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ON THE MEROMORPHIC SOLUTIONS OF CERTAIN NONLINEAR DIFFERENCE EQUATIONS

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Abstract. In this article, we investigate the meromorphic solutions of certain non-linear difference equations using Tumura-Clunie theorem and also provide examples which satisfy our results.

1. Introduction and Main Results

Meromorphic solutions of complex differential equations and complex difference equations plays a prominent role in the field of Complex analysis. Solutions of such equations admits several ways of approach, but recently solutions of complex differential or difference equations by Nevanlinna theory techniques has become a subject of great interest.

The Clunie lemma and Tumura-Clunie type theorems were efficient tool in finding the solutions of complex differential or difference equations.

In this article, we solve certain complex non-linear difference equations using Tumura-Clunie type theorems. We assume that the reader is familiar with the basic notions of Nevanlinna's Value distribution theory [see [8],[9]].

In [7], Anupama J.Patil proved the following the result

Theorem A. No trancendental meromorphic function f with $N(r,f) = S(r,f)$ will satisfy an equation of the form

$$a_1(z)P(f)\Pi(f) + a_2(z)\Pi(f) + a_3(z) \equiv 0$$

where $a_1(z)(\not\equiv 0)$, $a_2(z)$ and $a_3(z)$ are small functions of f,

$$P(f) = b_n f^n + b_{n-1}f^{n-1} + \ldots + b_1 f + b_0$$

where n is a positive integer, $b_n(\not\equiv 0), b_{n-1}, \ldots, b_0$ are small functions of f and $\Pi(f)$ is a differential polynomial in f i.e,

$$\Pi(f) = \sum_{i=1}^{n} \alpha_i(z) f^{n_{i0}}(f')^{n_{i1}}(f'')^{n_{i2}} \ldots (f^{(m)})^{n_{im}}$$

In this paper, we obtain two main results by considering difference function $f(z+c)$ and difference polynomial in place of $\Pi(f)$ in Theorem A.

Theorem 1.1 No transcendental meromorphic function f of finite order ρ with $N(r,f) = O(r^{\rho-1+\epsilon}) + S(r,f)$ will satisfy the non-linear difference equation of the form

$$a_1(z)P(f)f(z+c) + a_2(z)f(z+c) + a_3(z) \equiv 0$$

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where $c \in C$, $a_1(z)(\neq 0)$, $a_2(z)$ and $a_3(z)(\neq 0)$ are small functions in the sense of $T(r, a_i) = O(r^{\rho-1+\epsilon}) + S(r, f), i = 1, 2, 3$ and

$$P(f) = b_n f^n + b_{n-1} f^{n-1} + \ldots + b_1 f + b_0$$

where n is a positive integer, $b_n(\not\equiv 0), b_{n-1}, \ldots, b_0$ are small functions in the sense of $T(r, b_j) = O(r^{\rho-1+\epsilon}) + S(r, f), j = 0, 1, 2, \ldots, n$.

Theorem 1.2 No transcendental meromorphic function f of finite order ρ with $N(r, f) + N(r, 1/f) = O(r^{\rho-1+\epsilon}) + S(r, f)$ will satisfy the difference equation of the form

$$a_1(z)P(f)\Pi(f) + a_2(z)\Pi(f) + a_3(z) \equiv 0$$

where $n \geq 1$, $a_1(z)(\neq 0)$, $a_2(z)$ and $a_3(z)(\neq 0)$ are small functions in the sense of $T(r, a_i) = O(r^{\rho-1+\epsilon}) + S(r, f), i = 1, 2, 3$ and

$$\Pi(f) = \sum_{\lambda} a_\lambda(z) f(z)^{l_0} f(z + c_1)^{l_1} \ldots f(z + c_\lambda)^{l_\lambda}$$

is a difference polynomial of degree n where $n = max \sum_{j=1}^{\lambda} l_j$ and $c_1, c_2, \ldots, c_\lambda$ are distinct values in C and $T(r, a_\lambda) = S(r, f)$

2. Some Lemmas

Lemma 2.1 [2]. Let f be a meromorphic function of finite order ρ, and suppose that

$$\Psi(z) = a_n(z) f(z)^n + \cdots + a_0(z)$$

has small meromorphic coefficients $a_j(z)$, $a_n \neq 0$ in the sense of $T(r, a_j) = O(r^{\rho-1+\epsilon}) + S(r, f)$. Moreover , assume that

$$\overline{N}(r, \frac{1}{\Psi}) + \overline{N}(r, f) = O(r^{\rho-1+\epsilon}) + S(r, f).$$

Then

$$\Psi = a_n \left(f + \frac{a_{n-1}}{n a_n} \right)^n.$$

Lemma 2.2 [8]. Suppose $f(z)$ is a meromorphic function in the complex plane and $P(z) = a_0 f^n + a_1 f^{n-1} + \cdots + a_n$, where $a_0(\not\equiv 0), a_1, \cdots, a_n$ are small functions of $f(z)$. Then

$$T(r, P(f)) = nT(r, f) + S(r, f)$$

Lemma 2.3 [1]. Let f be a meromorphic function with exponent of convergence of poles $\lambda(\frac{1}{f}) = \lambda < +\infty$, $\eta \neq 0$ be fixed, then for each $\epsilon > 0$,

$$N(r, f(z + \eta)) = N(r, f) + O(r^{\rho-1+\epsilon}) + O(logr).$$

Lemma 2.4 [3,4]. Let $f(z)$ be a meromorphic function of finite order σ and let c be a fixed non-zero complex constant. Then for each $\epsilon > 0$, we have

$$m\left(r, \frac{f(z + c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z + c)}\right) = O(r^{\sigma-1+\epsilon})$$

Lemma 2.5 [1]. Let $f(z)$ be a meromorphic function of finite order σ, and let η be a fixed non-zero complex number, then for each $\epsilon > 0$, we have

$$T(r, f(z + \eta)) = T(r, f) + O(r^{\sigma-1+\epsilon}) + O(logr).$$

Lemma 2.6 [3]. Let $f(z)$ be a non-constant meromorphic solution of

$$f(z)^n P(z,f) = Q(z,f),$$

where $P(z,f)$ and $Q(z,f)$ are difference polynomials in $f(z)$, and let $\delta < 1$ and $\epsilon > 0$. If the degree of $Q(z,f)$ as a polynomial in $f(z)$ and its shifts is at most n, then

$$m(r, P(z,f)) = o\left(\frac{T(r+|c|, f)^{1+\epsilon}}{r^{\delta}}\right) + o(T(r,f))$$

for all r outside of a possible exceptional set with finite logarithmic measure.

Proof of Theorems

Proof of Theorem 1.1 We prove this theorem by contradiction.
We first consider the case $n \geq 2$. Suppose there exists a transcendental meromorphic function $f(z)$ of finite order ρ with

$$N(r,f) = O(r^{\rho-1+\epsilon}) + S(r,f) \tag{1}$$

satisfying

$$a_1(z)P(f)f(z+c) + a_2(z)f(z+c) + a_3(z) \equiv 0 \tag{2}$$

i.e $a_1\left[b_n f^n + b_{n-1}f^{n-1} + \cdots + b_1 f + b_0\right] f(z+c) + a_2 f(z+c) + a_3 \equiv 0$

$$\Longrightarrow a_1 b_n f^n f(z+c) + P_1(f)f(z+c) + a_3 \equiv 0 \tag{3}$$

where $P_1(f) = a_1 b_{n-1} f^{n-1} + \cdots + a_1 b_1 f + a_1 b_0 + a_2$
By our assumption (1) and Lemma 2.3, we have

$$N(r, f(z+c)) = O(r^{\rho-1+\epsilon}) + S(r,f) \tag{4}$$

Now (3) can be written as

$$a_1 b_n f^n + P_1(f) \equiv -\frac{a_3}{f(z+c)}$$

Consider

$$H(z) \equiv f^n + \frac{P_1(f)}{a_1 b_n} \equiv -\frac{a_3}{a_1 b_n f(z+c)} \tag{5}$$

From (4) and (5), we write

$$\begin{aligned} \overline{N}\left(r, \frac{1}{H}\right) \leq N\left(r, \frac{1}{H}\right) &= N\left(r, \frac{-a_1 b_n f(z+c)}{a_3}\right) \\ &= O(r^{\rho-1+\epsilon}) + S(r,f) \end{aligned}$$

With this and by our assumption, we have

$$\overline{N}\left(r, \frac{1}{H}\right) + \overline{N}(r,f) = O(r^{\rho-1+\epsilon}) + S(r,f)$$

Now applying Lemma 2.1, we get

$$H(z) = \left(f(z) + \frac{b_{n-1}}{n b_n}\right)^n \tag{6}$$

From (5) and (6), we have

$$\left(f(z)+\frac{b_{n-1}}{nb_n}\right)^n \equiv -\frac{a_3}{a_1 b_n f(z+c)}$$
$$\Longrightarrow \left(f(z)+\frac{b_{n-1}}{nb_n}\right)^n f(z+c) \equiv -\frac{a_3}{a_1 b_n}$$

Thus

$$T\left(r,\left(f(z)+\frac{b_{n-1}}{nb_n}\right)^n f(z+c)\right) = T\left(r,\frac{a_3}{a_1 b_n}\right)$$

$$\Longrightarrow T\left(r,\left(f(z)+\frac{b_{n-1}}{nb_n}\right)^n f(z+c)\right) = O(r^{\rho-1+\epsilon})+S(r,f) \tag{7}$$

Using (4), we write

$$\begin{aligned} N\left(r,\frac{f(z+c)}{f(z)}\right) &\leq N(r,f(z+c))+N\left(r,\frac{1}{f(z)}\right) \\ &= N\left(r,\frac{1}{f(z)}\right)+O(r^{\rho-1+\epsilon})+S(r,f) \end{aligned} \tag{8}$$

Now, using Lemma 2.4 and (8), we get

$$\begin{aligned} T\left(r,\frac{f(z+c)}{f(z)}\right) &= m\left(r,\frac{f(z+c)}{f(z)}\right)+N\left(r,\frac{f(z+c)}{f(z)}\right) \\ &= N\left(r,\frac{1}{f(z)}\right)+O(r^{\rho-1+\epsilon})+S(r,f) \\ &\leq T(r,f)+O(r^{\rho-1+\epsilon})+S(r,f) \end{aligned} \tag{9}$$

Now by the first fundamental theorem of Nevanlinna and from (7) and (9), we have

$$\begin{aligned} T\left(r,f\left(f(z)+\frac{b_{n-1}}{nb_n}\right)^n\right) &= T\left(r,\frac{1}{f\left(f(z)+\frac{b_{n-1}}{nb_n}\right)^n}\right)+O(1) \\ \leq\ T\left(r,\frac{f(z+c)}{f(z)}\right)&+T\left(r,\frac{1}{f(z+c)\left(f(z)+\frac{b_{n-1}}{nb_n}\right)^n}\right)+O(1) \\ &\leq T(r,f)+O(r^{\rho-1+\epsilon})+S(r,f) \end{aligned} \tag{10}$$

On the other hand, using Lemma 2.2, we write

$$T\left(r,f\left(f(z)+\frac{b_{n-1}}{nb_n}\right)^n\right) = (n+1)T(r,f)+S(r,f) \tag{11}$$

Thus from (10) and (11), we get

$$\begin{aligned} (n+1)T(r,f)+S(r,f) &\leq T(r,f)+O(r^{\rho-1+\epsilon})+S(r,f) \\ nT(r,f) &\leq O(r^{\rho-1+\epsilon})+S(r,f) \end{aligned}$$

which is contradiction. Thus our assumption is false.
Next, we shall consider the case $n = 1$.
If $n = 1$, then (2) becomes

$$\begin{aligned} a_1(z)(b_1(z)f(z) + b_0)f(z+c) + a_2(z)f(z+c) + a_3(z) &\equiv 0 \\ \Longrightarrow a_1b_1f(z)f(z+c) + a_1b_0f(z+c) + a_2f(z+c) &\equiv -a_3 \\ \Longrightarrow [a_1b_1f(z) + (a_1b_0 + a_2)]\, f(z+c) &\equiv -a_3 \end{aligned}$$

$$\Longrightarrow \left[f(z) + \frac{(a_1b_0 + a_2)}{a_1b_1}\right] f(z+c) \equiv -\frac{a_3}{a_1b_1} \tag{12}$$

Degree of $-\frac{a_3}{a_1b_1}$ is zero and the degree of the term $\left[f(z) + \frac{(a_1b_0+a_2)}{a_1b_1}\right]$ is one. Hence by Lemma 2.6, we get

$$m(r, f(z+c)) = o\left(\frac{T(r+|c|, f)^{1+\epsilon}}{r^{\delta}}\right) + S(r, f),$$

where $\delta < 1$ and $\epsilon > 0$, which holds for all r outside of a possible exceptional set with finite logarithmic measure. Thus using (4), we write

$$T(r, f(z+c)) \quad \leq \quad O(r^{\rho-1+\epsilon}) + S(r, f)$$

Now, we write (12) as

$$\left[f(z) + \frac{(a_1b_0 + a_2)}{a_1b_1}\right] \equiv -\frac{a_3}{a_1b_1f(z+c)}$$

Thus

$$\begin{aligned} T\left(r, f(z) + \frac{(a_1b_0 + a_2)}{a_1b_1}\right) &\equiv T\left(r, -\frac{a_3}{a_1b_1f(z+c)}\right) \\ T(r, f) &\equiv O(r^{\rho-1+\epsilon}) + S(r, f), \end{aligned}$$

which is again a contradiction. Thus our assumption is false. Hence the theorem.

Example : Let $f(z) = 2z^2+1$ and $\rho(f(z)) = 0$ (finite order) with $N(r, f) = S(r, f)$. Consider $P(f) = f(z) + 1$. Then (2) becomes

$$\begin{aligned} a_1(z)(f(z) + 1)f(z+c) + a_2(z)f(z+c)a_3(z) &\equiv 0 \\ \Longrightarrow f(z) + \left(\frac{a_1(z) + a_2(z)}{a_1(z)}\right) &\equiv -\frac{a_3(z)}{a_1(z)f(z+c)} \\ \Longrightarrow T\left(r, f(z) + \left(\frac{a_1(z) + a_2(z)}{a_1(z)}\right)\right) &= T\left(r, -\frac{a_3(z)}{a_1(z)f(z+c)}\right) \end{aligned}$$

Applying Nevanlinna's first fundamental theorem and using Lemma 2.2 and 2.5, we obtain

$$\begin{aligned} T(r, f) + S(r, f) &= T(r, f(z+c)) + O(r^{\rho-1+\epsilon}) + S(r, f) \\ \Longrightarrow T(r, f) + S(r, f) &= T(r, f) + O(r^{\rho-1+\epsilon}) + S(r, f) \end{aligned}$$

Remarks:
1. In Theorem 1.1, $a_2(z)$ may or may not be zero.
If $a_2(z) \neq 0$, we can proceed as in the proof of Theorem 1.1.
If $a_2(z) = 0$, then (2) becomes $a_1(z)P(f)f(z+c)+a_3(z) \equiv 0 \Longrightarrow a_1(z)P(f) \equiv \frac{-a_3(z)}{f(z+c)}$ we can proceed as in the proof of Theorem 1.1.
So, in both the cases, we obtain a non-transcendental meromorphic solution $f(z)$

of finite order ρ with $N(r,f) = O(r^{\rho-1+\epsilon}) + S(r,f)$ will satisfy the non-linear difference equation of the form

$$a_1(z)P(f)f(z+c) + a_2(z)f(z+c) + a_3(z) \equiv 0.$$

2. In Theorem 1.1, $a_3(z) \neq 0$. If $a_3(z) = 0$, then (2) becomes

$$\begin{aligned} a_1(z)P(f)f(z+c) + a_2(z)f(z+c) &\equiv 0 \\ \implies a_1(z)P(f) &\equiv -a_2(z) \\ \implies P(f) &\equiv \frac{-a_2(z)}{a_1(z)} \end{aligned}$$

Thus

$$T(r, P(f)) = T\left(r, \frac{-a_2(z)}{a_1(z)}\right)$$

Using Lemma 2.2, we get $nT(r,f) + S(r,f) = O(r^{\rho-1+\epsilon}) + S(r,f)$, which is contradiction.
Similarly,, if $a_1(z) = 0$ we obtain a contradiction. Hence $a_1(z) \neq 0$.

Proof of Theorem 1.2 We prove this theorem also by contradiction method. We first consider the case $n \geq 2$. Suppose, there exists a transcendental meromorphic function $f(z)$ of finite order ρ with

$$N(r,f) = O(r^{\rho-1+\epsilon}) + S(r,f) \tag{13}$$

satisfying the equation

$$a_1(z)P(f)\Pi(f) + a_2(z)\Pi(f) + a_3(z) \equiv 0 \tag{14}$$

i.e $a_1\left[b_n f^n + b_{n-1}f^{n-1} + \cdots + b_1 f + b_0\right]\Pi(f) + a_2\Pi(f) + a_3 \equiv 0$

$$\implies a_1 b_n f^n \Pi(f) + P_1(f)\Pi(f) + a_3 \equiv 0 \tag{15}$$

where $P_1(f) = a_1 b_{n-1} f^{n-1} + \cdots + a_1 b_1 f + a_1 b_0 + a_2$
We have difference polynomial as

$$\begin{aligned} \Pi(f) &= \sum_{\lambda} a_\lambda(z) f(z)^{l_0} f(z+c_1)^{l_1} \ldots f(z+c_\lambda)^{l_\lambda} \\ &= f(z)^n \sum_{\lambda} \frac{a_\lambda(z) f(z)^{l_0} f(z+c_1)^{l_1} \ldots f(z+c_\lambda)^{l_\lambda}}{f(z)^n} \\ &= f(z)^n \sum_{\lambda} a_\lambda(z) \left(\frac{f(z+c_1)}{f(z)}\right)^{l_1} \left(\frac{f(z+c_2)}{f(z)}\right)^{l_2} \ldots \left(\frac{f(z+c_\lambda)}{f(z)}\right)^{l_\lambda} \end{aligned}$$

By Lemma 2.3 and (13), we get

$$N\left(r, \frac{f(z+c_i)}{f(z)}\right) \leq N\left(r, f(z+c_i)\right) + N\left(r, \frac{1}{f(z)}\right), i = 1, 2, \ldots \lambda$$

$$= O(r^{\rho-1+\epsilon}) + S(r,f) \tag{16}$$

Combining this with the assumption that $T(r, a_\lambda) = S(r,f)$, we obtain that

$$N\left(r, \Pi(f)\right) = O(r^{\rho-1+\epsilon}) + S(r,f) \tag{17}$$

Now (15) can be written as

$$f(z)^n + \frac{P_1(f)}{a_1 b_n} \equiv -\frac{a_3}{a_1 b_n \Pi(f)} \equiv \psi(z) \qquad (say) \tag{18}$$

From (17) and (18), we have

$$N\left(r, \frac{1}{\psi(z)}\right) \equiv N\left(r, -\frac{a_1 b_n \Pi(f)}{a_3}\right) = O(r^{\rho-1+\epsilon}) + S(r,f) \tag{19}$$

Since

$$\psi(z) = f(z)^n + \frac{b_{n-1}}{b_n} f(z)^{n-1} + \frac{b_{n-2}}{b_n} f(z)^{n-2} + \ldots + \frac{b_0}{b_n} + \frac{a_2}{b_n}$$

By assumption and (19), we write

$$\begin{aligned} \overline{N}\left(r, \frac{1}{\psi}\right) + \overline{N}(r,f) &\leq N\left(r, \frac{1}{\psi}\right) + N(r,f) \\ &= O(r^{\rho-1+\epsilon}) + S(r,f) \end{aligned}$$

Then applying the Lemma 2.1, we get

$$\psi(z) = \left[f(z) + \frac{b_{n-1}}{n b_n}\right]^n \tag{20}$$

From (18) and (20), we have

$$\begin{aligned} \left[f(z) + \frac{b_{n-1}}{n b_n}\right]^n &\equiv -\frac{a_3}{a_1 b_n \Pi(f)} \\ \Longrightarrow \left[f(z) + \frac{b_{n-1}}{n b_n}\right]^n \Pi(f) &\equiv -\frac{a_3}{a_1 b_n} \end{aligned}$$

Thus

$$T\left(r, \left[f(z) + \frac{b_{n-1}}{n b_n}\right]^n \Pi(f)\right) = T\left(r, -\frac{a_3}{a_1 b_n}\right) = O(r^{\rho-1+\epsilon}) + S(r,f) \tag{21}$$

Consider

$$\frac{\Pi(f)}{f(z)^n} = \sum_{\lambda} a_\lambda \left[\frac{f(z+c_1)}{f(z)}\right]^{l_1} \left[\frac{f(z+c_2)}{f(z)}\right]^{l_2} \cdots\cdots \left[\frac{f(z+c_\lambda)}{f(z)}\right]^{l_\lambda}$$

So

$$m\left(r, \frac{\Pi(f)}{f(z)^n}\right) = \sum_{\lambda} \left[m(r, a_\lambda) + \sum_{i=1}^{\lambda} l_i m\left(r, \frac{f(z+c_i)}{f(z)}\right)\right]$$

using Lemma 2.4 and $m(r, a_\lambda) = S(r,f)$, we have

$$m\left(r, \frac{\Pi(f)}{f(z)^n}\right) = O(r^{\rho-1+\epsilon}) + S(r,f)$$

Using (16) with this, we obtain

$$T\left(r, \frac{\Pi(f)}{f(z)^n}\right) = O(r^{\rho-1+\epsilon}) + S(r,f) \tag{22}$$

Now, by the first fundamental theorem of Nevanlinna and from (21) and (22), we have

$$\begin{aligned}
T\left(r, f(z)^n\left[f(z)+\frac{b_{n-1}}{nb_n}\right]^n\right) &= T\left(r, \frac{1}{f(z)^n\left[f(z)+\frac{b_{n-1}}{nb_n}\right]^n}\right)+O(1)\\
&\leq T\left(r, \frac{\Pi(f)}{f(z)^n}\right)+T\left(r, \frac{1}{\Pi(f)\left[f(z)+\frac{b_{n-1}}{nb_n}\right]^n}\right)+O(1)\\
&= O(r^{\rho-1+\epsilon})+S(r,f) \qquad (23)
\end{aligned}$$

By Lemma 2.2, we have

$$T\left(r, f(z)^n\left[f(z)+\frac{b_{n-1}}{nb_n}\right]^n\right)=2nT(r,f)+S(r,f) \qquad (24)$$

Thus from (23) and (24), we get

$$2nT(r,f)+S(r,f)\leq O(r^{\rho-1+\epsilon})+S(r,f)$$

which is contradiction. Hence, our assumption is false.
Now we shall consider the case when $n=1$.
If $n=1$, then (14) becomes

$$a_1(z)(b_1f+b_0)\Pi(f)+a_2(z)\pi(f)+a_3(z)\equiv 0$$

$$\left[f(z)+\frac{a_1b_0+a_2}{a_1b_1}\right]\Pi(f)\equiv -\frac{a_3}{a_1b_1} \qquad (25)$$

The degree of $-\frac{a_3}{a_1b_1}$ is zero and the degree of the term $\left[f(z)+\frac{a_1b_0+a_2}{a_1b_1}\right]$ is one. Hence applying Lemma 2.6 to (25), we write

$$m(r,\Pi(f))=o\left(\frac{T(r+|c|,f)^{1+\epsilon}}{r^{\delta}}\right)+S(r,f), \qquad (26)$$

where $\delta<1$ and $\epsilon>0$, which holds for all r outside of a possible exceptional set with finite logarithmic measure. Thus adding (17) and (26), we write

$$T(r,\Pi(f))=O(r^{\rho-1+\epsilon})+S(r,f) \qquad (27)$$

Now (25)can be written as

$$\left[f(z)+\frac{(a_1b_0+a_2)}{a_1b_1}\right]\equiv -\frac{a_3}{a_1b_1\Pi(f)}$$

Thus by (27), we have

$$\begin{aligned}
T\left(r, f(z)+\frac{(a_1b_0+a_2)}{a_1b_1}\right) &\equiv T\left(r, -\frac{a_3}{a_1b_1\Pi(f)}\right)\\
\Longrightarrow T(r,f) &= O(r^{\rho-1+\epsilon})+S(r,f)
\end{aligned}$$

which is again a contradiction. Thus our assumption is false. Hence the theorem.

Example: Let $f(z)=2z^2+z+1$ and $\rho(f(z))=0$ with $N(r,f)+N(r,1/f)=$

$S(r,f)$.
Consider $P(f) = f^2(z)+1$ and $\Pi(f) = f(z)f(z+c)$. Then (2) becomes

$$\begin{aligned} a_1(z)(f^2(z)+1)(f(z)f(z+c)) + a_2(z)(f(z)f(z+c)) + a_3(z) &\equiv 0 \\ \Longrightarrow f^2(z) + \left(\frac{a_1(z)+a_2(z)}{a_1(z)}\right) &\equiv -\frac{a_3(z)}{a_1(z)f(z)f(z+c)} \\ \Longrightarrow T\left(r, f^2(z) + \left(\frac{a_1(z)+a_2(z)}{a_1(z)}\right)\right) &= T\left(r, -\frac{a_3(z)}{a_1(z)f(z)f(z+c)}\right) \end{aligned}$$

Applying Nevanlinna's first fundamental theorem and using Lemma 2.2 and 2.5, we obtain

$$\begin{aligned} 2T(r,f) + S(r,f) &= T(r,f) + T(r,f(z+c)) + O(r^{\rho-1+\epsilon}) + S(r,f) \\ \Longrightarrow 2T(r,f) + S(r,f) &= 2T(r,f) + O(r^{\rho-1+\epsilon}) + S(r,f) \end{aligned}$$

Remarks:
1. In Theorem 1.2, $a_2(z)$ may or may not be zero. In both the cases, we obtain a non-transcendental meromorphic solution.
2. If $a_1(z) = 0$ and $a_3(z) = 0$, we obtain a contradiction. Hence $a_1(z) \neq 0$ and $a_3(z) \neq 0$.
3. Lemma 2.3, 2.4 and 2.5 fails for meromorphic function of infinite order. Thus Theorem 1.1 and 1.2 are not true for infinite order.

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