] \subset C$.

The Mann iteration process [27] is defined by the sequence $\{x_n\}$,

$$\begin{cases} x_1 \in C, \\ \quad x_{n+1} = (1-\alpha_n) x_n + \alpha_n T x_n, \quad n \geq 1, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ is a sequence in (0,1).

Further, the Ishikawa iteration process [17] is defined by the sequence $\{x_n\}$,

$$\left\{\begin{aligned} & x_1 \in C, \\ & \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ & \qquad y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 1, \end{aligned}\right. \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0,1). This iteration process reduces to the Mann iteration process when $\beta_n = 0$ for all $n \geq 1$.

In 2007, Agarwal, O'Regan and Sahu [1] introduced the S-iteration process in a Banach space,

$$\left\{\begin{aligned} & x_1 \in C, \\ & \quad x_{n+1} = (1-\alpha_n)T x_n + \alpha_n T y_n, \\ & \qquad y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 1, \end{aligned}\right. \tag{1.7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0,1). Note that (1.3) is independent of (1.2) (and hence (1.1)). They showed that their process is independent of those of Mann and Ishikawa and converges faster than both of these (see [[1], Proposition 3.1]).

We now modify (1.7) in a CAT(0) space as follows.

Let C be a nonempty closed convex subset of a complete CAT(0) space X and $T\colon C \to C$ be a mapping. Suppose that $\{x_n\}$ is a sequence generated iteratively by

$$\left\{\begin{aligned} & x_1 \in C, \\ & \quad x_{n+1} = (1-\alpha_n)T x_n \oplus \alpha_n T y_n, \\ & \qquad y_n = (1-\beta_n)x_n \oplus \beta_n T x_n, \quad n \geq 1, \end{aligned}\right. \tag{1.8}$$

where and throughout the paper $\{\alpha_n\}$, $\{\beta_n\}$ are the sequences such that $0 \leq \alpha_n, \beta_n \leq 1$ for all $n \geq 1$.

We recall the following definitions in a metric space (X, d). A mapping $T\colon X \to X$ is called an a-contraction if

$$d(Tx, Ty) \quad \leq \quad a\, d(x, y) \text{ for all } x,\ y \in X, \tag{1.9}$$

where $a \in (0, 1)$.

The mapping T is called Kannan mapping [18] if there exists $b \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \quad \leq \quad b\,[d(x, Tx) + d(y, Ty)] \text{ for all } x,\ y \in X. \tag{1.10}$$

The mapping T is called Chatterjea mapping [10] if there exists $c \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \quad \leq \quad c\,[d(x, Ty) + d(y, Tx)] \text{ for all } x,\ y \in X. \tag{1.11}$$

In 1972, Zamfirescu [32] obtained the following interesting fixed point theorem.

Theorem Z. Let (X, d) be a complete metric space and $T\colon X \to X$ a mapping for which there exists the real number a, b and c satisfying $a \in (0,1),\ b,\ c \in (0, \frac{1}{2})$ such that for any pair $x,\ y \in X$, at least one of the following conditions holds:

(Z_1) $d(Tx, Ty) \leq a\, d(x, y)$,

(Z_2) $d(Tx, Ty) \leq b\,[d(x, Tx) + d(y, Ty)]$,

(Z_3) $d(Tx, Ty) \leq c\,[d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n,\ n = 0, 1, 2, \ldots$$

converges to p for any arbitrary but fixed $x_0 \in X$.

The conditions $(Z_1) - (Z_3)$ can be written in the following equivalent form

$$\begin{aligned} d(Tx, Ty) \quad &\leq \quad h \max\Big\{ d(x, y),\ \frac{d(x, Tx) + d(y, Ty)}{2}, \\ &\qquad\qquad \frac{d(x, Ty) + d(y, Tx)}{2} \Big\}, \end{aligned} \tag{1.12}$$

$\forall\, x,\ y \in X$ and $0 < h < 1$, has been obtained by Ciric [11] in 1974.

A mapping satisfying (1.12) is called Ciric quasi-contraction. It is obvious that each of the conditions $(Z_1) - (Z_3)$ implies (1.12).

An operator T satisfying the contractive conditions $(Z_1) - (Z_3)$ in the theorem Z is called Z-operator.

In 2000, Berinde [3] introduced a new class of operators on a normed space E satisfying

$$\|Tx - Ty\| \leq \delta\, \|x - y\| + L\, \|Tx - x\|\,, \quad (*)$$

for any $x,\ y \in E$, $0 \leq \delta < 1$ and $L \geq 0$.

He proved that this class is wider than the class of Zamfirescu operators and used the Mann iteration process to approximate fixed points of this class of operators in a normed space given in the form of following theorem.

Theorem B. Let C be a nonempty closed convex subset of a normed space E. Let $T\colon C \to C$ be an operator satisfying $(*)$. Let $\{x_n\}_{n=0}^{\infty}$ be defined by: for $x_1 = x \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ given by

$$x_{n+1} \quad = \quad (1 - b_n)x_n + b_n T x_n,\ n \geq 0,$$

where $\{b_n\}$ is a sequence in [0,1]. If $F(T) \neq \emptyset$ and $\sum_{n=1}^{\infty} b_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.

In this paper, inspired and motivated by [1, 32], we study S-iteration process and establish strong convergence theorem to approximate the fixed point for Ciric quasi contractive operator in the framework of CAT(0) spaces.

We need the following useful lemmas to prove our main result in this paper.

Lemma 1.1. (See [28]) Let X be a CAT(0) space.

(i) For $x,\, y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x,\, y]$ such that

$$d(x,\, z) = t\, d(x,\, y) \quad \text{and} \quad d(y,\, z) = (1-t)\, d(x,\, y). \quad (A)$$

We use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (A).

(ii) For $x,\, y \in X$ and $t \in [0,1]$, we have

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z).$$

Lemma 1.2. (See [4]) Let $\{p_n\}_{n=0}^{\infty}$, $\{q_n\}_{n=0}^{\infty}$, $\{r_n\}_{n=0}^{\infty}$ be sequences of nonnegative numbers satisfying the following condition:

$$p_{n+1} \leq (1-s_n)p_n + q_n + r_n,\ \forall\, n \geq 0,$$

where $\{s_n\}_{n=0}^{\infty} \subset [0,1]$. If $\sum_{n=0}^{\infty} s_n = \infty$, $\lim_{n\to\infty} q_n = O(s_n)$ and $\sum_{n=0}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} p_n = 0$.

2. **Strong convergence theorem in CAT(0) Space**

In this section, we establish strong convergence result of modified S-iteration process to approximate a fixed point for Ciric quasi contractive operator in the framework of CAT(0) spaces.

Theorem 2.1. Let C be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon C \to C$ be an operator satisfying the condition (1.12). Let $\{x_n\}$ be defined by the iteration scheme (1.8). If $\sum_{n=0}^{\infty} \alpha_n\beta_n = \infty$, then $\{x_n\}$ converges strongly to the unique fixed point of T.

proof. By Theorem Z, we know that T has a unique fixed point in C, say u. Consider $x,\, y \in C$. Since T is a operator satisfying (1.12), then if

$$\begin{aligned} d(Tx,Ty) &\leq \frac{h}{2}[d(x,Tx)+d(y,Ty)] \\ &\leq \frac{h}{2}[d(x,Tx)+d(y,x)+d(x,Tx)+d(Tx,Ty)], \end{aligned}$$

implies

$$\Big(1-\frac{h}{2}\Big)d(Tx,Ty) \leq \frac{h}{2}d(x,y) + hd(x,Tx),$$

which yields (using the fact that $0 < h < 1$)

$$d(Tx,Ty) \leq \Big(\frac{h/2}{1-h/2}\Big)d(x,y) + \Big(\frac{h}{1-h/2}\Big)d(x,Tx). \tag{2.1}$$

If

$$\begin{aligned} d(Tx,Ty) &\leq \frac{h}{2}[d(x,Ty)+d(y,Tx)] \\ &\leq \frac{h}{2}[d(x,Tx)+d(Tx,Ty)+d(y,x)+d(x,Tx)], \end{aligned}$$

implies

$$\Big(1-\frac{h}{2}\Big)d(Tx,Ty) \leq \frac{h}{2}\, d(x,y)+h\, d(x,Tx),$$

which also yields (using the fact that $0<h<1$)

$$d(Tx,Ty) \leq \Big(\frac{h/2}{1-h/2}\Big)d(x,y)+\Big(\frac{h}{1-h/2}\Big)d(x,Tx). \tag{2.2}$$

Denote

$$\delta = \max\Big\{h,\ \frac{h/2}{1-h/2}\Big\} = h,$$

$$L = \max\Big\{\frac{h}{1-h/2}, \frac{h}{1-h/2}\Big\} = \frac{h}{1-h/2}.$$

Thus, in all cases,

$$\begin{aligned} d(Tx,Ty) &\leq \delta\, d(x,y)+L\, d(x,Tx) \\ &= h\, d(x,y)+\Big(\frac{h}{1-h/2}\Big)d(x,Tx). \end{aligned} \tag{2.3}$$

holds for all $x,\, y \in C$.

Also from (1.12) with $y=u=Tu$, we have

$$\begin{aligned} d(Tx,u) &\leq h\max\Big\{d(x,u),\ \frac{d(x,Tx)}{2},\ \frac{d(x,u)+d(u,Tx)}{2}\Big\} \\ &\leq h\max\Big\{d(x,u),\ \frac{d(x,Tx)}{2},\ \frac{d(x,u)+d(u,Tx)}{2}\Big\} \\ &\leq h\max\Big\{d(x,u),\ \frac{d(x,u)+d(u,Tx)}{2},\ \frac{d(x,u)+d(u,Tx)}{2}\Big\}. \end{aligned} \tag{2.4}$$

Since for non-negative real numbers a and b, we have

$$\frac{a+b}{2} \leq \max\{a,\, b\}. \tag{2.5}$$

Using (2.5) in (2.4), we have

$$d(Tx,u) \leq h\, d(x,u). \tag{2.6}$$

Now (2.6) gives

$$d(Tx_n,u) \leq h\, d(x_n,u) \tag{2.7}$$

and

$$d(Ty_n,u) \leq h\, d(y_n,u). \tag{2.8}$$

Using (1.8), (2.8) and Lemma 1.1(ii), we have

$$
\begin{aligned}
d(y_n, u) &= d((1-\beta_n)x_n \oplus \beta_n T x_n, u) \\
&\leq (1-\beta_n)d(x_n, u) + \beta_n d(Tx_n, u) \\
&\leq (1-\beta_n)d(x_n, u) + h\beta_n d(x_n, u) \\
&= [1-(1-h)\beta_n]d(x_n, u). \qquad (2.9)
\end{aligned}
$$

Now using (1.8), (2.7), (2.9) and Lemma 1.1(ii), we have

$$
\begin{aligned}
d(x_{n+1}, u) &= d((1-\alpha_n)Tx_n \oplus \alpha\alpha_n T y_n, u) \\
&\leq (1-\alpha_n)d(Tx_n, u) + \alpha_n d(Ty_n, u) \\
&\leq (1-\alpha_n)h\, d(x_n, u) + h\alpha_n d(y_n, u) \\
&\leq (1-\alpha_n)h\, d(x_n, u) + h\alpha_n[1-(1-h)\beta_n]d(x_n, u) \\
&= [1-(1-h)\alpha_n\beta_n]h\, d(x_n, u) \\
&\leq [1-(1-h)\alpha_n\beta_n]d(x_n, u) \\
&= [1-A_n]d(x_n, u), \qquad (2.10)
\end{aligned}
$$

Where $A_n = (1-h)\alpha_n\beta_n$. Since $0 < h < 1$; α_n, $\beta_n \in [0,1]$ and by assumption of the theorem $\sum_{n=0}^{\infty} \alpha_n\beta_n = \infty$, it follows that $\sum_{n=0}^{\infty} A_n = \infty$. Hence, by Lemma 1.2, we get that $\lim_{n\to\infty} d(x_n, u) = 0$. Therefore $\{x_n\}$ converges strongly to a fixed point of T.

To show uniqueness of the fixed point u, assume that u_1, $u_2 \in F(T)$ and $u_1 \neq u_2$.

Applying (1.12) and using the fact that $0 < h < 1$, we obtain

$$
\begin{aligned}
d(u_1, u_2) &= d(Tu_1, Tu_2) \\
&\leq h \max\Big\{d(u_1, u_2),\ \frac{d(u_1, Tu_1) + d(u_2, Tu_2)}{2}, \\
&\qquad \frac{d(u_1, Tu_2) + d(u_2, Tu_1)}{2}\Big\} \\
&= h \max\Big\{d(u_1, u_2),\ \frac{d(u_1, u_1) + d(u_2, u_2)}{2}, \\
&\qquad \frac{d(u_1, u_2) + d(u_2, u_1)}{2}\Big\} \\
&= h \max\Big\{d(u_1, u_2),\ 0,\ d(u_1, u_2)\Big\} \\
&\leq h\, d(u_1, u_2) \\
&< d(u_1, u_2), \ \text{since } 0 < h < 1,
\end{aligned}
$$

which is a contradiction. Therefore $u_1 = u_2$. Thus $\{x_n\}$ converges strongly to the unique fixed point of T. This completes the proof.

The contraction condition (1.9) makes T continuous function on X while this is not the case with contractive conditions (1.10), (1.11) and (2.3).

The contractive conditions (1.10) and (1.11) both included in the class of Zamfirescu operators and so their convergence theorems for modified S-iteration process are obtained in Theorem 2.1 in the setting of CAT(0) space.

Remark 2.1. Our result extends the corresponding result of [29] to the case of modified S-iteration process and from uniformly convex Banach space to the setting of CAT(0) spaces.

Remark 2.2. Theorem 2.1 also extends Theorem B to the case of modified S-iteration process and from normed space to the setting of CAT(0) spaces.

3. Application to contraction of integral type

Theorem 3.1. Let C be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon C \to C$ be an operator satisfying the following condition:

$$\int_0^{d(Tx,Ty)} \mu(t)dt \quad \leq \quad h \int_0^{\max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}} \mu(t)dt \tag{3.1}$$

for all $x, y \in X$ and $0 < h < 1$, where $\mu\colon [0,+\infty) \to [0,+\infty)$ is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0,+\infty)$, nonnegative, and such that for each $\varepsilon > 0$, $\int_0^\varepsilon \mu(t)dt > 0$. Let $\{x_n\}$ be defined by the iteration process (1.8). If $\sum_{n=0}^{\infty} \alpha_n\beta_n = \infty$, then $\{x_n\}$ converges strongly to the unique fixed point of T.

Proof. The proof of Theorem 3.1 follows from Theorem 2.1 by taking $\mu(t) = 1$ over $[0,+\infty)$ since the contractive condition of integral type transforms into a general contractive condition (1.12) not involving integrals. This completes the proof.

Example 3.1. Let $X = \{0, 1, 2, 3, 4\}$ and d be the usual metric of reals. Let $T\colon X \to X$ be given by

$$\begin{cases} Tx = 3, & \text{if} \quad x = 0 \\ \quad\; = 2, & \text{otherwise.} \end{cases}$$

Again let $\mu\colon [0,+\infty) \to [0,+\infty)$ be given by $\mu(t) = 1$ for all $t \in [0,+\infty)$. Then $\mu\colon [0,+\infty) \to [0,+\infty)$ is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0,+\infty)$, nonnegative, and such that for each $\varepsilon > 0$, $\int_0^\varepsilon \mu(t)dt > 0$.

Let us take $x = 0$, $y = 1$. Then from condition (3.1), we have

$$\begin{aligned} 1 = \int_0^{d(Tx,Ty)} \mu(t)dt \quad &\leq \quad h \int_0^{\max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}} \mu(t)dt \\ &= \quad h \max\{1, 2, 2\} \end{aligned}$$

which implies $h \geq \frac{1}{2}$. Now if we take $0 < h < 1$, then condition (3.1) is satisfied and 2 is of course a unique fixed point of T.

The following corollary is a special case of Theorem 3.1.

Corollary 3.1. Let C be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon C \to C$ be an operator satisfying the following condition:

$$\int_0^{d(Tx,Ty)} \mu(t)dt \quad \leq \quad h \int_0^{d(x,y)} \mu(t)dt \tag{3.2}$$

for all $x, y \in X$ and $h \in (0,1)$, where $\mu\colon [0,+\infty) \to [0,+\infty)$ is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0,+\infty)$, nonnegative, and such that for each $\varepsilon > 0$, $\int_0^\varepsilon \mu(t)dt > 0$. Let $\{x_n\}$ be defined by the iteration process (1.8). If $\sum_{n=0}^\infty \alpha_n\beta_n = \infty$, then $\{x_n\}$ converges strongly to the unique fixed point of T.

Condition (3.2) is called Branciari [5] contractive condition of integral type.

Putting $\mu(t) = 1$ in the condition (3.2), we get Banach contraction condition.

Proof of corollary 3.1. The proof of corollary 3.1 immediately follows from Theorem 2.1 by taking $\mu(t) = 1$ over $[0,+\infty)$ and

$$\max\Big\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\Big\} = d(x,y)$$

since the contractive condition of integral type transforms into a general contractive condition (1.9) not involving integrals. This completes the proof.

Example 3.2. Let X be the real line with the usual metric d and suppose $C = [0,1]$. Define $T\colon C \to C$ by $\mathrm{Tx} = \frac{\mathrm{x}+1}{2}$ for all $x, y \in C$. Obviously T is self-mapping with a unique fixed point 1. Again let $\mu\colon [0,+\infty) \to [0,+\infty)$ be given by $\mu(t) = 1$ for all $t \in [0,+\infty)$. Then $\mu\colon [0,+\infty) \to [0,+\infty)$ is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0,+\infty)$, nonnegative, and such that for each $\varepsilon > 0$, $\int_0^\varepsilon \mu(t)dt > 0$.

If $x, y \in [0,1]$, then we have

$$d(Tx,Ty) = \Big|\frac{x-y}{2}\Big|.$$

Let us take $x = 0$, $y = 1$. Then from condition (3.2), we have

$$\frac{1}{2} = \int_0^{d(Tx,Ty)} \mu(t)dt \quad \leq \quad h.1 = h\int_0^{d(x,y)} \mu(t)dt$$

which implies $h \geq \frac{1}{2}$. Now if we take $0 < h < 1$, then condition (3.2) is satisfied and 1 is of course a unique fixed point of T.

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