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Essential Bipolar Fuzzy Ideals in Semigroups

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Abstract. In this paper, we give the concepts of essential bipolar fuzzy ideals in semigroups. We discuss the basic properties and relationships between essential bipolar fuzzy ideals and essential ideals in semigroups Finally, we extend to 0-essential bipolar fuzzy ideals in semigroups.

1. Introduction

The theory for dealing with uncertainty, fuzzy set theory, was discovered by Zadeh in 1965 [12], which it has applied in many areas such as medical science, robotics, computer science, information science, control engineering, measure theory, logic, set theory, topology etc. In 2000, Lee [6] developed theory of fuzzy set to theory of bipolar fuzzy set which function from intrval $[-1,0]\cup[0,1]$. The theory of bipolar set applied in information affects the effectiveness and efficiency of decision making. It is used in decision-making problems, organization problems, economic problems, and evaluation, risk management, environmental and social impact assessments. Later in 2012, S.K. Majumder [2] studies bipolar fuzzy set in Γ-semigroups and integration properties of bipolar fuzzy ideals in Γ-semigroups In 1971 U. Medhi et al. [7] was introduced Essential fuzzy ideals of ring. In 2013, U. Medhi and H.K. Saikia [8] discussed the concept of T-fuzzy essential ideals and studied the properties of T-fuzzy essential ideals. In 2017 S. Wani and K. Pawar [11] extend the concept of essential ideals in semigroups go to ternary semiring and studied essential ideals in ternary semiring. In 2020, S. Baupradist et al. [1]

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studied essential ideals and essential fuzzy ideals in semigroups. Together with 0-essential ideals and 0-essential fuzzy ideals in semigroups. In 2022 T. Gaketem et al. [10] studied essential bi-ideals and fuzzy essential bi-ideals in semigroups. Moreover, T. Gaketem and A. Iampan [3, 4] used knowledge of essential ideals in semigroups go to studied essential ideals in UP-algebra. In this papar, we used knowledge of essential fuzzy ideals in semigroups go study in bipolar valued fuzzy ideal in semigroup and we investigate it properties. Moreover, we characterize essential bipolar valued fuzzy ideals and 0-essential bipolar valued fuzzy ideals of semigroups.

2. Preliminaries

In this section, we review concepts basic definitions and the theorem used to prove all result in the next section.

A non-empty subset $\mathfrak{I}$ of a semigroup $\mathfrak{S}$ is called a *subsemigroup* of $\mathfrak{S}$ if $\mathfrak{I}^2 \subseteq \mathfrak{I}$.
A non-empty subset $\mathfrak{I}$ of a semigroup $\mathfrak{S}$ is called a *left (right)* ideal of $\mathfrak{S}$ if $\mathfrak{S}\mathfrak{I} \subseteq \mathfrak{I}$ ($\mathfrak{I}\mathfrak{S} \subseteq \mathfrak{I}$). An *ideal* $\mathfrak{I}$ of a semigroup $\mathfrak{S}$ is a non-empty subset which is both a left ideal and a right ideal of $\mathfrak{S}$. An *essential ideal* $\mathfrak{I}$ of a semigroup $\mathfrak{S}$ if $\mathfrak{I}$ is an ideal of $\mathfrak{S}$ and $\mathfrak{I} \cap \mathfrak{J} \neq \emptyset$ for every ideal $\mathfrak{J}$ of $\mathfrak{S}$.

We see that for any $\zeta_1, \zeta_2 \in [0, 1]$, we have

$$\zeta_1 \vee \zeta_2 = \max\{\zeta_1, \zeta_2\} \quad \text{and} \quad \zeta_1 \wedge \zeta_2 = \min\{\zeta_1, \zeta_2\}.$$

A fuzzy set ζ of a non-empty set $\mathfrak{T}$ is function from $\mathfrak{T}$ into unit closed interval $[0, 1]$ of real numbers, i.e., $\zeta : \mathfrak{T} \to [0, 1]$.

For any two fuzzy sets ζ and ϱ of a non-empty set $\mathfrak{T}$, define $\geq, =, \wedge$, and $\vee$ as follows:

(1) $\zeta \geq \varrho \Leftrightarrow \zeta(k) \geq \varrho(k)$ for all $k \in \mathfrak{T}$,
(2) $\zeta = \varrho \Leftrightarrow \zeta \geq \varrho$ and $\varrho \geq \zeta$,
(3) $(\zeta \wedge \varrho)(k) = \min\{\zeta(k), \varrho(k)\} = \zeta(k) \wedge \varrho(k)$ for all $k \in \mathfrak{T}$,
(4) $(\zeta \vee \varrho)(k) = \max\{\zeta(k), \varrho(k)\} = \zeta(k) \vee \varrho(k)$ for all $k \in \mathfrak{T}$.

For the symbol $\zeta \leq \varrho$, we mean $\varrho \geq \zeta$.

For any element $\mathfrak{k}$ in a semigroup $\mathfrak{S}$, define the set $F_{\mathfrak{k}}$ by

$$F_{\mathfrak{k}} := \{(\mathfrak{y}, \mathfrak{z}) \in \mathfrak{S} \times \mathfrak{S} \mid \mathfrak{k} = \mathfrak{y}\mathfrak{z}\}.$$

For two fuzzy sets ζ and ϱ on a semigroup $\mathfrak{S}$, define the product $\zeta \circ \varrho$ as follows: for all $\mathfrak{k} \in \mathfrak{S}$,

$$(\zeta \circ \varrho)(\mathfrak{k}) = \begin{cases} \bigvee_{(\mathfrak{y},\mathfrak{z}) \in F_{\mathfrak{k}}} \{\zeta(\mathfrak{y}) \wedge \varrho(\mathfrak{z})\} & \text{if} \quad F_{\mathfrak{k}} \neq \emptyset, \\ 0 & \text{if} \quad F_{\mathfrak{k}} = \emptyset. \end{cases}$$

The following definitions are types of fuzzy subsemigroups on semigroups.

Definition 2.1. *[9] A fuzzy set ζ of a semigroup $\mathfrak{S}$ is said to be a fuzzy ideal of $\mathfrak{S}$ if $\zeta(uv) \geq \zeta(u) \vee \zeta(v)$ for all $u, v \in \mathfrak{S}$.*

Definition 2.2. *[1] An* essential fuzzy ideal ζ *of a semigroup* $\mathfrak{S}$ *if* ζ *is a nonzero fuzzy ideal of* $\mathfrak{S}$ *and* $\zeta \wedge \varrho \neq 0$ *for every nonzero fuzzy ideal* ϱ *of* $\mathfrak{S}$.

Now, we reivew definition of bipolar valued fuzzy set and basic properties used in next section.

Definition 2.3. *[6] Let* $\mathfrak{S}$ *be a non-empty set. A bipolar fuzzy set (BF set)* ζ *on* $\mathfrak{S}$ *is an object having the form*

$$\zeta := \{(k, \zeta^p(k), \zeta^n(k)) \mid k \in \mathfrak{S}\},$$

where $\zeta^p : \mathfrak{S} \to [0,1]$ *and* $\zeta^n : \mathfrak{S} \to [-1,0]$.

Remark 2.1. *For the sake of simplicity we shall use the symbol* $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ *for the BF set* $\zeta = \{(k, \zeta^p(k), \zeta^n(k)) \mid k \in \mathfrak{S}\}$.

The following example of a BF set.

Example 2.1. *Let* $S = \{21, 22, 23...\}$. *Define* $\zeta^p : S \to [0,1]$ *is a function*

$$\zeta^p(u) = \begin{cases} 0 & \textit{if } u \textit{ is old number} \\ 1 & \textit{if } u \textit{ is even number} \end{cases}$$

and $\zeta^n : S \to [-1,0]$ *is a function*

$$\zeta^n(u) = \begin{cases} -1 & \textit{if } u \textit{ is old number} \\ 0 & \textit{if } u \textit{ is even number.} \end{cases}$$

Then $\zeta = (S; \zeta^p, \zeta^n)$ *is a BF set.*

For BF sets $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ and $\varrho = (\mathfrak{S}; \varrho^p, \varrho^n)$, define products $\zeta^p \circ \varrho^p$ and $\zeta^n \circ \varrho^n$ as follows: For $u \in \mathfrak{S}$

$$(\zeta^p \circ \varrho^p)(k) = \begin{cases} \bigvee\limits_{(y,z)\in F_k} \{\zeta^p(y) \wedge \varrho^p(z)\} & \text{if } k = yz \\ 0 & \text{if otherwise.} \end{cases}$$

and

$$(\zeta^n \circ \varrho^n)(k) = \begin{cases} \bigwedge\limits_{(y,z)\in F_k} \{\zeta^n(y) \vee \varrho^n(z)\} & \text{if } k = yz \\ 0 & \text{if otherwise.} \end{cases}$$

Definition 2.4. *[2] Let* $\mathfrak{I}$ *be a non-empty set of a semigroup* $\mathfrak{S}$. *A positive characteristic function and a negative characteristic function are respectively defined by*

$$\lambda^p_{\mathfrak{I}} : \mathfrak{S} \to [0,1], k \mapsto {}^p_{\mathfrak{I}}(u) := \begin{cases} 1 & k \in \mathfrak{I}, \\ 0 & k \notin \mathfrak{I}, \end{cases}$$

and

$$\lambda^n_{\mathfrak{J}} : \mathfrak{S} \to [-1,0], k \mapsto {}^n_I(k) := \begin{cases} -1 & k \in \mathfrak{J}, \\ 0 & k \notin \mathfrak{J}. \end{cases}$$

Remark 2.2. *For the sake of simplicity we shall use the symbol $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda^p_{\mathfrak{J}}, \lambda^n_{\mathfrak{J}})$ for the BF set $_{\mathfrak{J}} := \{(k, \lambda^p_{\mathfrak{J}}(k), \lambda^n_{\mathfrak{J}}(k)) \mid k \in \mathfrak{J}\}$.*

Definition 2.5. *[2] A BF set $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ on a semigroup $\mathfrak{S}$ is called a BF ideal on $\mathfrak{S}$ if it satisfies the following conditions: $\zeta^p(uv) \geq \zeta^p(u) \vee \zeta^p(v)$ and $\zeta^n(uv) \leq \zeta^n(v) \wedge \zeta^n(u)$ for all $u, v \in S$.*

The following theorems are true.

Theorem 2.1. *[2] Let $\mathfrak{K}$ be a nonempty subset of semigroup $\mathfrak{S}$. Then $\mathfrak{K}$ is an ideal of $\mathfrak{S}$ if and only if characteristic function $\lambda_{\mathfrak{K}} = (\mathfrak{S}; \lambda^p_{\mathfrak{K}}, \lambda^n_{\mathfrak{K}})$ is a BF ideal of $\mathfrak{S}$.*

Theorem 2.2. *[2] Let $\mathfrak{L}$ and $\mathfrak{J}$ be subsets of a non-empty set $\mathfrak{S}$. Then the following holds.*

(1) $\lambda^p_{\mathfrak{L}\cap\mathfrak{J}} = \lambda^p_{\mathfrak{L}} \wedge \lambda^p_{\mathfrak{J}}$.
(2) $\lambda^n_{\mathfrak{L}\cup\mathfrak{J}} = \lambda^n_{\mathfrak{L}} \vee \lambda^n_{\mathfrak{J}}$.
(3) $\lambda^p_{\mathfrak{L}} \circ \lambda^p_{\mathfrak{J}} = \lambda^p_{\mathfrak{L}\mathfrak{J}}$.
(4) $\lambda^n_{\mathfrak{L}} \circ \lambda^n_{\mathfrak{J}} = \lambda^n_{\mathfrak{L}\mathfrak{J}}$.

Let $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ be a BF set of a non-empty of $\mathfrak{S}$. Then the *support* of ζ instead of $\text{supp}(\zeta) = \{u \in S \mid \zeta(u) \neq 0\}$ where $\zeta^p(u) \neq 0$ and $\zeta^n(u) \neq 0$ for all $u \in \mathfrak{S}$.

Theorem 2.3. *Let $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ be a nonzero BF set of a semigroup $\mathfrak{S}$. Then $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ is a BF ideal of $\mathfrak{S}$ if and only if $\text{supp}(\zeta)$ is an ideal of $\mathfrak{S}$.*

Proof. Supposet that $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ is a BF ideal of $\mathfrak{S}$ and let $u, v \in \mathfrak{S}$, with
$u, v \in \text{supp}(\zeta)$ Then $\zeta^p(u) \neq 0$, $\zeta^p(v) \neq 0$ and $\zeta^n(u) \neq 0$, $\zeta^n(v) \neq 0$.
Since $\zeta = (S; \zeta^p, \zeta^n)$ is a BF ideal of $\mathfrak{S}$ we have $\zeta^p(uv) \geq \zeta^p(\mathfrak{u}) \vee \zeta^p(\mathfrak{v})$ and $\zeta^n(uv) \leq \zeta^n(\mathfrak{u}) \wedge \zeta^n(\mathfrak{v})$
Thus, $\zeta^p(uv) \neq 0$ and $\zeta^n(uv) \neq 0$. It implies that $uv \in \text{supp}(\zeta)$. Hence, $\text{supp}(\zeta)$ is an ideal of $\mathfrak{S}$.

Conversely, suppose that $\text{supp}(\zeta)$ is an ideal of $\mathfrak{S}$ and let $u, v, \in \mathfrak{S}$.

If $u, v \in \text{supp}(\zeta)$, then $uv \in \text{supp}(\zeta)$. Thus $\zeta^p(v) \neq 0$ and $\zeta^p(uv) \neq 0$.
Hence $\zeta^p(uv) \geq \zeta^p(u) \vee \zeta^p(v)$.

If $u \notin \text{supp}(\zeta)$ or $v \notin \text{supp}(\zeta)$ then $\zeta^p(uv) \geq \zeta^p(u) \vee \zeta^p(v)$.

Similarly, we can show that, $\zeta^n(uv) \leq \zeta^p(u) \wedge \zeta^n(v)$.

Thus, $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ is a BF ideal of $\mathfrak{S}$. □

3. Essential Bipolar Valued Fuzzy Ideals in a Semigroup.

Definition 3.1. *An* essential BF ideal *$\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ of a semigroup $\mathfrak{S}$ if $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ is a nonzero BF ideal of $\mathfrak{S}$ and $\zeta^p \wedge \varrho^p \neq 0$ and $\zeta^n \vee \varrho^n \neq 0$ for every nonzero BF ideal $\varrho = (\mathfrak{S}; \varrho^p, \varrho^n)$ of $\mathfrak{S}$.*

Theorem 3.1. *Let $\mathfrak{I}$ be an ideal of a semigroup $\mathfrak{S}$. Then $\mathfrak{I}$ is an essential ideal of $\mathfrak{S}$ if and only if $\lambda_{\mathfrak{I}} = (\mathfrak{S}; \lambda^p_{\mathfrak{I}}, \lambda^n_{\mathfrak{I}})$ is an essential BF ideal of $\mathfrak{S}$.*

Proof. Suppose that $\mathfrak{I}$ is an essential ideal of $\mathfrak{S}$ and let $\varrho = (\mathfrak{S}; \varrho^p, \varrho^n)$ be a nonzero BF ideal of $\mathfrak{S}$. Then by Theorem 2.3 $\mathrm{supp}(\varrho)$ is an ideal of $\mathfrak{S}$. Since $\mathfrak{I}$ is an essential ideal of $\mathfrak{S}$ we have $\mathfrak{I}$ is an ideal of $\mathfrak{S}$. Thus $\mathfrak{I} \cap \mathrm{supp}(\varrho) \neq \emptyset$. So there exists $u \in \mathfrak{I} \cap \mathrm{supp}(\varrho)$. Since $\mathfrak{I}$ is an ideal of $\mathfrak{S}$ we have $\lambda_{\mathfrak{I}} = (\mathfrak{S}; \lambda^p_{\mathfrak{I}}, \lambda^n_{\mathfrak{I}})$ is a BF ideal of $\mathfrak{S}$. Since $\varrho = (\mathfrak{S}; \varrho^p, \varrho^n)$ is a nonzero BF ideal of $\mathfrak{S}$ we have $(\lambda^p_{\mathfrak{I}} \wedge \varrho^p)(u) \neq 0$ and $(\lambda^n_{\mathfrak{I}} \vee \varrho^n)(u) \neq 0$ Thus, $\lambda^p_{\mathfrak{I}} \wedge \varrho^p \neq 0$ and $\lambda^n_{\mathfrak{I}} \vee \varrho^n \neq 0$. Therefore, $\lambda_{\mathfrak{I}} = (\mathfrak{S}; \lambda^p_{\mathfrak{I}}, \lambda^n_{\mathfrak{I}})$ is an essential BF ideal of $\mathfrak{S}$.

Conversely, assume that $\lambda_{\mathfrak{I}} = (\mathfrak{S}; \lambda^p_{\mathfrak{I}}, \lambda^n_{\mathfrak{I}})$ is an essential BF ideal of $\mathfrak{S}$ and let $\mathfrak{J}$ be an ideal of $\mathfrak{S}$. Then $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda^p_{\mathfrak{J}}, \lambda^n_{\mathfrak{J}})$ is a nonzero BF ideal of $\mathfrak{S}$.
Since $\lambda_{\mathfrak{I}} = (\mathfrak{S}; \lambda^p_{\mathfrak{I}}, \lambda^n_{\mathfrak{I}})$ is an essential BF ideal of $\mathfrak{S}$ we have $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda^p_{\mathfrak{J}}, \lambda^n_{\mathfrak{J}})$ is a BF ideal of $\mathfrak{S}$. Thus, $\lambda^p_{\mathfrak{I}} \wedge \lambda^p_{\mathfrak{J}} \neq 0$ and $\lambda^n_{\mathfrak{I}} \vee \lambda^n_{\mathfrak{J}} \neq 0$ So by Theorem 2.2, $\lambda^p_{\mathfrak{I}\cap\mathfrak{J}} \neq 0$ and $\lambda^n_{\mathfrak{I}\cup\mathfrak{J}} \neq 0$. Hence, $\mathfrak{I} \cap \mathfrak{J} \neq \emptyset$. Therefore, $\mathfrak{I}$ is an essential ideal of $\mathfrak{S}$. □

Theorem 3.2. *Let $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ be a nonzero BF ideal of a semigroup $\mathfrak{S}$. Then $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ is an essential BF ideal of $\mathfrak{S}$ if and only if $\mathrm{supp}(\zeta)$ is an essential ideal of $\mathfrak{S}$.*

Proof. Assume that $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ is an essential BF ideal of $\mathfrak{S}$ and let $\mathfrak{J}$ be an ideal of $\mathfrak{S}$. Then by Theorem 2.1, $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda^p_{\mathfrak{J}}, \lambda^n_{\mathfrak{J}})$ is a BF ideal of $\mathfrak{S}$. Since $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ is an essential BF ideal of $\mathfrak{S}$ we have $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ is a BF ideal of $\mathfrak{S}$. Thus, $\zeta^p \wedge \lambda^p_{\mathfrak{J}} \neq 0$ and $\zeta^n \vee \lambda^n_{\mathfrak{J}} \neq 0$. So there exists $u \in \mathfrak{S}$ such that $(\zeta^p \wedge \lambda^p_{\mathfrak{J}})(u) \neq 0$ and $(\zeta^n \vee \lambda^n_{\mathfrak{J}})(u) \neq 0$. It implies that $\zeta^p(u) \neq 0$, $\lambda^p_{\mathfrak{J}}(u) \neq 0$ and $\zeta^n(u) \neq 0$, $\lambda^n_{\mathfrak{J}}(u) \neq 0$. Hence, $u \in \mathrm{supp}(\zeta) \cap \mathfrak{J}$ so $\mathrm{supp}(\zeta) \cap \mathfrak{J} \neq \emptyset$. Therefore, $\mathrm{supp}(\zeta)$ is an essential ideal of $\mathfrak{S}$.

Conversely, assume that $\mathrm{supp}(\zeta)$ is an essential ideal of $\mathfrak{S}$ and let $\varrho = (\mathfrak{S}; \varrho^p, \varrho^n)$ be a nonzero BF ideal of $\mathfrak{S}$. Then by Thoerem 2.3 $\mathrm{supp}(\varrho)$ is an ideal of $\mathfrak{S}$. Since $\mathrm{supp}(\zeta)$ is an essential ideal of $\mathfrak{S}$ we have $\mathrm{supp}(\zeta)$ is an ideal of $\mathfrak{S}$. Thus $\mathrm{supp}(\zeta) \cap \mathrm{supp}(\varrho) \neq \emptyset$. So, there exists $u \in \mathrm{supp}(\zeta) \cap \mathrm{supp}(\varrho)$. It implies that $\zeta^p(u) \neq 0$, $\zeta^n(u) \neq 0$ and $\varrho^p(u) \neq 0$, $\varrho^n(u) \neq 0$. for all $u \in \mathfrak{S}$. Hence, $(\zeta^p \wedge \varrho^p)(u) \neq 0$ and $(\zeta^n \vee \varrho^n)(u) \neq 0$ for all $u \in \mathfrak{S}$. Therefore, $\zeta^p \wedge \varrho^p \neq 0$ and $\zeta^n \vee \varrho^n \neq 0$. We conclude that $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ is an essential BF ideal of $\mathfrak{S}$. □

Theorem 3.3. *Let $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ be an essential BF ideal of a semigroup $\mathfrak{S}$. If $\varrho = (\mathfrak{S}; \varrho^p, \varrho^n)$ is a BF ideal of $\mathfrak{S}$ such that $\zeta^p \leq \varrho^p$ and $\zeta^n \geq \varrho^n$, then $\varrho = (\mathfrak{S}; \varrho^p, \varrho^n)$ is also an essential BF ideal of $\mathfrak{S}$.*

Proof. Let $\varrho = (\mathfrak{S}; \varrho^p, \varrho^n)$ is a BF ideal of $\mathfrak{S}$ such that $\zeta^p \leq \varrho^p$ and $\zeta^n \geq \varrho^n$ and let $\xi = (\mathfrak{S}; \xi^p, \xi^n)$ be any BF ideal of $\mathfrak{S}$. Since $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ is an essential BF ideal of $\mathfrak{S}$ we have $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ is a BF ideal of $\mathfrak{S}$. Thus $\zeta^p \wedge \xi^p \neq 0$ and $\zeta^n \vee \xi^n \neq 0$. So $\varrho^p \wedge \xi^p \neq 0$ and $\varrho^n \vee \xi^n \neq 0$. Hence $\varrho = (\mathfrak{S}; \varrho^p, \varrho^n)$ is an essential BF ideal of $\mathfrak{S}$. □

Next, we study the intersection and union of BF sets as define.
Let $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ and $\varrho = (\mathfrak{S}; \varrho^p, \varrho^n)$ are BF sets of a semigroup $\mathfrak{S}$.

Define $\zeta \cap \varrho = (\zeta^p \cap \varrho^p, \zeta^n \cap \varrho^n)$ where $(\zeta^p \cap \varrho^p)(k) = \zeta^p(k) \wedge \varrho^p(k)$ and $(\zeta^n \cap \varrho^n)(k) = \zeta^n(k) \vee \varrho^n(k)$ for all $k \in \mathfrak{S}$.

Define $\zeta \cup \varrho = (\zeta^p \cup \varrho^p, \zeta^n \cup \varrho^n)$ where $(\zeta^p \cup \varrho^p)(k) = \zeta^p(k) \vee \varrho^p(k)$ and $(\zeta^n \cup \varrho^n)(k) = \zeta^n(k) \wedge \varrho^n(k)$ for all $k \in \mathfrak{S}$.

Theorem 3.4. *Let $\zeta_1 = (\mathfrak{S}; \zeta_1^p, \zeta_1^n)$ and $\zeta_2 = (\mathfrak{S}; \zeta_2^p, \zeta_2^n)$ be essential BF ideals of a semigroup $\mathfrak{S}$. Then $\zeta_1 \cup \zeta_2$ and $\zeta_1 \cap \zeta_2$ are essential BF ideals of $\mathfrak{S}$.*

Proof. By Theorem 3.3, we have $\zeta_1 \cup \zeta_2$ is an essential BF ideal of $\mathfrak{S}$.

Since $\zeta_1 = (\mathfrak{S}; \zeta_1^p, \zeta_1^n)$ and $\zeta_2 = (\mathfrak{S}; \zeta_2^p, \zeta_2^n)$ are essential BF ideals of $\mathfrak{S}$ we have $\zeta_1 = (\mathfrak{S}; \zeta_1^p, \zeta_1^n)$ and $\zeta_2 = (\mathfrak{S}; \zeta_2^p, \zeta_2^n)$ are BF ideals of $\mathfrak{S}$. Thus $\zeta_1 \cap \zeta_2$ is a BF ideal of $\mathfrak{S}$. Let $\xi = (\mathfrak{S}; \xi^p, \xi^n)$ be a BF ideal of $\mathfrak{S}$. Then $\zeta_1^p \wedge \xi^p \neq 0$ and $\zeta_1^n \vee \xi^n \neq 0$. Thus there exists $u \in \mathfrak{S}$ such that $(\zeta_1^p \wedge \xi^p)(u) \neq 0$ and $(\zeta_1^n \vee \xi^n)(u) \neq 0$. So $\zeta_1^p(u) \neq 0$, $\zeta_1^n(u) \neq 0$ and $\xi^p(u) \neq 0$ and $\xi^n(u) \neq 0$. Since $\zeta_2^p \neq 0$ and $\zeta_2^n \neq 0$ and let $v \in \mathfrak{S}$ such that $\zeta_2^p(v) \neq 0$ and $\zeta_2^n(v) \neq 0$. Since $\zeta_1 = (\mathfrak{S}; \zeta_1^p, \zeta_1^n)$ and $\zeta_2 = (\mathfrak{S}; \zeta_2^p, \zeta_2^n)$ are BF ideals of $\mathfrak{S}$ we have $\zeta_1^p(uv) \geq \zeta_1^p(u) \wedge \zeta_1^p(v) \geq 0$, $\zeta_1^n(uv) \leq \zeta_1^n(u) \vee \zeta_1^n(v) \leq 0$ and $\zeta_2^p(uv) \geq \zeta_2^p(u) \wedge \zeta_2^p(v) \geq 0$, $\zeta_2^n(uv) \leq \zeta_2^n(u) \vee \zeta_2^n(v) \leq 0$. Thus $(\zeta_1^p \wedge \zeta_2^p)(uv) = \zeta_1^p(uv) \wedge \zeta_2^p(uv) \neq 0$ and $(\zeta_1^n \vee \zeta_2^n)(uv) = \zeta_1^n(uv) \vee \zeta_2^n(uv) \neq 0$. Since $\xi = (\mathfrak{S}; \xi^p, \xi^n)$ is a BF ideal of $\mathfrak{S}$ and $\xi^p(u) \neq 0$ and $\xi^n(u) \neq 0$ we have $\xi^p(uv) \neq 0$ and $\xi^n(uv) \neq 0$ for all $u, v \in \mathfrak{S}$. Thus $((\zeta_1^p \wedge \zeta_2^p) \wedge \xi^p)(uv) \neq 0$ and $((\zeta_1^n \vee \zeta_2^n) \vee \xi^n)(uv) \neq 0$. Hence $((\zeta_1^p \wedge \zeta_2^p) \wedge \xi^p) \neq 0$ and $((\zeta_1^n \vee \zeta_2^n) \vee \xi^n) \neq 0$. Therefore, $\zeta_1 \cap \zeta_2$ is an essential BF ideal of $\mathfrak{S}$. □

Definition 3.2. *[1] An essential ideal $\mathfrak{I}$ of a semigroup $\mathfrak{S}$ is called*

(1) *a* minimal *if for every essential ideal of $\mathfrak{J}$ of $\mathfrak{S}$ such that $\mathfrak{J} \subseteq \mathfrak{I}$, we have $\mathfrak{J} = \mathfrak{I}$,*
(2) *a* prime *if $uv \in \mathfrak{I}$ implies $u \in \mathfrak{I}$ or $v \in \mathfrak{I}$,*
(3) *a* semiprime *if $u^2 \in \mathfrak{I}$ implies $u \in \mathfrak{I}$, for all $u, v \in \mathfrak{S}$.*

Example 3.1. *[1] Let $\mathfrak{S}$ be a semigroup with zero. Then $\{0\}$ is a unique minimal essential ideal of $\mathfrak{S}$, since $\{0\}$ is an eseential ideal of $\mathfrak{S}$.*

Definition 3.3. *An essential BF ideal $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ of a semigroup $\mathfrak{S}$ is called*

(1) *a* minimal *if for every essential BF ideal of $\varrho = (\mathfrak{S}; \varrho^p, \varrho^n)$ of $\mathfrak{S}$ such that $\zeta^p \leq \varrho^p$ and $\zeta^n \geq \varrho^n$, we have* $\text{supp}(\zeta) = \text{supp}(\varrho)$,
(2) *a* prime *if $\zeta^p(uv) \leq \zeta^p(u) \vee \zeta^p(v)$ and $\zeta^n(uv) \geq \zeta^n(u) \wedge \zeta^n(v)$,*
(3) *a* semiprime *if $\zeta^p(u^2) \leq \zeta^p(u)$ and $\zeta^n(u^2) \geq \zeta^n(u)$, for all $u, v \in \mathfrak{S}$.*

Theorem 3.5. *Let $\mathfrak{I}$ be a non-empty subset of a semigroup $\mathfrak{S}$. Then the following statement holds.*

(1) *$\mathfrak{I}$ is a minimal essential ideal of $\mathfrak{S}$ if and only if $\lambda_{\mathfrak{I}} = (\mathfrak{S}; \lambda_{\mathfrak{I}}^p, \lambda_{\mathfrak{I}}^n)$ is a minimal essential BF ideal of $\mathfrak{S}$,*
(2) *$\mathfrak{I}$ is a prime essential ideal of $\mathfrak{S}$ if and only if $\lambda_{\mathfrak{I}} = (\mathfrak{S}; \lambda_{\mathfrak{I}}^p, \lambda_{\mathfrak{I}}^n)$ is a prime essential BF ideal of $\mathfrak{S}$,*

(3) *$\mathfrak{J}$ is a semiprime essential ideal of S if and only if $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda^p_{\mathfrak{J}}, \lambda^n_{\mathfrak{J}})$ is a semiprime essential BF ideal of $\mathfrak{S}$.*

Proof. (1) Suppose that $\mathfrak{J}$ is a minimal essential ideal of $\mathfrak{S}$. Then $\mathfrak{J}$ is an essential ideal of $\mathfrak{S}$. By Theorem 3.1, $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda^p_{\mathfrak{J}}, \lambda^n_{\mathfrak{J}})$ is an essential BF ideal of $\mathfrak{S}$. Let $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ be an essential BF ideal of $\mathfrak{S}$ such that $\zeta^p \leq \lambda^p_{\mathfrak{J}}$ and $\zeta^n \geq \lambda^n_{\mathfrak{J}}$. Then $\mathrm{supp}(\zeta) \subseteq \mathrm{supp}(\lambda_{\mathfrak{J}})$. Thus, $\mathrm{supp}(\zeta) \subseteq \mathrm{supp}(\lambda_{\mathfrak{J}}) = \mathfrak{J}$. Hence, $\mathrm{supp}(\zeta) \subseteq \mathfrak{J}$. Since $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ is an essential BF ideal of $\mathfrak{S}$ we have $\mathrm{supp}(\zeta)$ is an essential ideal of $\mathfrak{S}$. By assumption, $\mathrm{supp}(\zeta) = \mathfrak{J} = \mathrm{supp}(\lambda_{\mathfrak{J}})$. Hence, $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda^p_{\mathfrak{J}}, \lambda^n_{\mathfrak{J}})$ is a minimal essential BF ideal of $\mathfrak{S}$.

Conversely, $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda^p_{\mathfrak{J}}, \lambda^n_{\mathfrak{J}})$ is a minimal essential BF ideal of $\mathfrak{S}$ and let $\mathfrak{B}$ be an essential ideal of $\mathfrak{S}$ such that $\mathfrak{B} \subseteq \mathfrak{J}$. Then $\mathfrak{B}$ is an ideal of $\mathfrak{S}$. Thus by Theorem 3.1, $\lambda_{\mathfrak{B}} = (\mathfrak{S}; \lambda^p_{\mathfrak{B}}, \lambda^n_{\mathfrak{B}})$ is an essential BF ideal of $\mathfrak{S}$ such that $\lambda^p_{\mathfrak{B}} \geq \lambda^p_{\mathfrak{J}}$ and $\lambda^n_{\mathfrak{B}} \leq \lambda^n_{\mathfrak{J}}$. So $\lambda_{\mathfrak{B}} = \lambda_{\mathfrak{J}}$. Hence $\mathfrak{B} = \mathrm{supp}(\lambda_{\mathfrak{B}}) = \mathrm{supp}(\lambda_{\mathfrak{J}}) = \mathfrak{J}$. Therefore $\mathfrak{J}$ is a minimal essential ideal of $\mathfrak{S}$.

(2) Suppose that $\mathfrak{J}$ is a prime essential ideal of $\mathfrak{S}$. Then $\mathfrak{J}$ is an essential ideal of $\mathfrak{S}$. Thus by Theorem 3.1 $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda^p_{\mathfrak{J}}, \lambda^n_{\mathfrak{J}})$ is an essential BF ideal of $\mathfrak{S}$. Let $u, v \in \mathfrak{S}$.

If $uv \in \mathfrak{J}$, then $u \in \mathfrak{J}$ or $v \in \mathfrak{J}$. Thus $\lambda^p_{\mathfrak{J}}(u) \vee \lambda^p_{\mathfrak{J}}(v) = 1 \geq \lambda^p_{\mathfrak{J}}(uv)$ and $\lambda^n_{\mathfrak{J}}(u) \wedge \lambda^n_{\mathfrak{J}}(v) = -1 \leq \lambda^n_{\mathfrak{J}}(uv)$.

If $uv \notin \mathfrak{J}$, then $\lambda^p_{\mathfrak{J}}(u) \vee \lambda^p_{\mathfrak{J}}(v) \geq \lambda^p_{\mathfrak{J}}(uv)$ and $\lambda^n_{\mathfrak{J}}(u) \wedge \lambda^n_{\mathfrak{J}}(v) \leq \lambda^n_{\mathfrak{J}}(uv)$.

Thus $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda^p_{\mathfrak{J}}, \lambda^n_{\mathfrak{J}})$ is a prime essential BF ideal of $\mathfrak{S}$.

Conversely, suppose that $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda^p_{\mathfrak{J}}, \lambda^n_{\mathfrak{J}})$ is a prime essential BF ideal of $\mathfrak{S}$. Then $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda^p_{\mathfrak{J}}, \lambda^n_{\mathfrak{J}})$ is an essential BF ideal. Thus by Theorem 3.1, $\mathfrak{J}$ is an essential ideal of $\mathfrak{S}$. Let $u, v \in \mathfrak{S}$. If $uv \in \mathfrak{J}$, then $\lambda^p_{\mathfrak{J}}(uv) = 1$ and $\lambda^n_{\mathfrak{J}}(uv) = -1$. By assumption, $\lambda^p_{\mathfrak{J}}(uv) \leq \lambda^p_{\mathfrak{J}}(u) \vee \lambda^p_{\mathfrak{J}}(v)$ and $\lambda^n_{\mathfrak{J}}(uv) \geq \lambda^n_{\mathfrak{J}}(u) \wedge \lambda^n_{\mathfrak{J}}(v)$. Thus $\lambda^p_{\mathfrak{J}}(u) \vee \lambda^p_{\mathfrak{J}}(v) = 1$ and $\lambda^n_{\mathfrak{J}}(u) \wedge \lambda^n_{\mathfrak{J}}(v) = -1$ so $u \in \mathfrak{J}$ or $v \in \mathfrak{J}$. Hence $\mathfrak{J}$ is a prime essential ideal of $\mathfrak{S}$.

(3) Suppose that $\mathfrak{J}$ is a semiprime essential ideal of $\mathfrak{S}$. Then $\mathfrak{J}$ is an essential ideal of $\mathfrak{S}$. Thus by Theorem 4.1, $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda^p_{\mathfrak{J}}, \lambda^n_{\mathfrak{J}})$ is an essential BF ideal of $\mathfrak{S}$. Let $u \in \mathfrak{S}$.

If $u^2 \in \mathfrak{J}$, then $u \in \mathfrak{J}$. Thus, $\lambda^p_{\mathfrak{J}}(u) = \lambda^p_{\mathfrak{J}}(u^2) = 1$ and $\lambda^n_{\mathfrak{J}}(u) = \lambda^n_{\mathfrak{J}}(u^2) = -1$. Hence, $\lambda^p_{\mathfrak{J}}(u^2) \leq \lambda^p_{\mathfrak{J}}(u)$ and $\lambda^n_{\mathfrak{J}}(u^2) \geq \lambda^n_{\mathfrak{J}}(u)$.

If $u^2 \notin \mathfrak{J}$, then $\lambda^p_{\mathfrak{J}}(u^2) = 0 \leq \lambda^p_{\mathfrak{J}}(u)$ and $\lambda^n_{\mathfrak{J}}(u^2) = 0 \geq \lambda^n_{\mathfrak{J}}(u)$.

Thus $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda^p_{\mathfrak{J}}, \lambda^n_{\mathfrak{J}})$ is a semiprime essential BF ideal of $\mathfrak{S}$.

Conversely, suppose that $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda^p_{\mathfrak{J}}, \lambda^n_{\mathfrak{J}})$ is a semiprime essential BF ideal of $\mathfrak{S}$. Then $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda^p_{\mathfrak{J}}, \lambda^n_{\mathfrak{J}})$ is an essential BF ideal of $\mathfrak{S}$. Thus by Theorem 4.1, $\mathfrak{J}$ is an essential ideal of $\mathfrak{S}$. Let $u \in \mathfrak{S}$ with $u^2 \in \mathfrak{J}$. Then $\lambda^p_{\mathfrak{J}}(u^2) = 1$ and $\lambda^n_{\mathfrak{J}}(u^2) = -1$. By assumption, $\lambda^p_{\mathfrak{J}}(u^2) \leq \lambda^p_{\mathfrak{J}}(u)$ and $\lambda^n_{\mathfrak{J}}(u^2) \geq \lambda^n_{\mathfrak{J}}(u)$. Thus $\lambda^p_{\mathfrak{J}}(u) = 1$ and $\lambda^n_{\mathfrak{J}}(u) = -1$ so $u \in \mathfrak{J}$. Hence, $\mathfrak{J}$ is a semiprime essential ideal of $\mathfrak{S}$.

$\square$

Theorem 3.6. *Let $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ be a minimal essential BF ideal of a semigroup $\mathfrak{S}$. If $\varrho = (\mathfrak{S}; \varrho^p, \varrho^n)$ is a BF ideal of $\mathfrak{S}$ such that $\zeta^p \leq \varrho^p$ and $\zeta^n \geq \varrho^n$, then $\varrho = (\mathfrak{S}; \varrho^p, \varrho^n)$ is also a minimal essential BF ideal of $\mathfrak{S}$.*

Proof. Let $\varrho = (\mathfrak{S}; \varrho^p, \varrho^n)$ is a BF ideal of $\mathfrak{S}$ such that $\zeta^p \leq \varrho^p$ and $\zeta^n \geq \varrho^n$ and let $\xi = (\mathfrak{S}; \xi^p, \xi^n)$ be any BF ideal of $\mathfrak{S}$. Since $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ is a minimal essential BF ideal of $\mathfrak{S}$ we have $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ is a BF ideal of $\mathfrak{S}$. Thus, $\zeta^p \wedge \xi^p \neq 0$ and $\zeta^n \vee \xi^n \neq 0$. So $\varrho^p \wedge \xi^p \neq 0$ and $\varrho^n \vee \xi^n \neq 0$. Hence, $\varrho = (\mathfrak{S}; \varrho^p, \varrho^n)$ is a minimal essential BF ideal of $\mathfrak{S}$. □

Corollary 3.1. *Let $\zeta_1 = (\mathfrak{S}; \zeta_1^p, \zeta_1^n)$ and $\zeta_2 = (\mathfrak{S}; \zeta_2^p, \zeta_2^n)$ be minimal essential BF ideals of a semigroup $\mathfrak{S}$. Then $\zeta_1 \cup \zeta_2$ is a minimal essential BF ideals of $\mathfrak{S}$.*

4. 0-Essential BF ideal.

In this section, we let $\mathfrak{S}$ be a semigroup with zero. begin we review the definition 0-essential ideal of $\mathfrak{S}$ as follows:

Definition 4.1. *[1] A nonzero ideal $\mathfrak{I}$ of a semigroup with zero $\mathfrak{S}$ is called a* 0-essential ideal *of $\mathfrak{S}$ if $\mathfrak{I} \cap \mathfrak{J} \neq \{0\}$ for every nonzero ideal of $\mathfrak{J}$ of $\mathfrak{S}$.*

Example 4.1. *[1] Let $(\mathbb{Z}_{12}, +)$ be semigroup. Then $\{0, 2, 4, 6, 8, 10\}$ and $\mathbb{Z}_{12}$ are 0-essential ideal of $\mathbb{Z}_{12}$.*

Definition 4.2. *A BF ideal $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ of a semigroup with zero $\mathfrak{S}$ is called a* nontrivial BF ideal *of $\mathfrak{S}$ if there exists a nonzero element $u \in \mathfrak{S}$ such that $\zeta^p(u) \neq 0$ and $\zeta^n(u) \neq 0$.*

We define the definition of 0-essential BF ideals of a semigroup with zero as follows:

Definition 4.3. *A* 0-essential BF ideal *$\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ of a semigroup with zero $\mathfrak{S}$ if $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ is a nonzero BF ideal of $\mathfrak{S}$ and supp$(\zeta \wedge \varrho) \neq \{0\}$ for every nonzero BF ideal $\varrho = (\mathfrak{S}; \varrho^p, \varrho^n)$ of $\mathfrak{S}$.*

Theorem 4.1. *Let $\mathfrak{I}$ be a nonzero ideal of a semigroup with zero $\mathfrak{S}$. Then $\mathfrak{I}$ is a 0-essential ideal of $\mathfrak{S}$ if and only if $\lambda_{\mathfrak{I}} = (\mathfrak{S}; \lambda_{\mathfrak{I}}^p, \lambda_{\mathfrak{I}}^n)$ is a 0-essential BF ideal of $\mathfrak{S}$.*

Proof. Suppose that $\mathfrak{I}$ is a 0-essential ideal of $\mathfrak{S}$ and let $\varrho = (\mathfrak{S}; \varrho^p, \varrho^n)$ be a nontrivial BF ideal of $\mathfrak{S}$. Then by Theorem 2.3, supp(ϱ) is a nonzero ideal of $\mathfrak{S}$. Since $\mathfrak{I}$ is a 0-essential ideal of $\mathfrak{S}$ we have $\mathfrak{I}$ is a nonzero ideal of $\mathfrak{S}$. Thus $\mathfrak{I} \cap$supp$(\varrho) \neq \{0\}$. So there exists $u \in \mathfrak{I} \cap$ supp(ϱ). Since $\mathfrak{I}$ is a nonzero ideal of $\mathfrak{S}$ we have $\lambda_{\mathfrak{I}} = (\mathfrak{S}; \lambda_{\mathfrak{I}}^p, \lambda_{\mathfrak{I}}^n)$ is a nonzero BF ideal of $\mathfrak{S}$. Since $\varrho = (\mathfrak{S}; \varrho^p, \varrho^n)$ is a nonzero BF ideal of $\mathfrak{S}$ we have supp$(\lambda_{\mathfrak{I}} \wedge \varrho)(u) \neq 0$. Thus, $\lambda_{\mathfrak{I}}^p \wedge \varrho^p \neq 0$ and $\lambda_{\mathfrak{I}}^n \vee \varrho^n \neq 0$. Therefore, $\lambda_{\mathfrak{I}} = (\mathfrak{S}; \lambda_{\mathfrak{I}}^p, \lambda_{\mathfrak{I}}^n)$ is a 0-essential BF ideal of $\mathfrak{S}$.

Conversely, assume that $\lambda_{\mathfrak{I}} = (\mathfrak{S}; \lambda_{\mathfrak{I}}^p, \lambda_{\mathfrak{I}}^n)$ is a 0-essential BF ideal of $\mathfrak{S}$ and let $\mathfrak{J}$ be a nonzero ideal of $\mathfrak{S}$. Then $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda_{\mathfrak{J}}^p, \lambda_{\mathfrak{J}}^n)$ is a nonzero BF ideal of $\mathfrak{S}$. Sicne $\lambda_{\mathfrak{I}} = (\mathfrak{S}; \lambda_{\mathfrak{I}}^p, \lambda_{\mathfrak{I}}^n)$ is a 0-essential BF ideal of $\mathfrak{S}$ we have $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda_{\mathfrak{J}}^p, \lambda_{\mathfrak{J}}^n)$ is a nontrivial BF ideal of $\mathfrak{S}$. Thus, supp$(\lambda_{\mathfrak{I}} \wedge \lambda_{\mathfrak{J}}) \neq \{0\}$.

So by Theorem 2.2, $\lambda^{p}_{\mathfrak{L}\cap\mathfrak{J}} \neq 0$ and $\lambda^{n}_{\mathfrak{L}\cup\mathfrak{J}} \neq 0$. Hence, $\mathfrak{I}\cap\mathfrak{J} \neq \{0\}$. Therefore $\mathfrak{I}$ is a 0-essential ideal of $\mathfrak{S}$. □

Theorem 4.2. *Let $\zeta = (\mathfrak{S};\zeta^p,\zeta^n)$ be a nonzero BF ideal of a semigroup with zero $\mathfrak{S}$. Then ζ is a 0-essential BF ideal of $\mathfrak{S}$ if and only if $\mathrm{supp}(\zeta)$ is a 0-essential ideal of $\mathfrak{S}$.*

Proof. Assume that $\zeta = (\mathfrak{S};\zeta^p,\zeta^n)$ is a 0-essential BF ideal of $\mathfrak{S}$ and let $\mathfrak{J}$ be a nontrivial ideal of $\mathfrak{S}$. Then by Theorem 2.1, $\lambda_{\mathfrak{J}} = (\mathfrak{S};\lambda^p_{\mathfrak{J}},\lambda^n_{\mathfrak{J}})$ is a nonzero BF ideal of $\mathfrak{S}$. Since $\zeta = (\mathfrak{S};\zeta^p,\zeta^n)$ is a 0-essential BF ideal of $\mathfrak{S}$ we have $\zeta = (\mathfrak{S};\zeta^p,\zeta^n)$ is a nonzero BF ideal of $\mathfrak{S}$. Thus $\zeta^p \wedge \lambda^p_{\mathfrak{J}} \neq 0$ and $\zeta^n \vee \lambda^n_{\mathfrak{J}} \neq 0$. So there exists a nonzero element $u \in \mathfrak{S}$ such that $(\zeta^p \wedge \lambda^p_{\mathfrak{J}})(u) \neq 0$ and $(\zeta^n \vee \lambda^n_{\mathfrak{J}})(u) \neq 0$. It implies that $\zeta^p(u) \neq 0$, $\zeta^n(u) \neq 0$ and $\lambda^p_{\mathfrak{J}}(u) \neq 0$, $\lambda^n_{\mathfrak{J}}(u) \neq 0$. Hence, $u \in \mathrm{supp}(\zeta)\cap\mathfrak{J}$ so $\mathrm{supp}(\zeta)\cap\mathfrak{J} \neq \{0\}$. Therefore $\mathrm{supp}(\zeta)$ is a 0-essential ideal of $\mathfrak{S}$.

Conversely, assume that $\mathrm{supp}(\zeta)$ is a 0-essential ideal of $\mathfrak{S}$ and let $\varrho = (\mathfrak{S};\varrho^p,\varrho^n)$ be a nonzero BF ideal of $\mathfrak{S}$. Then by Theorem 2.3 $\mathrm{supp}(\varrho)$ is a nontrivial zero ideal of $\mathfrak{S}$. Since $\mathrm{supp}(\zeta)$ is a 0-essential ideal of $\mathfrak{S}$ we have $\mathrm{supp}(\zeta)$ is a nonzero ideal of $\mathfrak{S}$. Thus $\mathrm{supp}(\zeta)\cap\mathrm{supp}(\varrho) \neq \{0\}$. So there exists $u \in \mathrm{supp}(\zeta)\cap\mathrm{supp}(\varrho)$, this implies that $\zeta^p(u) \neq 0$, $\zeta^n(u) \neq 0$ and $\varrho^p(u) \neq 0$, $\varrho^n(u) \neq 0$ for all $u \in \mathfrak{S}$. Hence, $(\zeta^p \wedge \varrho^p)(u) \neq 0$ and $(\zeta^n \vee \varrho^n)(u) \neq 0$ for all $u \in \mathfrak{S}$. Therefore, $\zeta^p \wedge \varrho^p \neq 0$ and $\zeta^n \vee \varrho^n \neq 0$. We conclude that $\zeta = (\mathfrak{S};\zeta^p,\zeta^n)$ is a 0-essential BF ideal of $\mathfrak{S}$. □

Theorem 4.3. *Let $\zeta = (\mathfrak{S};\zeta^p,\zeta^n)$ be a 0-essential BF ideal of a semigroup $\mathfrak{S}$. If $\varrho = (\mathfrak{S};\varrho^p,\varrho^n)$ is a BF ideal of $\mathfrak{S}$ such that $\zeta^p \leq \varrho^p$ and $\zeta^n \geq \varrho^n$, then $\varrho = (\mathfrak{S};\varrho^p,\varrho^n)$ is also a 0-essential BF ideal of $\mathfrak{S}$.*

Proof. Let $\varrho = (\mathfrak{S};\varrho^p,\varrho^n)$ is a BF ideal of $\mathfrak{S}$ such that $\zeta^p \leq \varrho^p$ and $\zeta^n \geq \varrho^n$ and let $\xi = (\mathfrak{S};\xi^p,\xi^n)$ be any nonzero BF ideal of $\mathfrak{S}$. Since $\zeta = (\mathfrak{S};\zeta^p,\zeta^n)$ is a 0-essential BF ideal of $\mathfrak{S}$ we have $\zeta = (\mathfrak{S};\zeta^p,\zeta^n)$ is a BF ideal of $\mathfrak{S}$. Thus $\mathrm{supp}(\zeta\wedge\xi) \neq 0$. So $\varrho^p \wedge \xi^p \neq 0$ and $\varrho^n \vee \xi^n \neq 0$. Hence $\varrho = (\mathfrak{S};\varrho^p,\varrho^n)$ is a 0-essential BF ideal of $\mathfrak{S}$. □

Theorem 4.4. *Let $\zeta_1 = (\mathfrak{S};\zeta^p_1,\zeta^n_1)$ and $\zeta_2 = (\mathfrak{S};\zeta^p_2,\zeta^n_2)$ be 0-essential BF ideals of a semigroup $\mathfrak{S}$. Then $\zeta_1\cup\zeta_2$ and $\zeta_1\cap\zeta_2$ are 0-essential BF ideals of $\mathfrak{S}$.*

Proof. By Theorem 4.3, we have $\zeta_1\cup\zeta_2$ is a 0-essential BF ideal of $\mathfrak{S}$.

Since $\zeta_1 = (\mathfrak{S};\zeta^p_1,\zeta^n_1)$ and $\zeta_2 = (\mathfrak{S};\zeta^p_2,\zeta^n_2)$ are 0-essential BF ideals of $\mathfrak{S}$ we have $\zeta_1 = (\mathfrak{S};\zeta^p_1,\zeta^n_1)$ and $\zeta_2 = (\mathfrak{S};\zeta^p_2,\zeta^n_2)$ are BF ideals of $\mathfrak{S}$. Thus $\zeta_1\cap\zeta_2$ is a BF ideal of $\mathfrak{S}$. Let $\xi = (\mathfrak{S};\xi^p,\xi^n)$ be a nontrival BF ideal of $\mathfrak{S}$. Since $\zeta_1 = (\mathfrak{S};\zeta^p_1,\zeta^n_1)$ is a BF ideal of S we havesupp(ζ_1) is an ideal of $\mathfrak{S}$. Thus $\mathrm{supp}(\zeta_1\wedge\xi) \neq \{0\}$. Thus there exists $u \in \mathfrak{S}$ such that $(\zeta^p_1 \wedge \xi^p)(u) \neq 0$ and $(\zeta^n_1 \vee \xi^n)(u) \neq 0$. Since $\zeta_2 = (\mathfrak{S};\zeta^p_2,\zeta^n_2)$ is a 0-essential BF ideal of $\mathfrak{S}$ we have $\mathrm{supp}(\zeta_2)$ is a 0-essential BF ideal of $\mathfrak{S}$. Thus, $\mathrm{supp}(\zeta_2\wedge\xi) \neq \{0\}$. So, there exists a nonzero element $v \in \mathrm{supp}(\zeta_2\wedge\xi)(u)$ implies $\zeta^p_2(v) \neq 0$ and $\zeta^n_2(v) \neq 0$. Since $\zeta_1 = (\mathfrak{S};\zeta^p_1,\zeta^n_1)$ and $\xi = (\mathfrak{S};\xi^p,\xi^n)$ are BF ideals of $\mathfrak{S}$ we have $\zeta^p_1(v) \geq \zeta^p_1(u)$, $\xi^p(v) \geq \xi^p(u)$ and $\zeta^n_1(v) \leq \zeta^n_1(u)$, $\xi^n(v) \leq \xi^n(u)$. So $((\zeta^p_1 \wedge \zeta^p_2) \wedge \xi^p)(v) \neq 0$ and $((\zeta^n_1 \vee \zeta^n_2) \vee \xi^n)(v) \neq 0$. Thus, $\mathrm{supp}((\zeta_1\wedge\zeta_2)\wedge\xi) \neq \{0\}$. Therefore, $\zeta_1\cap\zeta_2$ is a 0-essential BF ideals of $\mathfrak{S}$. □

Definition 4.4. *[1] A* 0*-essential ideal* $\mathfrak{I}$ *of a semigroup with zero* $\mathfrak{S}$ *is called*

(1) *a* minimal *if for every* 0*-essential ideal of* $\mathfrak{J}$ *of* $\mathfrak{S}$ *such that* $\mathfrak{J} \subseteq \mathfrak{I}$*, we have* $\mathfrak{J} = \mathfrak{I}$,

(2) *a* prime *if* $uv \in \mathfrak{I}$ *implies* $u \in \mathfrak{I}$ *or* $v \in \mathfrak{I}$,

(3) *a* semiprime *if* $u^2 \in \mathfrak{I}$ *implies* $u \in \mathfrak{I}$*, for all* $u, v \in \mathfrak{S}$.

Example 4.2. *Let* $(\mathbb{Z}_{12}, +)$ *be a semigroup with zero. Then* $\{0, 2, 4, 6, 8, 10\}$ *is a minimal* 0*-essential ideal of* S*.*

Definition 4.5. *A* 0*-essential BF ideal* $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ *of a semigroup* $\mathfrak{S}$ *is called*

(1) *a* minimal *if for every* 0*-essential BF ideal of* $\varrho = (\mathfrak{S}; \varrho^p, \varrho^n)$ *of* $\mathfrak{S}$ *such that* $\zeta^p \leq \varrho^p$ *and* $\zeta^n \geq \varrho^n$*, we have* $\mathrm{supp}(\zeta) = \mathrm{supp}(\varrho)$,

(2) *a* prime *if* $\zeta^p(uv) \leq \zeta^p(u) \vee \zeta^p(v)$ *and* $\zeta^n(uv) \geq \zeta^n(u) \wedge \zeta^n(v)$,

(3) *a* semiprime *if* $\zeta^p(u^2) \leq \zeta^p(u)$ *and* $\zeta^n(u^2) \geq \zeta^n(u)$*, for all* $u, v \in \mathfrak{S}$.

Theorem 4.5. *Let* $\mathfrak{I}$ *be a non-empty subset of a semigroup* $\mathfrak{S}$*. Then the following statement holds.*

(1) $\mathfrak{I}$ *is a minimal* 0*-essential ideal of* $\mathfrak{S}$ *if and only if* $\lambda_{\mathfrak{I}} = (\mathfrak{S}; \lambda^p_{\mathfrak{I}}, \lambda^n_{\mathfrak{I}})$ *is a minimal* 0*-essential BF ideal of* $\mathfrak{S}$,

(2) $\mathfrak{I}$ *is a prime* 0*-essential ideal of* $\mathfrak{S}$ *if and only if* $\lambda_{\mathfrak{I}} = (\mathfrak{S}; \lambda^p_{\mathfrak{I}}, \lambda^n_{\mathfrak{I}})$ *is a prime* 0*-essential BF ideal of* $\mathfrak{S}$,

(3) $\mathfrak{I}$ *is a semiprime* 0*-essential ideal of* S *if and only if* $\lambda_{\mathfrak{I}} = (\mathfrak{S}; \lambda^p_{\mathfrak{I}}, \lambda^n_{\mathfrak{I}})$ *is a semiprime* 0*-essential BF ideal of* $\mathfrak{S}$.

Proof. (1) Suppose that $\mathfrak{I}$ is a minimal 0-essential ideal of $\mathfrak{S}$. Then $\mathfrak{I}$ is a 0-essential ideal of $\mathfrak{S}$. By Theorem 4.1, $\lambda_{\mathfrak{I}} = (\mathfrak{S}; \lambda^p_{\mathfrak{I}}, \lambda^n_{\mathfrak{I}})$ is a 0-essential BF ideal of $\mathfrak{S}$. Let $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ be a 0-essential BF ideal of $\mathfrak{S}$ such that $\zeta^p \leq \lambda^p_{\mathfrak{I}}$ and $\zeta^n \geq \lambda^n_{\mathfrak{I}}$. Then $\mathrm{supp}(\zeta) \subseteq \mathrm{supp}(\lambda_{\mathfrak{I}})$. Thus $\mathrm{supp}(\zeta) \subseteq \mathrm{supp}(\lambda_{\mathfrak{I}}) = \mathfrak{I}$. Thus $\mathrm{supp}(\zeta) \subseteq \mathfrak{I}$. Since $\zeta = (\mathfrak{S}; \zeta^p, \zeta^n)$ is a 0-essential BF ideal of $\mathfrak{S}$ we have $\mathrm{supp}(\zeta)$ is a 0-essential ideal of $\mathfrak{S}$. By assumption, $\mathrm{supp}(\zeta) = \mathfrak{I} = \mathrm{supp}(\lambda_{\mathfrak{I}})$. Hence, $\lambda_{\mathfrak{I}} = (\mathfrak{S}; \lambda^p_{\mathfrak{I}}, \lambda^n_{\mathfrak{I}})$ is a minimal 0-essential BF ideal of $\mathfrak{S}$.

Conversely, $\lambda_{\mathfrak{I}} = (\mathfrak{S}; \lambda^p_{\mathfrak{I}}, \lambda^n_{\mathfrak{I}})$ is a minimal 0-essential BF ideal of $\mathfrak{S}$ and let $\mathfrak{B}$ be a 0-essential ideal of $\mathfrak{S}$ such that $\mathfrak{B} \subseteq \mathfrak{I}$. Then $\mathfrak{B}$ is an ideal of $\mathfrak{S}$. Thus by Theorem 4.1, $\lambda_{\mathfrak{B}} = (\mathfrak{S}; \lambda^p_{\mathfrak{B}}, \lambda^n_{\mathfrak{B}})$ is an essential BF ideal of $\mathfrak{S}$ such that $\lambda^p_{\mathfrak{B}} \geq \lambda^p_{\mathfrak{I}}$ and $\lambda^n_{\mathfrak{B}} \leq \lambda^n_{\mathfrak{I}}$. So $\lambda_{\mathfrak{B}} = \lambda_{\mathfrak{I}}$. Hence $\mathfrak{B} = \mathrm{supp}(\lambda_{\mathfrak{B}}) = \mathrm{supp}(\lambda_{\mathfrak{I}}) = \mathfrak{I}$. Therefore $\mathfrak{I}$ is a minimal 0-essential ideal of $\mathfrak{S}$.

(2) Suppose that $\mathfrak{I}$ is a prime 0-essential ideal of $\mathfrak{S}$. Then $\mathfrak{I}$ is a 0-essential ideal of $\mathfrak{S}$. Thus by Theorem 4.1 $\lambda_{\mathfrak{I}} = (\mathfrak{S}; \lambda^p_{\mathfrak{I}}, \lambda^n_{\mathfrak{I}})$ is a 0-essential BF ideal of $\mathfrak{S}$. Let $u, v \in \mathfrak{S}$.

If $uv \in \mathfrak{I}$, then $u \in \mathfrak{I}$ or $v \in \mathfrak{I}$. Thus $\lambda^p_{\mathfrak{I}}(u) \vee \lambda^p_{\mathfrak{I}}(v) = 1 \geq \lambda^p_{\mathfrak{I}}(uv)$ and $\lambda^n_{\mathfrak{I}}(u) \wedge \lambda^n_{\mathfrak{I}}(v) = -1 \leq \lambda^n_{\mathfrak{I}}(uv)$.

If $uv \notin \mathfrak{I}$, then $\lambda^p_{\mathfrak{I}}(u) \vee \lambda^p_{\mathfrak{I}}(v) \geq \lambda^p_{\mathfrak{I}}(uv)$ and $\lambda^n_{\mathfrak{I}}(u) \wedge \lambda^n_{\mathfrak{I}}(v) \leq \lambda^n_{\mathfrak{I}}(uv)$.

Thus $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda^p_{\mathfrak{J}}, \lambda^n_{\mathfrak{J}})$ is a prime 0-essential BF ideal of $\mathfrak{S}$.

Conversely, suppose that $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda^p_{\mathfrak{J}}, \lambda^n_{\mathfrak{J}})$ is a prime 0-essential BF ideal of $\mathfrak{S}$. Then $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda^p_{\mathfrak{J}}, \lambda^n_{\mathfrak{J}})$ is a 0-essential BF ideal. Thus by Theorem 4.1, $\mathfrak{J}$ is a 0-essential ideal of $\mathfrak{S}$. Let $u, v \in \mathfrak{S}$. If $uv \in \mathfrak{J}$, then $\lambda^p_{\mathfrak{J}}(uv) = 1$ and $\lambda^n_{\mathfrak{J}}(uv) = -1$. By assumption $\lambda^p_{\mathfrak{J}}(uv) \leq \lambda^p_{\mathfrak{J}}(u) \vee \lambda^p_{\mathfrak{J}}(v)$ and $\lambda^n_{\mathfrak{J}}(uv) \geq \lambda^n_{\mathfrak{J}}(u) \wedge \lambda^n_{\mathfrak{J}}(v)$. Thus $\lambda^p_{\mathfrak{J}}(u) \vee \lambda^p_{\mathfrak{J}}(v) = 1$ and $\lambda^n_{\mathfrak{J}}(u) \wedge \lambda^n_{\mathfrak{J}}(v) = -1$ so $u \in \mathfrak{J}$ or $v \in \mathfrak{J}$. Hence $\mathfrak{J}$ is a prime 0-essential ideal of $\mathfrak{J}$.

(3) Suppose that $\mathfrak{J}$ is a semiprime 0-essential ideal of $\mathfrak{S}$. Then $\mathfrak{J}$ is a 0-essential ideal of $\mathfrak{S}$. Thus by Theorem 4.1, $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda^p_{\mathfrak{J}}, \lambda^n_{\mathfrak{J}})$ is a 0-essential BF ideal of S. Let $u \in \mathfrak{S}$.

If $u^2 \in \mathfrak{J}$, then $u \in \mathfrak{J}$ Thus $\lambda^p_{\mathfrak{J}}(u) = \lambda^p_{\mathfrak{J}}(u^2) = 1$ and $\lambda^n_{\mathfrak{J}}(u) = \lambda^n_{\mathfrak{J}}(u^2) = -1$. Hence, $\lambda^p_{\mathfrak{J}}(u^2) \leq \lambda^p_{\mathfrak{J}}(u)$ and $\lambda^n_{\mathfrak{J}}(u^2) \geq \lambda^n_{\mathfrak{J}}(u)$.

If $u^2 \notin \mathfrak{J}$, then $\lambda^p_{\mathfrak{J}}(u^2) = 0 \leq \lambda^p_{\mathfrak{J}}(u)$ and $\lambda^n_{\mathfrak{J}}(u^2) = 0 \geq \lambda^n_{\mathfrak{J}}(u)$.

Thus, $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda^p_{\mathfrak{J}}, \lambda^n_{\mathfrak{J}})$ is a semiprime 0-essential BF ideal of $\mathfrak{S}$.

Conversely, suppose that $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda^p_{\mathfrak{J}}, \lambda^n_{\mathfrak{J}})$ is a semiprime 0-essential BF ideal of $\mathfrak{S}$. Then $\lambda_{\mathfrak{J}} = (\mathfrak{S}; \lambda^p_{\mathfrak{J}}, \lambda^n_{\mathfrak{J}})$ is a 0-essential BF ideal. Thus by Theorem 4.1, $\mathfrak{J}$ is a 0-essential ideal of $\mathfrak{S}$. Let $u \in \mathfrak{S}$ with $u^2 \in \mathfrak{J}$ Then $\lambda^p_{\mathfrak{J}}(u^2) = 1$ and $\lambda^n_{\mathfrak{J}}(u^2) = -1$. By assumption, $\lambda^p_{\mathfrak{J}}(u^2) \leq \lambda^p_{\mathfrak{J}}(u)$ and $\lambda^n_{\mathfrak{J}}(u^2) \geq \lambda^n_{\mathfrak{J}}(u)$. Thus $\lambda^p_{\mathfrak{J}}(u) = 1$ and $\lambda^n_{\mathfrak{J}}(u) = -1$ so $u \in \mathfrak{J}$. Hence $\mathfrak{J}$ is a semiprime 0-essential ideal of $\mathfrak{S}$.

□

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