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Large Fractional Linear Type Differential Equations

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Abstract. This paper aims to handle some types of fractional differential equations with a fractional-order values $\beta > 1$. In particular, we propose a novel analytical solution called an atomic solution for certain fractional linear type differential equations as well as for some other types of partial differential equations with fractional-order values exceeding one. Some examples are provided to validate our findings.

1. Introduction

One of the most principal connections between pure and applied mathematics is differential equations in their three main types; ordinary, partial and fractional differential equations. A large number of applications that arise naturally in many fields of science and engineering might be described by fractional differential equations. Their solutions and investigations provide a remarkable growth to several mathematical approaches, see e.g. [1–5].

In 2014, R. Khalil introduced in [8] a newly scheme for dealing differential equations in their two types; ordinary and fractional ones. Such novel scheme relies on the theory of tensor product of Banach spaces, which can be employed for the aim of obtaining the so-called atomic solutions for ordinary/fractional differential equations. It is commonly known that most of fractional (ordinary/partial) differential equations, which were treated in literature, are handled with fractional-order values between 0 and 1. Hence obtained many interesting results can be. By taking this notion into account. From

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this point of view, the object of this paper is to study $\alpha-$fractional (ordinary/partial) differential equations for fractional-order value greater than one, i.e. $\beta > 1$. Accordingly, many notable theoretical results are proved and then used to some given fractional differential equations.

The rest of this paper is arrange as follows: Section 2 recalls some basic facts and preliminaries related to the fractional calculus and atomic solution. Section 3 provides the main results of this work. Section 4 illustrates some applications of the derived theoretical results, followed by Section 5 that summarizes the whole conclusions of this work.

2. Preliminaries

In this section, some needed basic facts and preliminaries in connection to the fractional calculus and the so-called atomic solution are recalled. In particular, the definition of the so-called α-conformable fractional derivative D^{α} is introduced , [8]. So, for $x \in E \subseteq (0, \infty)$, we have:

$$D^{\alpha} f(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon x^{1-\alpha}) - f(x)}{\epsilon}.$$

If the above limit exists, then $D^{\alpha} f(x)$ is called the α-conformable fractional derivative of f at x. As a result, if f is α-differentiable on $(0, r)$ for some $r > 0$ and $\lim\limits_{\epsilon \to 0^+} D^{\alpha} f(x)$ exists, then we define

$$D^{\alpha} f(0) = \lim_{x \to 0} D^{\alpha} f(x).$$

For $\alpha \in (0, 1]$ and $f,\ g$ are α-differentiable at a point t, one can easily see that the conformable fractional derivative satisfies the following properties:

$$\begin{aligned}
&(i)\ D^{\alpha}(bf + cg) = bD^{\alpha}(f) + cD^{\alpha}(g), \text{ for all } b, c \in \mathbb{R},\\
&(ii)\ D^{\alpha}(\lambda) = 0, \text{for all constant functions } f(t) = \lambda,\\
&(iii)\ D^{\alpha}(fg) = fD^{\alpha}(g) + gD^{\alpha}(f),\\
&(iv)\ D^{\alpha}(\tfrac{f}{g}) = \tfrac{gD^{\alpha}(f) - fD^{\alpha}(g)}{g^2}, g(t) \neq 0.
\end{aligned}$$

In this regard, we list below the conformable fractional derivatives for certain functions:

$$\begin{aligned}
&(i)\ D^{\alpha}(t^p) = pt^{p-\alpha},\\
&(ii)\ D^{\alpha}(\sin(\tfrac{1}{\alpha} t^{\alpha})) = \cos(\tfrac{1}{\alpha} t^{\alpha}),\\
&(iii)\ D^{\alpha}(\cos(\tfrac{1}{\alpha} t^{\alpha})) = -\sin(\tfrac{1}{\alpha} t^{\alpha}),\\
&(iv)\ D^{\alpha}(e^{\frac{1}{\alpha} t^{\alpha}}) = e^{\frac{1}{\alpha} t^{\alpha}}.
\end{aligned}$$

On letting $\alpha = 1$ in the above derivatives, we get their corresponding classical rules for ordinary derivatives. Further, one should notice that a function could be α-conformable differentiable at a point but it is not differentiable at that point. For example, if one takes $f(t) = 2\sqrt{t}$, then $D^{\frac{1}{2}}(f)(0) = 1$, while for the classical fractional derivative, $D^1(f)(0)$ does not exist. For more overview about fractional calculus and its applications, we refer to the references [6–12].

Many differential equations might be transformed to their corresponding fractional-order forms. These equations can have many applications in many branches of science and engineering, see [13–15].

It is commonly known that the main technique used to solve many partial differential equations is using Fourier series. From this point of view, fractional Fourier series was introduced in [16]. Such a concept proved to be very fruitful in solving fractional partial differential equations. But sometimes it is not possible to use separation of variables technique to deal with these equations. Here comes the concept of atomic solution, which would help us to face this problem. This concept coupled with how we can use it in solving fractional (ordinary/partial) differential equations will be provided in the following content.

Definition 2.1. *(Atomic solution) Let X and Y be two Banach spaces and X^* be the dual of X. Assume $x \in X$ and $y \in Y$. The operator $T : X^* \to Y$ defined by:*

$$T(x^*) = x^*(x)y,$$

is a bounded one rank linear operator. Then, we write $x \otimes y$ for T. Such operators are called atoms, which are among the main ingredients in the theory of tensor products. Besides, the atoms are used in theory of best approximation in Banach spaces.

In what follow, we recall one of the most known results that we shall need in our investigation. This result was extensively utilized in many research papers, see [17–20].

Lemma 2.1. *If the sum of two atoms is an atom, then either the first components are dependent or the second ones are dependent. In other words, if $x_1 \otimes y_1 + x_2 \otimes y_2 = x_3 \otimes y_3$, then either $x_1 = x_2 = x_3$.*

3. Main Results

For $n < \beta < n+1$, the β-fractional derivatives of the function $f : [0, \infty) \to \mathbb{R}$ was defined in [8] as

$$D^{\beta} f(x) = \lim_{\epsilon \to 0} \frac{f^{[\beta]-1}(x + \epsilon x^{[\beta]-\alpha}) - f^{[\beta]-1}(x)}{\epsilon}.$$

This turned out to be equivalent to the following assertion:

$$D^{\beta}(f(x)) = x^{[\beta]-\beta} f^{[\beta]}(x). \tag{3.1}$$

It is worth mentioning that if f is $(n+1)$-differentiable and $n < \beta < n+1$, then we have:

$$D^{\beta}(f(x)) = x^{n+1-\alpha} f^{(n+1)}(x) = x^{n+1-\alpha} D_x^{n+1} f(x),$$

for $n < \beta < n+1$. Now, if one lets $1 < \beta < 2$, then $\beta = 1+\alpha$, where $0 < \alpha < 1$. Hence, $D^{\beta} f = D^{1+\alpha} f$. Similarly, if $n < \beta < n+1$, then $\beta = n+\alpha$, where $0 < \alpha < 1$, and so, $D^{\beta} f = D^{n+\alpha} f$, for $0 < \alpha < 1$.

Proposition 3.1. *Let $n < \beta < n+1$ and $\alpha = \beta - n$. Then we have:*

$$\begin{aligned} D^{\beta} f(x) &= D^{\alpha} D^n f(x) = D^{\alpha} f^{(n)}(x) \\ &= x^{1-\alpha} f^{(n+1)}(x). \end{aligned}$$

Proof. To prove this result, it should be noted that

$$\begin{aligned} D^{\beta} f(x) &= D^{n+\alpha} f(x) \\ &= x^{(n+1)-\beta} f^{(n+1)}(x). \end{aligned}$$

But, from identity (3.1), one can write:

$$\begin{aligned} D^{\beta} f(x) &= x^{(n+1)-(n+\alpha)} f^{(n+1)}(x) \\ &= x^{1-\alpha} f^{(n+1)}(x). \end{aligned}$$

□

Definition 3.1. *An equation of the form:*

$$a_n D^{n+\alpha} y + a_{n-1} D^{n-1+\alpha} y + \cdots + a_1 D^{\alpha} y + a_0 x^{1-\alpha} y = 0 \tag{3.2}$$

is called an α-fractional linear equation of order n with $0 < \alpha < 1$.

In the following content, we state and prove the main result of this section, which would be about the general solution of an α-fractional linear type equation.

Theorem 3.1. *Let $0 < \alpha < 1$ and*

$$a_n D^{n+\alpha} y + a_{n-1} D^{n-1+\alpha} y + \cdots + a_1 D^{\alpha} y + a_0 x^{1-\alpha} y = 0 \tag{3.3}$$

be an α-fractional linear type equation of order n. Let $r_0, r_1, r_2, \cdots, r_n, r_{n+1}$ be the roots of the equation

$$a_{n+1} r^{n+1} + a_n r^n + \cdots + a_1 r + a_0 = 0.$$

Then by assuming no repetition of the roots, the general solution of (3.3) has the form:

$$y = c_0 y_0 + \cdots + c_n y_n + c_{n+1} y_{n+1},$$

where $y_k = e^{r_k x}$.

Proof. By Proposition 3.1, equation (3.2) can be written in the form:

$$a_n D^{\alpha}(D^n y) + \cdots + a_1 D^{\alpha} y + a_0 x^{1-\alpha} y = 0.$$

This is just of the form:

$$a_n x^{1-\alpha} y^{(n+1)} + \cdots + a_1 x^{1-\alpha} y' + a_0 x^{1-\alpha} y = 0. \tag{3.4}$$

Thus (3.3) becomes:

$$a_{n+1} y^{(n+1)} + a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0.$$

Actually, the above equation represents an ordinary linear differential equation. So, the associated characteristic equation is of the form:

$$a_{n+1} r^{n+1} + a_n r^n + \cdots + a_1 r + a_0 = 0. \tag{3.5}$$

Solving the above equation to get the roots $r_0, r_1, \cdots, r_n, r_{n+1}$. Consequently, if all the roots are distinct and real, then there are $(n+1)$ solutions that have the forms:

$$y_0 = e^{r_0 x}, y_1 = e^{r_1 x}, \cdots, y_{n+1} = e^{r_{n+1} x}.$$

Therefore, the general solution of the α-fractional linear type equation (3.3) is:

$$y_g = c_0 y_0 + \cdots + c_n y_n + c_{n+1} y_{n+1}.$$

□

Remark 3.1. *If a root is repeated in 3.5, then as is known in the theory of ordinary differential equations, some of the solutions are multiplied by some factors.*

4. Applications

In this part, we intend to provide a solution of a fractional ordinary differential equation, and the atomic solution of a fractional partial differential equation.

Example 4.1. *Let us have the following fractional differential equation:*

$$y^{(\frac{3}{2})} + 2y^{(\frac{1}{2})} - 3\sqrt{x}y = 0. \tag{4.1}$$

Hence, we have, $\alpha = \frac{1}{2}$ *and* $n = 1$. *In other words, (3.4) becomes:*

$$D^{\frac{1}{2}} y' + 2\sqrt{x}y' - 3\sqrt{x}y = 0.$$

So, we have:

$$\sqrt{x}y'' + 2\sqrt{x}y' - 3\sqrt{x}y = 0.$$

This is equivalent to say:

$$y'' + 2y' - 3y = 0. \tag{4.2}$$

The characteristic equation of (4.1) is then of the form:

$$r^2 + 2r - 3 = 0,$$

with the roots $r_1 = -3$ *and* $r_2 = 1$. *Hence, we obtain:*

$$y_g = c_1 e^{-3x} + c_2 e^x. \tag{4.3}$$

One can easily check that (4.3) is the general solution of (4.1).

In the following content, we aim to find the atomic solution of a certain fractional partial differential equation of order $\alpha > \frac{3}{2}$.

Example 4.2. *For $x > 0$ and $y > 0$, consider the following fractional partial differential equation:*

$$D_x^{\frac{3}{2}} D_y^{\frac{3}{2}} u + D_x^{\frac{1}{2}} D_y^{\frac{1}{2}} u = \sqrt{xy}u, \tag{4.4}$$

with the following initial conditions

$$u(0,0) = 1, \frac{\partial u}{\partial x}(0,0) = 1, \frac{\partial u}{\partial y}(0,0) = 1. \tag{4.5}$$

If one wants to solve the above problem with the use of our proposed approach, we firs assume:

$$U(x,y) = P(x)Q(y).$$

Now, conditions (4.5) yield the following condition on P and Q:

$$P'(0) = P(0) = 1 \text{ and } Q'(0) = Q(0) = 1. \tag{4.6}$$

This means:

$$P^{\frac{3}{2}}(x)Q^{\frac{3}{2}}(y) + P^{\frac{1}{2}}(x)Q^{\frac{1}{2}}(y) = \sqrt{xy}P(x)Q(y).$$

From the analysis of $D^{n+\alpha}u$, we get:

$$\sqrt{x}P''(x)\sqrt{y}Q''(y) + \sqrt{x}P'(x)\sqrt{y}Q'(y) = \sqrt{xy}P(x)Q(y).$$

Hence, we have:

$$P''(x)Q''(y) + P'(x)Q'(y) = P(x)Q(y). \tag{4.7}$$

With the use of Lemma 2.1, we have two cases:

Case (a): $P''(x) = P'(x) = P(x)$.

Case (b): $Q''(x) = Q'(x) = Q(x)$.

Now for Case (a), we have the following three situations:

(1) $P''(x) = P'(x)$.

(2) $P''(x) = P(x)$.

(3) $P'(x) = P(x)$.

Observe that situation 1 can be solved to get $P(x) = c_1 + c_2e^x$. But from conditions (4.6, we get $P(x) = e^x$. In the same regard, solving both situations 2 and 3 gives also the same solution $P(x) = e^x$. Now substituting this result in (4.5), we get:

$$e^xQ''(y) + e^xQ'(y) = e^xQ(y).$$

Immediately, we have:

$$Q''(y) + Q'(y) - Q(y) = 0.$$

So, we have $r^2 + r - 1 = 0$ with the roots $r_1 = \frac{-1+\sqrt{5}}{2}$ and $r_2 = \frac{-1-\sqrt{5}}{2}$. Now, from conditions (4.6), we get:

$$Q(y) = (1+\sqrt{5})e^{r_1y} - \sqrt{5}e^{r_2y}.$$

Therefore, our proposed atomic solutions has the form:

$$U(x,y) = P(x)Q(y) = e^x((1+\sqrt{5})e^{r_1 y} - \sqrt{5}e^{r_2 y}).$$

Since equation (4.6) is symmetric in P and Q, then Case (b) gives the following atomic solution:

$$U(x,y) = ((1+\sqrt{5})e^{r_1 x} - \sqrt{5}e^{r_2 x})e^y. \tag{4.8}$$

However, in order to see how the atomic solution (4.8) of equation (4.4) looks like, we plot such solution in Figure 4.2.

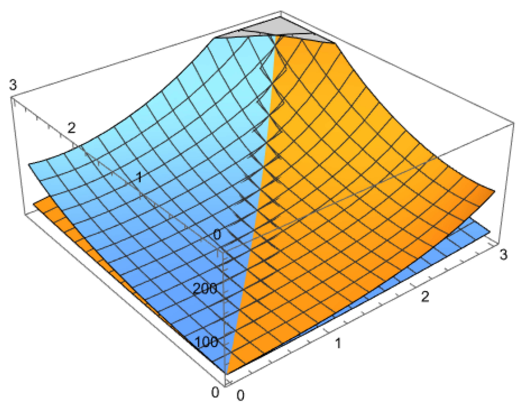

Figure 1. The atomic solution (4.8) of equation (4.4).

5. Conclusions

This paper has successfully introduced a new analytical method for handling the study of a class of α-fractional (ordinary/partial) differential equations with fractional-order value greater than one via atomic solutions method. The theory of tensor product of Banach spaces coupled with some properties of atoms operators have been utilized for achieving such a notion.

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References

[1] I.M. Batiha, S. Alshorm, A. Ouannas, S. Momani, O.Y. Ababneh, M. Albdareen, Modified Three-Point Fractional Formulas with Richardson Extrapolation, Mathematics. 10 (2022), 3489. https://doi.org/10.3390/math10193489.

[2] I.M. Batiha, S. Alshorm, I.H. Jebril, M.A. Hammad, A Brief Review about Fractional Calculus, Int. J. Open Probl. Comput. Math. 15 (2022), 39-56.

[3] I.M. Batiha, A. Obeidat, S. Alshorm, A. Alotaibi, H. Alsubaie, S. Momani, M. Albdareen, F. Zouidi, S.M. Eldin, H. Jahanshahi, A Numerical Confirmation of a Fractional-Order COVID-19 Model's Efficiency, Symmetry. 14 (2022), 2583. https://doi.org/10.3390/sym14122583.

[4] I.M. Batiha, O.Y. Ababneh, A.A. Al-Nana, W.G. Alshanti, S. Alshorm, S. Momani, A Numerical Implementation of Fractional-Order PID Controllers for Autonomous Vehicles, Axioms. 12 (2023), 306. https://doi.org/10.3390/axioms12030306.

[5] I.H. Jebril, I.M. Batiha, On the Stability of Commensurate Fractional-Order Lorenz System, Progr. Fract. Differ. Appl. 8 (2022), 401-407. https://doi.org/10.18576/pfda/080305.

[6] T. Abdeljawad, On Conformable Fractional Calculus, J. Comput. Appl. Math. 279 (2015), 57-66. https://doi.org/10.1016/j.cam.2014.10.016.

[7] M. Adil Khan, Y.M. Chu, A. Kashuri, R. Liko, G. Ali, Conformable Fractional Integrals Versions of Hermite-Hadamard Inequalities and Their Generalizations, J. Funct. Spaces. 2018 (2018), 6928130. https://doi.org/10.1155/2018/6928130.

[8] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A New Definition of Fractional Derivative, J. Comput. Appl. Math. 264 (2014), 65-70. https://doi.org/10.1016/j.cam.2014.01.002.

[9] E. Mahmoudi, The Beta Generalized Pareto Distribution With Application to Lifetime Data, Math. Computers Simul. 81 (2011), 2414-2430. https://doi.org/10.1016/j.matcom.2011.03.006.

[10] M.A. Hammad, Conformable Fractional Martingales and Some Convergence Theorems, Mathematics. 10 (2021), 6. https://doi.org/10.3390/math10010006.

[11] M.A. Hammad, M.A. Horani, A. Shmasenh, R. Khalil, Reduction of Order of Fractional Differential Equations, J. Math. Comput. Sci. 8 (2018), 683-688. https://doi.org/10.28919/jmcs/3806.

[12] A. Dababneh, B. Albarmawi, M.A. Hammad, A. Zraiqat, T. Hamadneh, Conformable Fractional Bernoulli Differential Equation with Applications, in: 2019 IEEE Jordan International Joint Conference on Electrical Engineering and Information Technology (JEEIT), IEEE, Amman, Jordan, 2019: pp. 421-424. https://doi.org/10.1109/JEEIT.2019.8717456.

[13] W.S. Chung, Fractional Newton Mechanics With Conformable Fractional Derivative, J. Comput. Appl. Math. 290 (2015), 150-158. https://doi.org/10.1016/j.cam.2015.04.049.

[14] M. Horani, M.A. Hammad, R. Khalil, Variation of Parameters for Local Fractional Nonhomogenous Linear-Differential Equations, J. Math. Computer Sci. 16 (2016), 147-153.

[15] W. Deeb, R. Khalil, Best approximation in $L(X,Y)$, Math. Proc. Camb. Phil. Soc. 104 (1988), 527-531. https://doi.org/10.1017/s0305004100065713.

[16] M.A. Hammad, R. Khalil, Fractional Fourier Series With Applications, Amer. J. Comput. Appl. Math. 4 (2014), 187-191.

[17] M. Al-Horani, R. Khalil, I. Aldarawi, Fractional Cauchy Euler Differential Equation, J. Comput. Anal. Appl. 28 (2020), 226-233.

[18] R. Khalil, Isometries of $Lp^* \widehat{\otimes} Lp$, Tam. J. Math. 16 (1985), 77-85.

[19] H. Al-Zoubi, A. Dabaneh, M. Al-Sabbagh, Ruled Surfaces of Finite II-Type, WSEAS Trans. Math. 18 (2019), 1-5.

[20] F. Bekraoui, M. Al-Horani, R. Khalil, Atomic Solution of Fractional Abstract Cauchy Problem of High Order in Banach Spaces, Eur. J. Pure Appl. Math. 15 (2022), 106-125.