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Powered Inverse Rayleigh Distribution Using DUS Transformation

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Abstract. This article reports an extension of powered inverse Rayleigh distribution via DUS transformation, named DUS-Powered Inverse Rayleigh (DUS-PIR) distribution. Some statistical properties of suggested distribution in particular, moments, mode, quantiles, order statistics, entropy and , inequality measures have been investigated extensively. To estimate the parameters, maximum likelihood estimation (MLE) is discussed. The model flexibility is validated by two real data.

1. Introduction

The accuracy and consistency of statistical analysis are extremely affected by the assumed probability model or distribution. As a result of this verity, in recent decades formulating new distributions becomes a basic conception in statistical theory; this is generally done by adding an extra parameter to the baseline distribution. For example, [1–5] and many more. The different transformation techniques have been used by the several authors. For example, DUS, sine, and MG transformations are reported by [6–8]. In all transformation exponential distribution is deemed as baseline distribution.

If $g(x)$ and $G(x)$ denote the probability density function (PDF) and cumulative density function (CDF) of a baseline lifetime distribution, then the PDF and CDF of a DUS-transformation are given as

$$f(x) = \frac{1}{e-1} g(x) e^{G(x)}, \quad x > 0, \tag{1.1}$$

$$F(x) = \frac{e^{G(x)} - 1}{e-1}, \quad x > 0. \tag{1.2}$$

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Hazard rate function (HRF) is.

$$h(x) = \frac{g(x)}{e^{-[G(x)-1]} - 1}. \tag{1.3}$$

Inverse Rayleigh (IR) distribution was introduced by [9]. IR distribution finds enormous applications in survival analysis. Various properties of IR distribution have been studied by [10]. Powered IR distribution was proposed by [11] through the powered transformation to increase its flexibility and applicability. [5] established and studied in detail the length powered IR distribution. [12] established the several recurrence relations from powered IR distribution.

A random variable (r.v.) X follows powered IR distribution, if its PDF and CDF are given, respectively by:

$$g(x;\alpha,\theta) = \frac{2\alpha}{\theta x^{2\alpha+1}} e^{-\frac{1}{\theta x^{2\alpha}}}, \quad \alpha,\theta > 0,\ x > 0. \tag{1.4}$$

$$G(x;\alpha,\theta) = e^{-\frac{1}{\theta x^{2\alpha}}}, \quad \alpha,\theta > 0,\ x > 0. \tag{1.5}$$

To modelling all kinds of data sets, no single distribution can be speculated as the best fit. Despite of existence many distributions in the literature. We are therefore induced to establish a new distribution via DUS transformation and named as DUS-Powered IR distribution.

The paper is framed as follows: In Section 2, DUS-Powered IR distribution is derived and graphically depicted. Several mathematical and statistical properties are established in Section 3. Also, the entropies and measures of inequality are addressed in Sections 3. The parameters estimation is obtained in Section 4. The model superiority is shown through two real data in Section 5. Section 6 reports the concluding remarks.

2. DUS- Powered IR Distribution

Now utilizing (1.4) and (1.5) into (1.1) and (1.2) respectively. We can obtain the CDF, PDF and HRF for DUS-PIR distribution as follows:

$$F(x;\alpha,\theta) = \frac{\exp\left(e^{-\frac{1}{\theta}x^{-2\alpha}}\right) - 1}{e-1}, \quad x > 0; \alpha,\theta > 0, \tag{2.1}$$

$$f(x;\alpha,\theta) = \frac{2\alpha x^{-(2\alpha+1)}}{(e-1)\theta} \exp\left(e^{-\frac{1}{\theta}x^{-2\alpha}}\right) e^{-\frac{1}{\theta}x^{-2\alpha}}, \tag{2.2}$$

$$h(x,\alpha,\theta) = \frac{2\alpha x^{-(2\alpha+1)} e^{-\frac{1}{\theta}x^{-2\alpha}}}{\theta\left[\exp\left(1 - e^{-\frac{1}{\theta}x^{-2\alpha}}\right) - 1\right]} \tag{2.3}$$

respectively.

The depiction of plots are shown in the following figures for fix parameters.

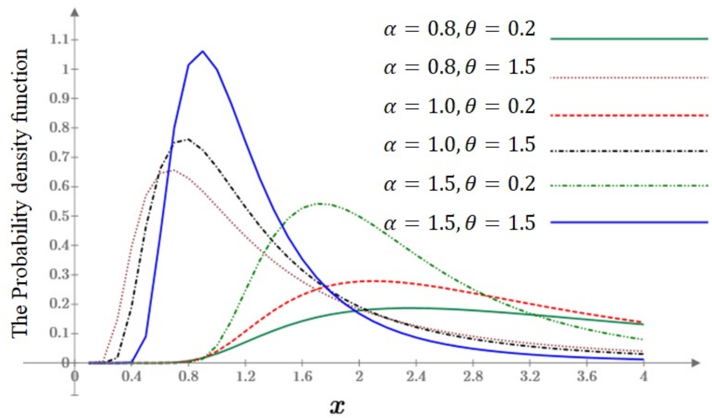

Figure 1. $f(x)$ for fix parameters.

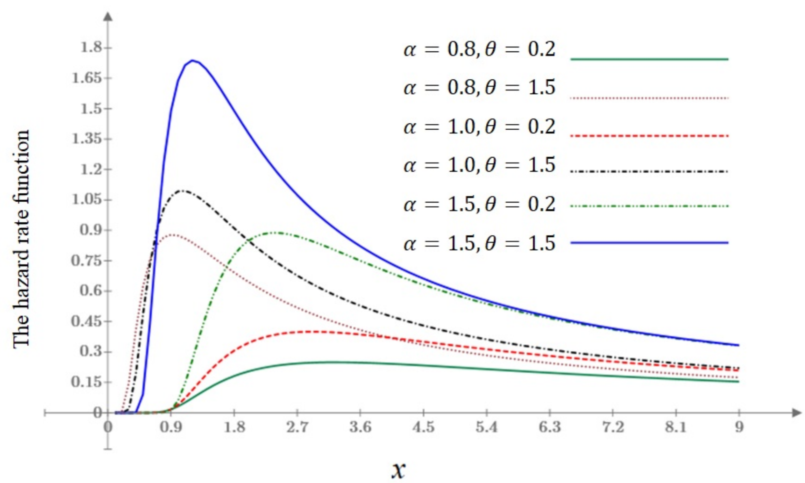

Figure 2. $h(x)$ for fix values of α and θ.

The depiction from Figures 1 and 2 are.

(i) The DUS-PIR distribution has unimodal,

(ii) The failure rate is increasing, then decreasing for fix values of parameters α and θ.

3. Some Statistical Properties

Some statistical properties of DUS-PIR distribution, including rth moments, quantile function, skewness, kurtosis, and order statistics are studied.

3.1. **The moments:** Let X ∼DUS-PIR distribution with parameters (α, θ), then the rth moment is given in Theorem 3.1.

Theorem 3.1. *The moments of DUS-PIR distribution is given as*

$$\mu_r' = \left(\frac{1}{e-1}\right)\sum_{k=0}^{\infty}\frac{1}{(k+1)!}\left(\frac{k+1}{\theta}\right)^{\frac{r}{2\alpha}}\Gamma\left(1-\frac{r}{2\alpha}\right). \tag{3.1}$$

Proof: The rth moment of the r.v. X is

$$\mu_r' = \int_0^\infty x^r f(x;\alpha,\theta)dx.$$

From (2.2), we have

$$\mu_r' = \int_0^\infty \frac{2\alpha}{(e-1)\theta}x^{r-(2\alpha+1)}e^{e^{-\frac{1}{\theta}x^{2\alpha}}}e^{-\frac{1}{\theta}x^{2\alpha}}dx,$$

Since $\theta > 0$, we have $e^{e^{-\frac{1}{\theta}x^{2\alpha}}} = \sum_{k=0}^{\infty}\frac{e^{-\frac{k}{\theta x^{2\alpha}}}}{k!}$, so

$$\mu_r' = \sum_{k=0}^{\infty}\frac{2\alpha}{k!(e-1)\theta}\int_0^\infty x^{r-(2\alpha+1)}e^{-\frac{(k+1)}{\theta}x^{-2\alpha}}dx \tag{3.2}$$

Let $u = \left(\frac{k+1}{\theta}\right)x^{-2\alpha}$, then Equation (3.2) reduces as

$$\mu_r' = \left(\frac{1}{e-1}\right)\sum_{k=0}^{\infty}\frac{1}{(k+1)!}\left(\frac{k+1}{\theta}\right)^{\frac{r}{2\alpha}}\Gamma\left(1-\frac{r}{2\alpha}\right).$$

3.2. **Mode:** Setting first derivative of (2.2) as follows.

$$f'(x) = \frac{2\alpha}{(e-1)\theta^2}x^{-2(2\alpha+1)}\exp\left(e^{-\frac{1}{\theta}x^{-2\alpha}}-\frac{1}{\theta}x^{-2\alpha}\right)\left[2\alpha-\theta(1+2\alpha)x^{2\alpha}+2\alpha e^{-\frac{1}{\theta}x^{-2\alpha}}\right] = 0. \tag{3.3}$$

Above equation does not possess analytic solution in x.

For a quick graphical solution of the mode, we sketch the plot of left-hand side of (3.3) at different values of α, θ as depicted in Figure 3.

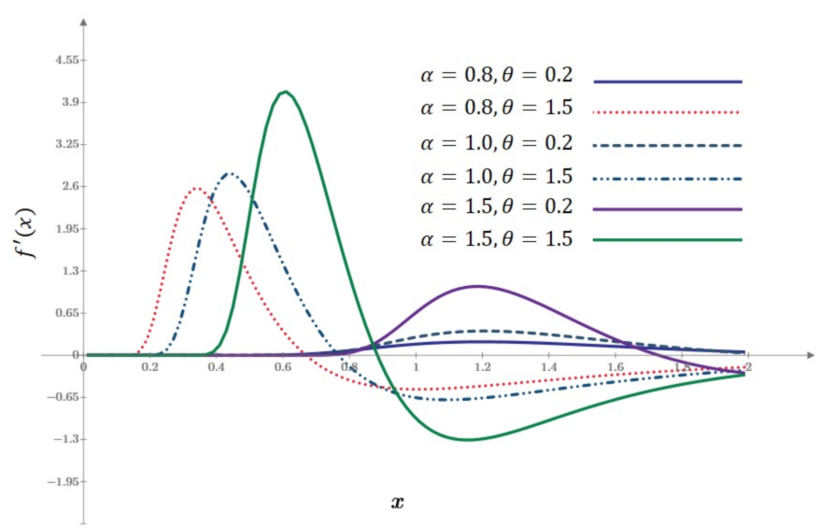

Figure 3. $f'(x)$ for selected values of α and θ.

It confirms from these plots that DUS-PIR distribution has one mode based on selected values of α and θ.

3.3. **Quantiles and Random Number Generation:** The quantile x_q,$(0 < q < 1)$, of DUS-PIR(α, θ) distribution can be attained, by employing the CDF in (2.1), in the given simple form.

$$x_q = \left\{ -\frac{1}{\theta \ln \left[\ln(q(e-1)+1) \right]} \right\}^{\frac{1}{2\alpha}}. \tag{3.4}$$

One of the good characteristics of the suggested distribution is that we can smoothly calculate its quantiles in simple as well as an explicit form.

To generate random sample with size $(n \geq 1)$ form DUS-PIR(α, θ) distribution, we can use (3.4) by generating n random values for q, where $q \sim U(0, 1)$.

To find the median, using the above equation for $q = 0.50$,

$$\text{Med.} = \left\{ -\frac{1}{\theta \ln \left[\ln(0.5(e+1)) \right]} \right\}^{\frac{1}{2\alpha}}.$$

The shapes of DUS-PIR distribution can be viewed by skewness and kurtosis. Utilizing the concept of quantiles, skewness and kurtosis are as follows, [13].

Bowley' skewness:

$$sk = \frac{x_{0.75} - 2x_{0.50} + x_{0.25}}{x_{0.75} - x_{0.25}}.$$

Moors' kurtosis:

$$ku = \frac{x_{0.875} + x_{0.375} - (x_{0.625} + x_{0.125})}{x_{0.75} - x_{0.25}}$$

3.4. **Order Statistics:** The rth order statistic (O.S.) $X_{(r)}$ based on ordered sample $(X_1 < X_2 < \cdots < X_n)$ from a continuous distribution having CDF $F_X(x)$ and PDF $f_X(x)$ is.

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} f_x(x) [F_x(x)]^{r-1} [1 - F_x(x)]^{n-r}, \quad r = 1, 2, \cdots, n. \tag{3.5}$$

So, the r^{th} order statistic from DUS-PIR distribution is

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} \frac{2\alpha x^{-(2\alpha+1)}}{(e-1)^n \theta} e^{\left(e^{-\frac{1}{\theta}x^{-2\alpha}} - \frac{1}{\theta}x^{-2\alpha}\right)} \left[e^{e^{-\frac{1}{\theta}x^{-2\alpha}}} - 1 \right]^{r-1} \left[e - e^{e^{-\frac{1}{\theta}x^{-2\alpha}}} \right]^{n-r} \tag{3.6}$$

Putting $r = 1$ and $r = n$ in (3.6), we can obtain PDF of smallest and largest (O.S.).

3.5. **Entropy:** Entropy helps to measure the uncertainty of the r.v. X. Some notable entropies are defined as follows.

Rényi entropy:

$$R_\delta(x) = \frac{1}{1-\delta}\log\left[\int_0^\infty f^\delta(x)dx\right], \quad \delta > 0 \quad \text{and } \delta \neq 1. \tag{3.7}$$

Tsallis entropy:

$$T_\delta(x) = \frac{1}{1-\delta}\left[\int_0^\infty f^\delta(x)dx - 1\right], \quad \delta > 0 \quad \text{and } \delta \neq 1. \tag{3.8}$$

Havrda and Charvat entropy (H-C)

$$HC_\delta(x) = \frac{1}{2^{1-\delta}-1}\left[\int_0^\infty f^\delta(x)dx - 1\right]. \tag{3.9}$$

Theorem 3.2. *If $X\sim$DUS-PIRD, then the Rényi Entropy of X is given as*

$$R_\delta(x) = \frac{1}{1-\delta}\log\left[\frac{1}{2\alpha}\left(\frac{2\alpha}{\theta(e-1)}\right)^\delta \Gamma\left(\frac{\delta(2\alpha+1)-1}{2\alpha}\right)\sum_{k=0}^\infty \frac{\delta^k}{k!}\left(\frac{\theta}{k+\delta}\right)^{\frac{\delta(2\alpha+1)-1}{2\alpha}}\right]. \tag{3.10}$$

Proof: From (2.2) into (3.7), we have

$$f^\delta(x) = \left(\frac{2\alpha}{\theta(e-1)x^{2\alpha+1}}\right)^\delta \sum_{k=0}^\infty \frac{\delta^k e^{-\frac{(k+\delta)}{\theta}x^{-2\alpha}}}{k!}$$

and

$$\int_0^\infty f^\delta(x)dx = \left(\frac{2\alpha}{\theta(e-1)}\right)^\delta \sum_{k=0}^\infty \frac{\delta^k}{k!}\int_0^\infty x^{-\delta(2\alpha+1)}e^{-\frac{(k+\delta)}{\theta}x^{-2\alpha}}dx$$

Let $u = \frac{k+\delta}{\theta}x^{-2\delta}$, then $x = \left(\frac{\theta}{k+\delta}\right)^{-\frac{1}{2\alpha}} u^{-\frac{1}{2\alpha}}$ and

$$\begin{aligned}\int_0^\infty f^\delta(x)dx &= \left(\frac{2\alpha}{\theta(e-1)}\right)^\delta \sum_{k=0}^\infty \frac{\delta^k}{k!}\left(\frac{\theta}{k+\delta}\right)^{\frac{\delta(2\alpha+1)-1}{2\alpha}} \frac{1}{2\alpha}\int_0^\infty u^{\frac{\delta(2\alpha+1)-1}{2\alpha-1}-1}e^{-u}du \\ &= \frac{1}{2\alpha}\left(\frac{2\alpha}{\theta(e-1)}\right)^\delta \sum_{k=0}^\infty \frac{\delta^k}{k!}\left(\frac{\theta}{k+\delta}\right)^{\frac{\delta(2\alpha+1)-1}{2\alpha}} \Gamma\left(\frac{\delta(2\alpha+1)-1}{2\alpha}\right).\end{aligned}$$

Therefore, the Renyi entropy is

$$R_\delta(x) = \frac{1}{1-\delta}\log\left[\frac{1}{2\alpha}\left(\frac{2\alpha}{\theta(e-1)}\right)^\delta \Gamma\left(\frac{\delta(2\alpha+1)-1}{2\alpha}\right)\sum_{k=0}^\infty \frac{\delta^k}{k!}\left(\frac{\theta}{k+\delta}\right)^{\frac{\delta(2\alpha+1)-1}{2\alpha}}\right].$$

Theorem 3.3. *If $X\sim$ DUS-PIRD(α,θ), the Tsallis entropy of X is*

$$T_\delta(x) = \frac{1}{1-\delta}\left[\frac{1}{2\delta}\left(\frac{2\alpha}{\theta(e-1)}\right)^\delta \sum_{k=0}^\infty \frac{\delta^k}{k!}\left(\frac{\theta}{k+\delta}\right)^{\frac{\delta(2\alpha+1)-1}{2\alpha}} \Gamma\left(\frac{\delta(2\alpha+1)-1}{2\alpha}\right) - 1\right]. \tag{3.11}$$

Proof: Proof is easy.

Theorem 3.4. *If $X \sim$DUS-PIRD(α,θ), the Havrda and Charvat entropy of X is*

$$HC_{\delta}(x)=\frac{1}{2^{1-\delta}-1}\left[\frac{1}{2\alpha}\left(\frac{2\alpha}{\theta(e-1)}\right)^{\delta}\sum_{k=0}^{\infty}\frac{\delta^k}{k!}\left(\frac{\theta}{k+\delta}\right)^{\frac{\delta(2\alpha+1)-1}{2\alpha}}\Gamma\left(\frac{\delta(2\alpha+1)-1}{2\alpha}\right)-1\right]. \quad (3.12)$$

Proof: Proof is easy.

3.6. **Bonferroni and Lorenz curves:** A model for inequality of wealth distribution was proposed by [14] and to measure the income inequality introduced by [15]. Both models are used in financial mathematics, insurance, and population studies. Bonferroni and Lorenz's curves are defined as:

$$B(p)=\frac{1}{p\mu}\int_0^q xf(x)dx, \qquad L(p)=\frac{1}{\mu}\int_0^q xf(x)dx. \quad (3.13)$$

From (2.2),

$$\int_0^q xf(x)dx=\sum_{k=0}^{\infty}\frac{2\alpha}{k!(e-1)\theta}\int_0^{\infty}x^{-2\alpha}e^{-\frac{(k+1)}{\theta}x^{-2\alpha}}dx$$

Let $u=\left(\frac{k+1}{\theta}\right)x^{-2\alpha}$, then

$$\begin{aligned}\int_0^q xf(x)dx &= \sum_{k=0}^{\infty}\frac{1}{k!(e-1)\theta}\left(\frac{\theta}{k+1}\right)^{1-\frac{1}{2\alpha}}\int_{\left(\frac{k+1}{\theta}\right)q^{-2\alpha}}^{\infty}u^{-\frac{1}{2\alpha}}e^{-u}dx\\ &= \frac{1}{e-1}\left(\frac{1}{\theta}\right)^{\frac{1}{2\alpha}}\sum_{k=0}^{\infty}\frac{1}{(k+1)!}(k+1)^{\frac{1}{2\alpha}}\Gamma\left(1-\frac{1}{2\alpha},\frac{(k+1)}{\theta}q^{-2\alpha}\right). \quad (3.14)\end{aligned}$$

From equations (3.1), when $r=1$ and (3.14) into (3.13), then the Bonferroni curve is given by

$$B(p)=\frac{1}{p\mu}\int_0^q xf(x)dx=\frac{\sum_{k=0}^{\infty}\frac{(k+1)^{\frac{1}{2\alpha}}}{(k+1)!}\Gamma\left(1-\frac{1}{2\alpha},\frac{(k+1)}{\theta}q^{-2\alpha)}\right)}{p\sum_{k=0}^{\infty}\frac{(k+1)^{\frac{1}{2\alpha}}}{(k+1)!}\Gamma\left(1-\frac{1}{2\alpha}\right)}. \quad (3.15)$$

The Lorenz curve is obtained as

$$L(p)=\frac{1}{\mu}\int_0^q xf(x)dx=\frac{\sum_{k=0}^{\infty}\frac{(k+1)^{\frac{1}{2\alpha}}}{(k+1)!}\Gamma\left(1-\frac{1}{2\alpha},\frac{(k+1)}{\theta}q^{-2\alpha)}\right)}{\sum_{k=0}^{\infty}\frac{(k+1)^{\frac{1}{2\alpha}}}{(k+1)!}\Gamma\left(1-\frac{1}{2\alpha}\right)}. \quad (3.16)$$

4. Estimation of Parameters

To understand the probabilistic model fully, estimating the unknown parameters for designated sample is a main procedure. Various estimation approaches under classical and Bayesian model are reported in literature. This section considers the estimation of DUS-PIR distribution via maximum likelihood approach based on complete data.

4.1. **Maximum Likelihood Estimation:** Let $x_1, x_2, \cdots, x_n$ random sample follows the DUS-PIR distribution. The likelihood function (L.F.) of (2.2) is

$$L(\alpha,\theta)=\prod_{i=1}^{n} f(x_i,\alpha,\theta)=\prod_{i=1}^{n}\left[\frac{2\alpha x^{-(2\alpha+1)}}{(e-1)\theta}\exp\left(e^{-\frac{1}{\theta}x^{-2\alpha}}\right)e^{-\frac{1}{\theta}x^{-2\alpha}}\right]. \tag{4.1}$$

The log-L.F.is. given by

$$LogL(\alpha,\theta)=-n\ln(e-1)+n\ln(2\alpha)-n\ln(\theta)-(2\alpha+1)\sum_{i=1}^{n}\ln(x_i)-\frac{1}{\theta}\sum_{i=1}^{n}x_i^{-2\alpha}+\sum_{i=1}^{n}e^{-\frac{1}{\theta}x_i^{-2\alpha}}. \tag{4.2}$$

The partial derivatives of (4.2) are as follows.

$$\begin{aligned}
\frac{\partial}{\partial\alpha}LogL(\alpha,\theta) &= \frac{n}{\alpha}-2\sum_{i=1}^{n}\ln(x_i)+\frac{2}{\theta}\sum_{i=1}^{n}x_i^{-2\alpha}\ln(x_i)+\frac{2}{\theta}\sum_{i=1}^{n}e^{-\frac{1}{\theta}x_i^{-2\alpha}}x_i^{-2\alpha}\ln(x_i),\\
\frac{\partial}{\partial\theta}LogL(\alpha,\theta) &= -\frac{n}{\theta}+\frac{1}{\theta^2}\sum_{i=1}^{n}x_i^{-2\alpha}+\frac{1}{\theta^2}\sum_{i=1}^{n}e^{-\frac{1}{\theta}x_i^{-2\alpha}}x_i^{-2\alpha}.
\end{aligned}$$

The MLEs of α and θ can be derived as follows.

$$\frac{n}{\alpha}-2\sum_{i=1}^{n}\ln(x_i)+\frac{2}{\theta}\sum_{i=1}^{n}x_i^{-2\alpha}\ln(x_i)+\frac{2}{\theta}\sum_{i=1}^{n}e^{-\frac{1}{\theta}x_i^{-2\alpha}}x_i^{-2\alpha}\ln(x_i)=0, \tag{4.3}$$

$$-\frac{n}{\theta}+\frac{1}{\theta^2}\sum_{i=1}^{n}x_i^{-2\alpha}+\frac{1}{\theta^2}\sum_{i=1}^{n}e^{-\frac{1}{\theta}x_i^{-2\alpha}}x_i^{-2\alpha}=0. \tag{4.4}$$

Equations (4.3) and (4.4) has no closed form. So, we shall use a numerical program system to find its solution with respect to α and θ.

4.2. **Asymptotic Confidence Interval:** We derive asymptotic confidence intervals of unknown parameters using variance- covariance matrix $\boldsymbol{V}$, which is the inverse Fisher information matrix. The ML estimators are asymptotically normally distributed with multivariate normal distribution, see, [16].

$$(\hat{\alpha},\hat{\theta})\sim N_2(\boldsymbol{\Theta},\boldsymbol{V}),$$

where $\boldsymbol{\Theta}=(\alpha,\theta)$ and $\boldsymbol{V}$ is given as follows

$$\boldsymbol{V}=\begin{pmatrix} -\frac{\partial^2 LogL}{\partial\alpha^2} & -\frac{\partial^2 LogL}{\partial\alpha\partial\theta}\\ -\frac{\partial^2 LogL}{\partial\alpha\partial\theta} & -\frac{\partial^2 LogL}{\partial\theta^2}\end{pmatrix}^{-1}_{\Theta\to\hat{\Theta}},$$

where,

$$\frac{\partial^2}{\partial\alpha^2}LogL(\alpha,\theta) = -\frac{n}{\alpha^2}-\frac{4}{\theta}\sum_{i=1}^{n}x_i^{-2\alpha}[\ln(x_i)]^2+\frac{4}{\theta^2}\sum_{i=1}^{n}e^{-\frac{1}{\theta}x_i^{-2\alpha}}x_i^{-2\alpha}[\ln(x_i)]^2 -\frac{4}{\theta}\sum_{i=1}^{n}e^{-\frac{1}{\theta}x_i^{-2\alpha}}x_i^{-2\alpha}[\ln(x_i)]^2, \tag{4.5}$$

$$\frac{\partial^2}{\partial\alpha\partial\theta}LogL(\alpha,\theta) = -\frac{2}{\theta^2}\sum_{i=1}^{n}x_i^{-2\alpha}\ln(x_i)+\frac{2}{\theta^3}\sum_{i=1}^{n}e^{-\frac{1}{\theta}x_i^{-2\alpha}}x_i^{-4\alpha}\ln(x_i) -\frac{2}{\theta^2}\sum_{i=1}^{n}e^{-\frac{1}{\theta}x_i^{-2\alpha}}x_i^{-2\alpha}\ln(x_i), \tag{4.6}$$

$$\frac{\partial^2}{\partial\theta^2}LogL(\alpha,\theta) = \frac{n}{\theta^2}-\frac{2}{\theta^3}\sum_{i=1}^{n}x_i^{-2\alpha}+\frac{1}{\theta^4}\sum_{i=1}^{n}e^{-\frac{1}{\theta}x_i^{-2\alpha}}x_i^{-4\alpha}-\frac{2}{\theta^3}\sum_{i=1}^{n}e^{-\frac{1}{\theta}x_i^{-2\alpha}}x_i^{-2\alpha}. \tag{4.7}$$

A $100(1-\delta)\%$ confidence interval for $\boldsymbol{\Theta}=(\alpha,\theta)$, can be approximated by

$$\hat{\alpha}\pm z_{\frac{\delta}{2}}\sqrt{var(\hat{\alpha})}, \qquad \text{and} \qquad \hat{\theta}\pm z_{\frac{\delta}{2}}\sqrt{var(\hat{\theta})}$$

where $z_{\frac{\delta}{2}}$ is upper $100\frac{\delta}{2}$ -th percentile of $N(0,1)$, and $var(\hat{\Theta}_i)$ is the diagonal i-th element in $\boldsymbol{V}$.

5. Practical Illustration

The main objective of any new distribution is to increase its adaptability and applicability, which makes it useful in several field of studies, particularly, in the fields concerning with lifetime analysis. This section depicts the usefulness of DUS-PIR distribution and compare with the Powered IR, Exponential transformed IR, Transmuted IR, Exponentiated IR, IR and Rayleigh distribution using two sets of data. For comparison some criteria such as,

- K-S. (Kolmogorov Smirnov) statistic,

$$K-S=\sup_x|F_m(x)-\hat{F}(x)|$$

- R^2: the determination coefficient,

$$R^2=\frac{\sum_{i=1}^{m}\left(\hat{F}(x_i)-\overline{F}\right)^2}{\sum_{i=1}^{m}\left(\hat{F}(x_i)-\overline{F}\right)^2+\sum_{i=1}^{m}\left(F_m(x_i)-\hat{F}(x_i)\right)^2},$$

- RMSE: the root mean square error

$$RMSE=\left[\frac{1}{m}\sum_{i=1}^{m}\left(F_m(x_i)-\hat{F}(x_i)\right)^2\right]^{1/2},$$

- A.I.C. (Akaike Information Criterion), [17].

$$AIC=2k-2\ell.$$

- A. I.C.C. (Akaike Information Criterion with Correction), [18].

$$AAIC = AIC + \frac{2k(k+1)}{m-k+1},$$

- B.I.C. (Bayesian Information Criterion), [19].

$$BIC = k\ln(m) - 2\ell,$$

- and H.Q.I.C. (Hannan-Quinn Information Criterion)

$$HQIC = 2k\ln[\ln(m)] - 2\ell,$$

have been used, where k and m stands for number of parameters and observed data, $\ell = LogL$, $\hat{F}(x)$ is estimated CDF and $F_m(x)$ is the empirical DF.

$$\bar{F}(x) = \frac{1}{m}\sum_{i=1}^{m}\hat{F}(x_i), \qquad F_m(x) = \frac{1}{m}\sum_{i=1}^{m} I\left(x_{(i)} \le x\right)$$

and

$$I\left(x_{(i)} \le x\right) = \begin{cases} 1, & \text{if } x_{(i)} \le x \\ 0, & \text{otherwise} \end{cases}$$

According to prevailing knowledge, the model with the lowest AIC, AAIC, BIC, HQIC and K-S value is considered as best fit for the data.

Dataset 1: The following data reported by [20]. It comprises thirty consecutive March precipitation (in inches) observations.

0.77	1.74	0.81	1.20	1.95	1.20	0.47	1.43	3.37	2.20	3.00	3.09	1.51	2.10	0.52
1.62	1.31	0.32	0.59	0.81	2.81	1.87	1.18	1.35	4.75	2.48	0.96	1.89	0.90	2.05

For the above considered data, we have extracted the values of MLEs of parameters, K-S test, and p-values in below table.

Table 1. MLEs, K-S statistics and p-value.

Models	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	K-S	p-value
DUS-PIR	0.860	1.332	–	0.14557	0.52380
PIR	0.775	0.975	–	0.15223	0.462057
ETIR	–	1.454	–	0.18935	0.207305
TIR	–	1.591	-0.67	0.18176	0.247695
EIR	0.731	1.456	–	0.19818	0.166981
IR	–	1.164	–	0.23956	0.053115
Rayleigh	–	3.773	–	0.35059	0.000843

The log–likelihood (ℓ), information criteria, RMSE and R^2 are reported below.

Table 2. The ℓ, Information Criteria, RMSE and R^2.

Models	ℓ	AIC	AICC	BIC	HQIC	RMSE	R^2
DUS-PIR	-41.238	86.4760	86.9210	89.2790	87.3730	0.054453	0.96024
PIR	-41.917	87.8340	88.2780	90.6360	88.7310	0.059373	0.95182
ETIR	-42.026	86.0530	86.1950	87.4540	86.5010	0.075709	0.93570
TER	-42.101	88.2020	88.6470	91.0050	89.0990	0.073716	0.94105
EIR	-136.04	276.081	276.525	278.883	276.977	0.078017	0.92284
IR	-44.137	90.2730	90.4160	91.6740	90.7210	0.107514	0.88381
Rayleigh	-38.924	79.8490	79.9910	81.2500	80.2970	0.201452	0.48597

Listed values in the Tables 1-2. It has been noticed that DUS-PIR distribution interprets a better fit among all lifetime distributions.

The variance-covariance matrix is given as

$$\boldsymbol{V} = \begin{pmatrix} 0.012 & 0.008 \\ 0.008 & 0.076 \end{pmatrix}.$$

Then the 95% confidence interval for α and θ for DUS-PIR distribution are (0.648, 1.073) and (0.790, 1.874), respectively. It is shown that the LF has a unique solution by Figure 4.

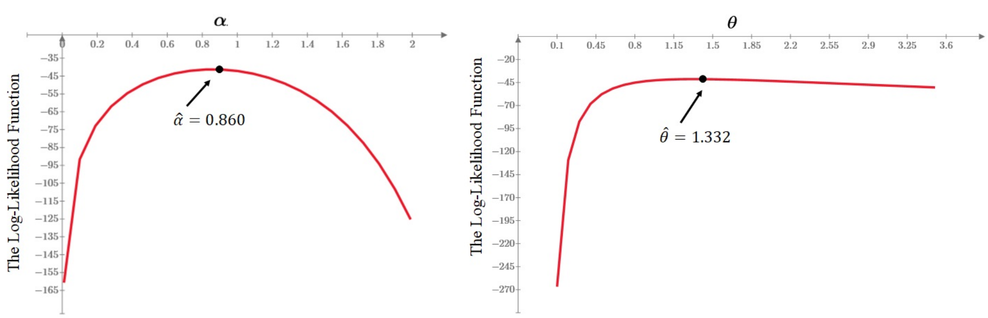

Figure 4. The profile of the log-LF of α and θ.

Dataset 2: The given data set is reported by [21]. It represents the survival times (in days) of 72 guinea pigs injected with different doses of tubercle bacilli.

2	24	34	44	54	57	60	61	65	70	76	84	95	109	129	146	233	297
15	32	38	48	54	58	60	62	67	72	76	85	96	110	131	175	258	341
22	32	38	52	55	58	60	63	68	73	81	87	98	121	143	175	258	341
24	33	43	53	56	59	60	65	70	75	83	91	99	127	146	211	263	376

Estimated values of parameters, test statistic and criterion are provided in the following table.

Table 3. MLEs, K-S statistics and p-value.

Models	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	K-S	p-value
DUS-PIR	0.782	0.003	–	0.18446	0.01290686
PIR	0.797	0.004	–	0.19755	5.1528E-24
ETIR	–	5.715×10^{-4}	–	0.20597	0.00371676
TIR	–	6.503×10^{-4}	-0.781	0.17999	0.01642093
EIR	0.616	6.555×10^{-4}	–	0.20997	0.00290489
IR	–	4.571×10^{-4}	–	0.25083	0.00017822
Rayleigh	–	1.628×10^{4}	–	0.97964	3.3351×10^{-62}

Table 4. The ℓ , Information Criteria, RMSE and R^2.

Models	ℓ	AIC	AICC	BIC	HQIC	RMSE	R^2
DUS-PIR	-394.466	792.932	793.106	797.485	794.744	0.068685	0.931008
PIR	-395.649	795.298	795.472	799.852	797.111	0.076096	0.913825
ETIR	-400.074	802.149	802.206	804.426	803.055	0.092092	0.899998
TIR	-398.920	801.839	802.013	806.392	803.652	0.078811	0.929423
EIR	-614.106	1232.00	1232.00	1237.00	1234.00	0.083047	0.899951
IR	-406.736	815.472	815.529	817.749	816.378	0.126351	0.831767
Rayleigh	-408.300	818.600	818.657	820.877	819.506	0.576828	5.726×10^{-05}

From Tables 3-4. It has been observed that DUS-PIR distribution suggests a better fit among all lifetime distributions for considered data. The variance-covariance matrix is given as

$$V = \begin{pmatrix} 0.004 & -7.853\times10^{-5} \\ -7.853\times10^{-5} & 1.680\times10^{-6} \end{pmatrix}.$$

Then the 95% confidence interval for α and θ for DUS-PIR distribution are (0.659, 0.905) and ($1.589\text{R}\times10^{-4}$, 0.005.), respectively. It is shown that the LF has a unique solution by Figure 5.

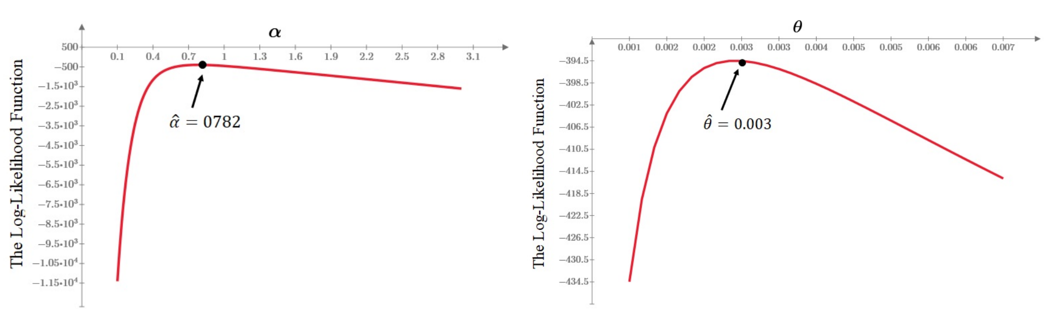

Figure 5. The profile of the log-LF of α and θ.

6. Conclusion

In this article, a new exponential transformed powered inverse Rayleigh distribution which includes unimodal behavior, and some of its basic properties are investigated. From the computation, it is confirmed that proposed distribution complies a better fitting to the datasets under consideration in terms of all the criteria.

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