Int. J. Anal. Appl. (2023), 21:53

International Journal of Analysis and Applications

Doubt m-Polar Fuzzy Sets Based on BCK-Algebras

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Abstract. Doubt m-polar subalgebras (ideals) were introduced and some properties were investigated. Also, doubt m-polar positive implicative (commutative) ideals were defined and related results were proved.

1. **Introduction**

The main problem in fuzzy mathematics is how to carry out the ordinary concepts to the fuzzy case. The difficulty lies in how to pick out the rational generalization from the large number of available approaches. It is worth noting that fuzzy ideals are different from ordinary ideals in the sense that one cannot say which BCK-algebra element belongs to the fuzzy ideal under consideration and which one does not. The concept of fuzzy sets was introduced by Zadeh [1]. Since then these ideas have been applied to other algebraic structures such as semigroups, groups, rings, modules, vector spaces and topologies. In 1991, Xi [5] applied the concept of fuzzy sets to BCK-algebras which are introduced by Imai and K.Iseki [2]. In [10], A.AL-Masarwah and A. Ghafur Ahmad introduced the concept of Doubt Bipolar fuzzy subalgebra and ideals in BCK/BCI Algebra. In this paper we introduced the notion of Doubt m-polar fuzzy subalgebras and ideals of BCK -algebras.Moreover, we define the notion of doubt m-polar fuzzy positive implicative (commutative) ideal of BCK -algebras, and investigate some related properties. We show that in a positive implicative(commutative) BCK-algebra, a fuzzy subset is a doubt m-polar fuzzy ideal if and only if it is a doubt m-polar fuzzy positive implicative ideal. We show that m-polar fuzzy subset of a BCK-algebra is a doubt m-polar fuzzy positive implicative

Received: Mar. 27, 2022.

2010 *Mathematics Subject Classification.* 08A72.

Key words and phrases. BCK-algebras; fuzzy subalgebras; m-polar fuzzy subalgebras; m-polar fuzzy ideals; doubt m-polar fuzzy subalgebras; doubt m-polar fuzzy ideals.

https://doi.org/10.28924/2291-8639-21-2023-53
ISSN: 2291-8639

(commutative) ideal if and only if the doubt σ-level cut set of this m-polar fuzzy subset is an doubt m-polar fuzzy positive implicative (commutative) ideal.

2. **Preliminaries**

First, We recall some elementary aspects which are used to present the paper. Throughout this paper, X always denotes a BCK- algebra without any specifications, and for details about the theory of these algebras we may refer to [2–4, 7].

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a BCK-algebra if it satisfies the following axioms for all $x, y, z \in X$:

I: $((x * y) * (x * z)) * (z * y) = 0,$

II: $(x * (x * y)) * y = 0,$

III: $x * x = 0,$

IV: $x * y = 0$ and $y * x = 0$ imply $x = y$.

V: $0 * x = 0.$

A partial ordering $\leq$ on a BCK-algebra X can be defined by $x \leq y$ if and only if $x * y = 0$.
Any BCK-algebra X satisfies the following axioms for all $x, y, z \in X$:

(I1) $x * 0 = x,$

(I2) $(x * y) * z = (x * z) * y,$

(I3) $x * y \leq x,$

(I4) $(x * y) * z \leq (x * z) * (y * z),$

(I5) $x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x.$

A BCK-algebra X is said to be positive implicative if it satisfies the following equality:

$$(\forall x, y, z \in X)((x * z) * (y * z) = (x * y) * z).$$

A BCK-algebra X is said to be commutative if it satisfies the following equality:

$$(\forall x, y \in X)(x \wedge y = y \wedge x),$$

where $x \wedge y = y \wedge x$.

Definition 2.1 [5]A non-empty subset I of a BCK-algebra X is called a subalgebra of X if $x * y \in I$ for any $x, y \in I$.

Definition 2.2 [5]A non-empty subset S of a BCK-algebra X is called an ideal of X if

(S1) $0 \in S,$

(S2) $x * y \in S$ and $y \in S$,then $x \in S$ for all $x, y \in X$.

Definition 2.3 [7] A non-empty subset S of a BCK-algebra X is called a positive implicative ideal of X if if it satisfies (S1) and

(S3) $(x * y) * z \in S$ and $y * z \in S$,then $x * z \in S$ for all $x, y \in X$.

Definition 2.4 [6] A non-empty subset S of a BCK-algebra X is called a commutative ideal of X if if it satisfies (S1) and $(x * y) * z \in S$ and $z \in S$,then

$$x * (y \wedge x) \in S \text{ for all } x, y \in X.$$

Lemma 2.5 [7]An ideal S of a BCK-algebra X is commutative if and only if the following assertion is valid

$$(\forall x, y \in X)(x * y \in S \Rightarrow x * (y \wedge x) \in S).$$

Definition 2.6 [1]*A fuzzy* set in a BCK-algebra X is a function $\mu : X \to [0, 1]$.

Definition 2.7 [5]A fuzzy set μ in a BCK-algebra X is called a fuzzy subalgebra of X if $\mu(x * y) \geq min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

Definition 2.8 [5]A fuzzy set μ in a BCK-algebra X is called a fuzzy ideal of X if $\mu(0) \geq \mu(x)$ and $\mu(x) \geq min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$.

introduced the definition of a doubt fuzzy subalgebra and a doubt fuzzy ideal in BCK-algebras, which are as follows:

Definition 2.9 [8]A fuzzy set $A = \{(x, \mu_A(x))|x \in X\}$ in X is called a doubt fuzzy subalgebra of X if $\mu_A(x * y) \leq max\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in X$.

Definition 2.10 [8]A fuzzy set $A = \{(x, \mu(x))|x \in X\}$ in X is called a doubt fuzzy ideal of X if $\mu_A(0) \leq \mu_A(x)$ and $\mu_A(x) \leq max\{\mu_A(x * y), \mu_A(y)\}$ for all $x, y \in X$.

The proposed work is done on $m-$polar fuzzy sets.The formal definition of An m-polar fuzzy set is given below:

Definition 2.11 [9]An m-polar fuzzy set Q on a non- empty set X is a mapping $Q : X \to [0, 1]^m$. The membership value of every element $x \in X$ is denoted by

$$Q(x) = (p_1 \circ Q(x), p_2 \circ Q(x), ..., p_m \circ Q(x)).$$

where $p_i \circ Q : [0, 1]^m \to [0, 1]$ is defined the i-th projection mapping.

Note that $[0, 1]^m$(m-th$-$power of $[0, 1]$) is considered as a poset with the point wise order $\leq$, where m is an arbitrary ordinal number (we make an appointment that $m = \{n \mid n < m\}$ when $m > 0$), $\leq$ is defined by $x \leq y \Leftrightarrow p_i(x) \leq p_i(y)$ for each $i \in m$ ($x, y \in [0, 1]^m$), and $p_i : [0, 1]^m \to [0, 1]$ is the i-th projection mapping ($i \in m$).
It is easy to see that $0 = (0, 0, ..., 0)$is the smallest value in $[0, 1]^m$ and
$1 = (1, 1, ..., 1)$ is the largest value in $[0, 1]^m$.

Definition 2.12 [11]An m-polar fuzzy set Q in X is called An m-polar fuzzy subalgebra of X if it satisfies the following conditions for all $x, y \in X$:

$$Q(x * y) \geq \inf\{Q(x), Q(y)\}.$$

Thats is $(\forall\ x, y \in X)$ $(p_i \circ Q(x * y) \geq \inf\{p_i \circ Q(x), p_i \circ Q(y)\})$ for each $i = 1, 2, ..., m$.

Definition 2.13 [11]An m-polar fuzzy set Q is called An m-polar fuzzy ideal of X if it satisfies the following conditions for all $x, y \in X$:

(F1) $Q(0) \geq Q(x)$,

(F2) $Q(x) \geq \inf\{Q(x * y), Q(y)\}$.

Thats is $(\forall\ x, y \in X)$ $(p_i \circ Q(x) \geq \inf\{p_i \circ Q(x * y), p_i \circ Q(y)\})$ for each $i = 1, 2, ..., m$.

Definition 2.14 [12] An m-polar fuzzy set Q in X is called a m-polar fuzzy positive implicative ideal If it satisfies (F1) and

(F3) $Q(x * z) \geq \inf\{Q((x * y) * z), Q(y * z)\}$ for all $x, y, z \in X$.

Thats is $(\forall\ x, y \in X)$ $(p_i \circ Q(x * y) \geq \inf\{p_i \circ Q(x), p_i \circ Q(y)\})$ for each $i = 1, 2, ..., m$.

Definition 2.15 [11]An m-polar fuzzy set Q in X is called a m-polar fuzzy commutative ideal If it satisfies (F1) and

(F4) $Q(x * (y \wedge x)) \geq \inf\{Q((x * y) * z), Q(z)\}$ for all $x, y, z \in X$.

Thats is $(\forall\ x, y \in X)$ $(p_i \circ Q(x * (y \wedge x)) \geq \inf\{p_i \circ Q((x * y) * z), p_i \circ Q(z)\})$ for each $i = 1, 2, ..., m$.

3. Doubt m-polar fuzzy subalgebras

In this section, we introduce doubt m-polar fuzzy subalgebras in BCK-algebras and investigate some of their properties.

Definition 3.1 Let Q be An m-polar fuzzy subset of X, then Q is called a doubt m-polar fuzzy subalgebra of X if it satisfies the following conditions:

$$(\text{for all } x, y \in X)(Q(x * y) \leq \sup\{Q(x), Q(y)\}).$$

Thats is $(\forall\ x, y \in X)$ $(p_i \circ Q(x * y) \leq \sup\{p_i \circ Q(x), p_i \circ Q(y)\})$ for each $i = 1, 2, ..., m$.

Example 3.2 Consider a BCK-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Defined a 4-polar fuzzy set $Q : X \to [0,1]^4$ by:

$$Q(x) = \left\{ \begin{array}{l} (0.1, 0.3, 0.4, 0.5), \text{if } x = 0, \\ (0.2, 0.4, 0.6, 0.7), \text{if } x = a, \\ (0.3, 0.5, 0.7, 0.8), \text{if } x = b, \\ (0.4, 0.6, 0.8, 0.9), \text{if } x = c. \end{array} \right\}$$

By routine calculation, we know that Q is a doubt m-polar fuzzy subalgebra of X.
For any m-polar fuzzy set Q on X and $\sigma = (\sigma_1, \sigma_2, ..., \sigma_m) \in [0,1]^m$, the set

$$Q_{[\sigma]} = \{x \in X : Q(x) \leq \sigma\},$$

is called the doubt σ -level cut set of Q and the set

$$Q^s_{[\sigma]} = \{x \in X : Q(x) < \sigma\},$$

is called the doubt strong σ -level cut set of Q.

Theorem 3.3 Let Q be an m-polar fuzzy set over X and $\sigma = (\sigma_1, \sigma_2, ..., \sigma_m) \in [0,1]^m$. If Q is a doubt m-polar fuzzy subalgebra of X, then the nonempty doubt σ -level cut set of Q is a subalgebra of X.

Proof. Suppose that Q is doubt m-polar fuzzy subalgebra of X and $Q_{[\sigma]} \neq \phi$. For any $x, y \in Q_{[\sigma]}$ we have $Q(x) \leq \sigma$ and $Q(y) \leq \sigma$. It follows from *Definition (3.1)* that

$$Q(x * y) \leq sup\{Q(x), Q(y)\} \leq \sigma,$$

Therefore, $x * y \in Q_{[\sigma]}$. Hence, $Q_{[\sigma]}$is a subalgebra of X. □

Theorem 3.4 Let Q be an m-polar fuzzy set over X and let $\sigma = (\sigma_1, \sigma_2, ..., \sigma_m) \in [0,1]^m$. If Q is a doubt m-polar fuzzy subalgebra of X, then the nonempty doubt strong σ -level cut set of Q is a subalgebra of X.

Proof. Suppose that Q is doubt m-polar fuzzy subalgebra of X and $Q^s_{[\sigma]} \neq \phi$. For any $x, y \in Q^s_{[\sigma]}$ we have $Q(x) < \sigma$and $Q(y) < \sigma$. It follows from *Definition (3.1)* that

$$Q(x * y) \leq \sup\{Q(x), Q(y)\} < \sigma,$$

Therefore, $x * y \in Q^s_{[\sigma]}$. Hence, $Q^s_{[\sigma]}$ is a subalgebra of X. □

Theorem 3.5 Let Q be an m-polar fuzzy set over X and $Q_{[\sigma]} \neq \phi$ is subalgebra of X for all $\sigma = (\sigma_1, \sigma_2, ..., \sigma_m) \in [0,1]^m$. Then Q is a doubt m-polar fuzzy subalgebra of X.

Proof. Assume the contrary, that there exist $a, b \in X$ such that

$$Q(a * b) > sup\{Q(a), Q(b)\}$$

Thus, there is $\sigma = (\sigma_1, \sigma_2, ..., \sigma_m) \in [0, 1]^m$ such that

$$Q(a * b) > \sigma \geq sup\{Q(a), Q(b)\}.$$

So, one can conclude that $a, b \in Q_{[\sigma]}$ andt $a * b \notin Q_{[\sigma]}$. But this contradicts that $Q_{[\sigma]}$ is subalgebra of X. Therefore, $Q(x * y) \leq sup\{Q(x), Q(y)\}$ for all $x, y \in X$. Hence, Q is a doubt m-polar fuzzy subalgebra of X. □

Theorem 3.6 Let Q be an m-polar fuzzy set over X and $Q^s_{[\sigma]} \neq \phi$ be subalgebra of X for all $\sigma = (\sigma_1, \sigma_2, ..., \sigma_m) \in [0, 1]^m$. Then Q is a doubt m-polar fuzzy subalgebra of X.

Proof. Suppose that there exist $a, b \in X$ such that

$$Q(a * b) > sup\{Q(a), Q(b)\}.$$

So, there exists $\sigma = (\sigma_1, \sigma_2, ..., \sigma_m) \in [0, 1]^m$ such that

$$Q(a * b) > \sigma > sup\{Q(a), Q(b)\}.$$

Consequently, $a, b \in Q^s_{[\sigma]}$ and $a * b \notin Q^s_{[\sigma]}$. But This contradics that $Q^s_{[\sigma]}$ is subalgebra of X. Hence, Q is a doubt m-polar fuzzy subalgebra of X. □

Proposition 3.7 If Q is a doubt m-polar fuzzy subalgebra of X, then

$$Q(0) \leq Q(x) \text{for all } x \in X.$$

Proof. For any $x \in X$, we have $Q(0) = Q(x * x) \leq sup\{Q(x), Q(x)\} = Q(x)$ for all $x \in X$. This completes the proof. □

Proposition 3.8 If every doubt m-polar fuzzy subalgebra Q of X satisfies

$$Q(x * y) \leq Q((y)$$

for all $x, y \in X$, then Q is constant.

Proof. Note that in a BCK-algebra X, $x * 0 = x$ for all $x \in X$, since $Q(x * y) \leq Q((y)$, we have$Q(x) = Q(x * 0) \leq Q(0)$, It follows from *Proposition (3.7)* that $Q(x) = Q(0)$ for all $x, y \in X$. Therefore, Q is constant. □

For elements x and y of a BCK-algebra X, let us write $x * y^n$ for $(...((x * y) * y) * ...) * y$ and $x^n * y$ for $x * (... * ((x * (x * y))...)$ where y and x occur n times respectively.

Proposition 3.9 Let Q be a doubt m-polar fuzzy subalgebra of X and $n \in N$. Then for any $x \in X$, we have

(1) $Q(x^n * x) \leq Q(x)$, if n is odd.
(2) $Q(x^n * x) = Q(x)$, if n is even.

Proof. 1. If n is odd, then $n = 2k - 1$ for some positive integer k. Let $x \in X$, then $Q(x * x) = Q(0) \leq Q(x)$. Now assume that $Q(x^{2k-1} * x) \leq Q(x)$ for some positive integer k. Then,

$$\begin{aligned} Q(x^{2(k+1)-1} * x) &= Q(x^{2k+1} * x) \\ &= Q(x^{2k-1} * (x * (x * x))) \\ &= Q(x^{2k-1} * (x * 0)) \\ &= Q(x^{2k-1} * x) \\ &\leq Q(x) \end{aligned}$$

This proves (1). Similarly, we can prove (2). □

4. **Doubt m-polar fuzzy ideals.**

In this section, we introduce the notions of doubt m-polar fuzzy ideals in BCK-algebras. Several fundamental properties and theorems related to these concepts are also studied and investigated.

Definition 4.1 An m-polar fuzzy set Q in X is called a doubt m-polar fuzzy ideal if it satisfies the following conditions for all $x, y \in X$:

(1) $Q(0) \leq Q(x)$,
(2) $Q(x) \leq \sup\{Q(x * y), Q(y)\}$.

Thats is $(\forall\, x, y \in X)$ $(p_i \circ Q(x) \leq \sup\{p_i \circ Q(x * y), p_i \circ Q(y)\})$ for each $i = 1, 2, ..., m$.

Example 4.2 Consider a BCK-algebra $X = \{0, a, b, c\}$ which is given in *Example (3.2)* defined a 4-polar fuzzy set $Q : X \to [0, 1]^4$ by:

$$Q(x) = \left\{ \begin{array}{c} (0.1, 0.2, 0.3, 0.4), \text{if } x = 0 \\ (0.3, 0.5, 0.6, 0.8), \text{if } x = a, b \\ (0.5, 0.6, 0.7, 0.8), \text{if } x = c \end{array} \right\}.$$

By routine calculation, we know that Q is a doubt m-polar fuzzy ideal of X.

Proposition 4.3 Let Q be a doubt m-polar fuzzy ideal of X. If $\leq$ is a partial ordering on X, then $Q(x) \leq Q(y)$for all $x, y \in X$such that $x \leq y$.

Proof. Assume that $\leq$ is a partial ordering on X defined by $x \leq y$ if and only if $x * y = 0$ for all $x, y \in X$. Then

$$\begin{aligned} Q(x) &\leq sup\{Q(x * y), Q(y)\} \\ &= sup\{Q(0), Q(y) \\ &= Q(y). \end{aligned}$$

This completes the proof. □

Proposition 4.4 Let Q be an m-polar fuzzy ideal of X. If Xsatisfies the following assertion:

$$(\forall x, y, z \in X)(x * y \leq z),$$

then $Q(x) \leq \sup\{Q(y), Q(z)\}$ for all $x, y, z \in X$.

Proof. Assume that $x * y \leq z$ that valid in X. Then

$$Q(x * y) \leq \sup\{Q((x * y) * z), Q(z\} = \sup\{Q(0), Q(z)\} = Q(z),$$

for all $x, y, z \in X$. It follows that

$$Q(x) \leq \sup\{Q(x * y), Q(y) \leq \sup\{Q(y), Q(z)\},$$

for all $x, y, z \in X$. This completes the proof. □

Proposition 4.5 Let Q be a doubt m-polar fuzzy ideal of X. Then

$$Q(x * y) \leq Q((x * y) * y) \Leftrightarrow Q((x * z) * (y * z)) \leq Q((x * y) * z),$$

for all $x, y, z \in X$.

Proof. Note that

$$\begin{aligned} ((x * (y * z)) * z) * z &= ((x * z) * (y * z)) * z \\ &\leq (x * y) * z \end{aligned}$$

for all $x, y, z \in X$. Assume that $Q(x * y) \leq Q((x * y) * y)$ for all $x, y, z \in X$. It follows from (I2) and *Proposition (4.3)* that

$$\begin{aligned} Q((x * z) * (y * z)) &= Q((x * (y * z)) * z) \\ &\leq Q(((x * (y * z)) * z) * z) \\ &\leq Q((x * y) * z), \end{aligned}$$

for all $x, y, z \in X$.

Conversely, suppose that

$$Q((x * z) * (y * z)) \leq Q((x * y) * z), \tag{4.1}$$

for all $x, y, z \in X$. If we substitute z for y in Equations (4.1). Then

$$\begin{aligned} Q(x*z) &= Q((x*z)*0) \\ &= Q((x*z)*(z*z)) \\ &\leq Q((x*z)*z) \end{aligned}$$

for all $x, z \in X$ by using (III) and (I1) □

Proposition 4.6 Let Q be a doubt m-polar fuzzy ideal of X. Then

$$Q(x*y) \leq \sup\{Q(x*z), Q(z*y)\},$$

for all $x, y, z \in X$.

Proof. Note that $((x*y)*(x*z)) \leq (z*y)$ for all $x, y, x \in X$. It follows from *Proposition (4.3)*, that

$$Q((x*y)*(x*z)) \leq Q(z*y).$$

Now, by *Definition (4.1)*, we have

$$\begin{aligned} Q(x*y) &\leq \sup\{Q((x*y)*(x*z)), Q(x*z)\} \\ &\leq \sup\{Q(x*z), Q(z*y)\}, \end{aligned}$$

for all $x, y, z \in X$. This completes the proof. □

Proposition 4.7 Let Q be a doubt m-polar fuzzy ideal of X. Then

$$Q(x*(x*y)) \leq Q(y),$$

for all $x, y \in X$.

Proof. Let Q be a doubt m-polar fuzzy ideal of X. Then for all $x, y \in X$, we have

$$\begin{aligned} Q(x*(x*y)) &\leq \sup\{Q((x*(x*y))*y), Q(y)\} \\ &= \sup\{Q((x*y)*(x*y)), Q(y)\} \\ &= \sup\{Q(0), Q(y)\} \\ &= Q(y). \end{aligned}$$

This completes the proof. □

Theorem 4.8 Let Q be a m-polar fuzzy set over X and let $\sigma \in [0,1]^m$. If Q is a doubt m-polar fuzzy ideal of X, then the nonempty doubt σ -level cut set of Q is an ideal of X.

Proof. Assume that $Q_{[\sigma]} \neq \phi$ for $\sigma \in [0,1]^m$. Clearly, $0 \in Q_{[\sigma]}$. Let $x * y \in Q_{[\sigma]}$ and $y \in Q_{[\sigma]}$. Then $Q(x*y) \leq \sigma$ and $Q(y) \leq \sigma$. It follows from *Definition (4.1)* that

$$Q(x) \leq \sup\{Q(x*y), Q(y)\} \leq \sigma.$$

So, $x \in Q_{[\sigma]}$. Therefore $Q_{[\sigma]}$ is an ideal of X. □

Theorem 4.9 Let Q be An m-polar fuzzy set over X and let $\sigma \in [0,1]^m$. If Q is a doubt m-polar fuzzy ideal of X, then the nonempty doubt strong σ -level cut set of Q is an ideal of X.

Proof. Assume that $Q^s_{[\sigma]} \neq \phi$ for $\sigma \in [0,1]^m$. Clearly, $0 \in Q^s_{[\sigma]}$. Let $x * y \in Q^s_{[\sigma]}$ and $y \in Q^s_{[\sigma]}$. Then $Q(x*y) < \sigma$ and $Q(y) < \sigma$. It follows from *Definition (4.1)* that

$$Q(x) \leq \sup\{Q(x*y), Q(y)\} < \sigma.$$

So, $x \in Q^s_{[\sigma]}$. Therefore $Q^s_{[\sigma]}$ is an ideal of X. □

Theorem 4.10 Let Q be a m-polar fuzzy set over X and assume that $Q_{[\sigma]} \neq \phi$ is an ideal of X for all $\sigma \in [0,1]^m$. Then Q is a doubt m-polar fuzzy ideal of X.

Proof. Assume that $Q_{[\sigma]} \neq \phi$ is an ideal of X for all $\sigma \in [0,1]^m$. If there exist $h \in X$ such that $Q(0) > Q(h)$ then $Q(0) > \sigma_h \geq Q(h)$, for some $\sigma_h \in [0,1]^m$. Then $0 \notin Q_{[\sigma_h]}$.Which is contradiction. Hence $Q(0) \leq Q(h)$, for all $x \in X$. Now, assume that there exist $h, q \in X$ such that $Q(h) > \sup\{Q(h*q), Q(q)\}$. Then there exist $\beta \in [0,1]^m$ such that $Q(h) > \beta \geq \sup\{Q(h*q), Q(q)\}$. It follow that $h*q \in Q_{[\beta]}$ and $q \in Q_{[\beta]}$, but $h \notin Q_{[\beta]}$. This is impossible, and so $Q(x) \leq \sup\{Q(x*y), Q(y)\}$, for all $x, y \in X$. Therefore, Q is a doubt m-polar fuzzy ideal of X. □

Theorem 4.11 Let Q be An m-polar fuzzy set over X and assume that $Q^s_{[\sigma]} \neq \phi$ is an ideal of X for all $\sigma \in [0,1]^m$. Then Q is a doubt m-polar fuzzy ideal of X.

Proof. Assume that $Q^s_{[\sigma]} \neq \phi$ is an ideal of X for all $\sigma \in [0,1]^m$. If there exist $h \in X$ such that $Q(0) > Q(h)$ then $Q(0) > \sigma_h > Q(h)$ for some $\sigma_h \in [0,1]^m$. Then $0 \notin Q^s_{[\sigma_h]}$, which is a contradiction. Hence $Q(0) \leq Q(h)$, for all $x \in X$. Now, assume that there exist $h, q \in X$ such that $Q(h) > \sup\{Q(h*q), Q(q)\}$. Then there exist $\beta \in [0,1]^m$ such that $Q(h) > \beta > \sup\{Q(h*q), Q(q)\}$. It follow that $h*q \in Q_{\beta}$ and $q \in Q^s_{[\beta]}$, but $h \notin Q^s_{[\beta]}$. This is impossible, and so $Q(x) \leq \sup\{Q(x*y), Q(y)\}$, for all $x, y \in X$. Therefore, Q is a doubt m-polar fuzzy ideal of X. □

Proposition 4.12 Let Q be a doubt m-polar fuzzy ideal of X. If the inequality $x*y \leq z$ holds in X, then $Q(x) \leq \sup\{Q(y), Q(z)\}$, for all $x, y, z \in X$.

Proof. Let Q be a doubt m-polar fuzzy ideal of X and let $x, y, z \in X$ be such that $x * y \leq z$. Then $(x * y) * z = 0$, and so

$$\begin{aligned} Q(x) &\leq \sup\{Q(x * y), Q(y)\} \\ &\leq \sup\{sup\{Q((x * y) * z), Q(z)\}, Q(y)\} \\ &= \sup\{sup\{Q(0), Q(z)\}, Q(y)\} \\ &= \sup\{Q(y), Q(z)\}. \end{aligned}$$

This completes the proof. □

Theorem 4.13 In a BCK-algebra X, every doubt m-polar fuzzy ideal of X is a doubt m-polar fuzzy subalgebra of X.

Proof. Let Q be a doubt m-polar fuzzy ideal of a BCK- algebra X. For any $x, y \in X$, we have

$$\begin{aligned} Q(x * y) &\leq \sup\{Q((x * y) * x), Q(x)\} \\ &= \sup\{Q((x * x) * y), Q(x)\} \\ &= \sup\{Q(0 * y), Q(x)\} \\ &= \sup\{Q(0), Q(x)\} \\ &\leq \sup\{Q(x), Q(y)\}. \end{aligned}$$

Hence, Q is a doubt m-polar fuzzy subalgebra of a BCK-algebra X. □

Example 4.14 In *Example (4.2)*, Q is a doubt m-polar fuzzy ideal of X, so that Q is a doubt m-polar fuzzy subalgebra of X.

The converse of *Theorem (4.13)* is not true in general as seen in the following example.

Example 4.15 The doubt m-polar fuzzy subalgebra Q in *Example (3.2)* is not a doubt m-polar fuzzy ideal of X, since

$$Q(b) = (0.3, 0.5, 0.7, 0.8) \nleq (0.2, 0.4, 0.6, 0.7) = \sup\{Q(b * a), Q(a)\}.$$

We give a condition for a doubt m-polar fuzzy subalgebra to be a doubt m-polar fuzzy ideal in a BCK-algebra.

Theorem 4.16 Let Q be a doubt m-polar fuzzy subalgebra of X. If the inequality $x * y \leq z$ holds in X, then Q is a doubt m-polar fuzzy ideal of X.

Proof. Let Q be a doubt m-polar fuzzy subalgebra of X. Then from *Proposition (4.5)*, $Q(0) \leq Q(x)$ for all $x \in X$. As $x * y \leq z$ holds in X, then from *Proposition (4.12)*, we get $Q(x) \leq \sup\{Q(y), Q(z)\}$ for all $x, y, z \in X$. Since $x * (x * y) \leq y$ for all $x, y \in X$, then $Q(x) \leq \sup\{Q(x * y), Q(y)\}$. Hence, Q is a doubt m-polar fuzzy ideal of X. □

Definition 4.17 Let $(X, *, 0)$and $(X^{'}, *^{'}, 0^{'})$ be two BCK-algebras, a homomorphism is a map $f : X \to X^{'}$satisfying $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

Definition 4.18 Let $f : X \to X^{'}$ be a homomorphism of BCK-algebras and let Q be a m-polar fuzzy set in $X^{'}$, then the m-polar fuzzy set Q_f in X define by $Q_f = Q \circ f$ $(i.e., Q_f(x) = Q(f(x)$ for all $x \in X)$ is called the preimage of Q under f.

Theorem 4.19 An onto homomorphic preimage of a doubt m-polar fuzzy ideal is a doubt m-polar fuzzy ideal.

Proof. Let $f : X \to X^{'}$ be an onto homomorphism of BCK-algebras, Q be a doubt m-polar fuzzy ideal in $X^{'}$, and Q_f be preimage of Q under f. For any $x^{'} \in X^{'}$ there exist $x \in X$ such that $f(x) = x^{'}$. We have

$$Q_f(0) = Q(f(0)) = Q(0^{'}) \leq Q(x^{'}) = Q(f(x)) = Q_f(x).$$

Let $x \in X$ and $y^{'} \in X^{'}$, then there exist $y \in X$ such that $f(y) = y^{'}$. We have

$$\begin{aligned} Q_f(x) &= Q(f(x)) \\ &\leq \sup\{Q(f(x) * y^{'}), Q(y^{'})\} \\ &= \sup\{Q(f(x) * f(y)), Q(f(y))\} \\ &= \sup\{Q(f(x * y)), Q(f(y))\} \\ &= \sup\{Q_f((x * y)), Q_f((y))\}. \end{aligned}$$

Hence, Q_f is a doubt m-polar fuzzy ideal of X. □

Proposition 4.20 Let Q be a doubt m-polar fuzzy ideal of X. Then the sets

$$J = \{x \in X : Q(x) = Q\,(0)\}$$

is an ideal of X.

Proof. Obviously, $0 \in J$. Hence, $J \neq \phi$. Now, let $x, y \in J$ such that $x * y, y \in J$. Then $Q(x * y) = Q(0) = Q(y)$.Now, $Q(x) \leq sup\{Q(x * y), Q(y)\} = Q(0)$ since Q be a doubt m-polar fuzzy ideal of X,$Q(0) \leq Q(x)$. Therefore, $Q(0) = Q(x)$. It follows that $x \in J$, for all $x, y \in X$. Therefore, J is an ideal of X. □

For any elements $\omega \in X$, we consider the sets:

$$X_\omega = \{x \in X \mid Q(x) \leq Q(\omega)\}.$$

Clearly, $\omega \in X_\omega$. So that X_ω is a nonempty set of X.

Theorem 4.21 Let ω be any element of X. If Q is a doubt m-polar fuzzy ideal of X, then X_ω is an ideal of X.

Proof. Clearly, $0 \in X_\omega$. Let $x, y \in X$ be such that $x * y \in X_\omega$ and $y \in X_\omega$. Then $Q(x * y) \leq Q(\omega), Q(y) \leq Q(\omega)$ It follows that from *Definition (4.1)*, that $Q(x) \leq sup\{Q(x * y), Q(y)\} \leq Q(\omega)$ Hence,$x \in X_\omega$. Therefore X_ω is an ideal of X. □

Theorem 4.22 Let $\omega \in X$ and let Q be An m-polar fuzzy set over X. Then If X_ω is an ideal of X, then the following assertion is valid for all $x, y, z \in X$

(A1) $Q(x) \geq \sup\{Q(y * z), Q(z)\} \Rightarrow Q(x) \geq Q(y)$,

If Q satisfies (A1) and

(A2) $Q(0) \leq Q(x)$

for all $x \in X$. Then X_ω is ideal for all $\omega \in Im(Q)$.

Proof. (1) Assume that X_ω is ideal of X for $\omega \in X$. Let $x, y, z \in X$ be such that $Q(x) \geq \sup\{Q(y * z), Q(z)\}$. Then $y * z \in X_\omega$ and $z \in X_\omega$ where $\omega = x$. Since X_ω is an ideal of X, it follows that $y \in X_\omega$ for $\omega = x$.Hence, $Q(y) \leq Q(\omega) = Q(x)$. (2) Let $\omega \in Im(Q)$ and suppose that Q satisfies (A1) and (A2).Clearly, $0 \in X_\omega$by (A2). Let $x, y \in X$ be such that $x * y \in X_\omega$ and $y \in X_\omega$. Then $Q(x * y) \leq Q(\omega)$ and $Q(y) \leq Q(\omega)$, which implies that $\sup\{Q(x * y), Q(y)\} \leq Q(\omega)$. It follows from *(A1)* that $Q(\omega) \geq Q(x)$. Thus, $x \in X_\omega$, and therefore X_ω is ideal of X. □

5. **Doubt m-polar fuzzy positive implicative ideals.**

In this section, we introduce doubt m-polar fuzzy positive Implicative ideals in BCK- algebra, several fundamental properties and theorems related to this concept are and theorems related to this concept are also studied and investigated.

Definition 5.1 An m-polar fuzzy set Q in X is called a doubt m-polar fuzzy positive implicative ideal If it satisfies the following conditions for all $x, y, z \in X$:

(1) $Q(0) \leq Q(x)$.
(2) $Q(x * z) \leq \sup\{Q((x * y) * z), Q(y * z)\}$.

Thats is $(\forall\, x, y \in X)$ $(p_i \circ Q(x * z) \leq \sup\{p_i \circ Q((x * y) * z), p_i \circ Q(y * z)\})$ for each $i = 1, 2, ..., m$.

Example 5.2 Consider a BCK- algebra $x = \{0, 1, 2, 3, 4\}$ with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	2
3	3	3	3	0	3
4	4	4	4	4	0

Defined a 4-polar fuzzy set $Q : X \to [0, 1]^4$by:

$$Q(x) = \left\{ \begin{array}{l} (0.3, 0.2, 0.1, 0.3), \text{if } x = 0, \\ (0.6, 0.3, 0.8, 0.6), \text{if } x = 1, \\ (0.5, 0.2, 0.6, 0.5), \text{if } x = 2, \\ (0.4, 0.2, 0.3, 0.4), \text{if } x = 3, \\ (0.7, 0.4, 0.9, 0.7), \text{if } x = 4. \end{array} \right\}$$

By routine calculation, we know that Q is a doubt m-polar fuzzy positive Implicative Ideal of X.

Theorem 5.3 Any doubt m-polar fuzzy positive implicative ideal of X is a doubt m- polar fuzzy ideal of X.

Proof. Let Q be a doubt m- polar fuzzy positive implicative ideal of X. Then $Q(0) \leq Q(x)$. By taking $z = 0$ in *Definition(5.1)* we have ,

$$Q(x) \leq sup\{Q(x * y), Q(y)\}.$$

Hence, Q is a doubt m-polar fuzzy ideal of X. □

The converse of *Theorem (5.3)* is not true in general as seen in the following example.

Example 5.4 Consider a BCK- algebra $x = \{0, f, j, l\}$ with the following Cayley table:

$*$	0	f	j	1
0	0	0	0	0
f	f	0	0	f
j	j	f	0	j
1	1	1	1	0

Defined a 4-polar fuzzy set $Q : X \to [0, 1]^3$ by:

$$Q(x) = \left\{ \begin{array}{c} (0.3, 0.3, 0.3), \text{if } x = 0, \\ (0.5, 0.5, 0.8), \text{if } x = f, j, \\ (0.3, 0.3, 0.3), \text{if } x = l. \end{array} \right\}$$

By routine calculation, we know that Q is a doubt m-polar fuzzy ideal of X. But it is not a doubt m-polar fuzzy positive Implicative Ideal of X. Since

$$\begin{aligned} Q(j * f) &= Q(f) \\ &= (0.5, 0.5, 0.8) \\ &\nleq sup(Q((j * f) * f), Q(f * f)\} \\ &= \sup\{Q(0), Q(0)\} \\ &= Q(0) \\ &= (0.3, 0.3, 0.3). \end{aligned}$$

We now give the condition for a doubt m-polar fuzzy ideal to be a doubt m-polar fuzzy positive implicative ideal of X.

Theorem 5.5 An m-polar fuzzy set of X is a doubt m-polar fuzzy positive implicative ideal of X if and only if it is a doubt m- polar fuzzy ideal of X and the following condition is valid for all $x, y \in X$.

$$Q(x * y) \leq Q((x * y) * y). \tag{5.1}$$

Proof. Suppose Q is a doubt m-polar fuzzy positive implicative ideal of X. By *Theorem (5.3)*, Q is a doubt m- polar fuzzy ideal of X. If z is replaced by y in *Definition (5.1)* , then

$$\begin{aligned} Q(x * y) &\leq \sup\{Q(x * y) * y), Q(y * y)\} \\ &= \sup\{Q((x * y) * y), Q(0)\} \\ &= Q((x * y) * y). \end{aligned}$$

For all $x, y \in X$.

Conversely, let Q be a doubt m-polar fuzzy ideal of X. Then, $Q(0) \leq Q(x)$ for all $x \in X$. Also, since

$$((x * z) * z) * (y * z) \leq (x * z) * y = (x * y) * z.$$

For all $x, y \in X$, it follow by *Proposition (5.3)* that

$$Q(((x * z) * z) * (y * z)) \leq Q((x * y) * z).$$

Now, by (5.1)

$$\begin{aligned} Q(x * z) &\leq Q((x * z) * z) \\ &\leq \sup\{Q(((x * z) * z) * (y * z)), Q((y * z)\} \\ &\leq \sup\{Q((x * y) * z), Q(y * z)\}. \end{aligned}$$

Hence, Q is a doubt m-polar fuzzy positive implicative ideal of X. □

Theorem 5.6 In positive implicative BCK-algebra X, every doubt m-polar fuzzy ideal is a doubt m-polar fuzzy positive implicative ideal.

Proof. Let Q be a doubt m-polar fuzzy ideal of a positive implicative BCK-algebra X, we have

$$\begin{aligned}((x*z)*((x*y)*z))*(y*z) &= ((x*z*)*(y*z))*((x*y)*z)\\ &= ((x*y)*z)*((x*y)*z)\\ &= 0.\end{aligned}$$

And so,

$$((x*z)*((x*y)*z))*(y*z)=0, \text{i.e.}, ((x*z)*((x*y)*z))\leq(y*z),$$

for all $x,y,z\in X$. Since Q is a doubt m-polar fuzzy ideal, it follow from *Proposition (4.4)* that $Q(x*z)\leq\sup\{Q((x*y)*z),Q(y*z)\}$. Hence, Q is a doubt m-polar fuzzy positive implicative ideal of X. □

Theorem 5.7 Let Q be An m-polar fuzzy set of a BCK- algebra X. Then Q is a doubt m-polar fuzzy positive implicative ideal of X if and only if it satisfies

$$(\forall\sigma\in[0,1]^m)(Q_{[\sigma]}\neq\phi\Rightarrow Q_{[\sigma]}\text{is a positive implicative ideal of } X \text{ for all } \sigma\in[0,1]^m).$$

Proof. Since Q is a doubt m-polar fuzzy positive implicative ideal of X, then Q is a doubt m- polar fuzzy ideal of X and so every σ-level cut set $Q_{[\sigma]}$ of Q is an ideal of X. Let $x,y,z\in X$ be such that $(x*y)*z\in Q_{[\sigma]}$ and $(y*z)\in Q_{[\sigma]}$. Then $Q((x*y)*z)\leq\sigma$ and $Q(y*z)\leq\sigma$. It follow that :

$$Q(x*z)\leq\sup\{Q((x*y)*z),Q(y*z)\}\leq\sigma.$$

So that $(x*z)\in Q_{[\sigma]}$. Hence $Q_{[\sigma]}$ is a positive implicative ideal of X.

Conversely, assume that $Q_{[\sigma]}\neq\phi$ is a positive implicative ideal of X for all $\sigma\in[0,1]^m$. If there exist $h\in X$ such that $Q(0)>Q(h)$, then $Q(0)>\sigma_h\geq Q(h)$ for some $\sigma_h\in[0,1]^m$. Then $0\notin Q_{[\sigma_h]}$, which is contradiction. Hence $Q(0)\leq Q(h)$, for all $x\in X$. Now, assume that there exist $h,k,q\in X$ such that

$$Q(h*q)>\sup\{Q((h*k)*q),Q(k*q)\}.$$

Then there exists $\beta\in[0,1]^m$ such that

$$Q(h*q)>\beta\geq\sup\{Q((h*k)*q),Q(k*q)\}.$$

It follow that $(h*k)*q\in Q_{[\beta]}$ and $k*q\in Q_{[\beta]}$, but $h*q\notin Q_{[\beta]}$. This is impossible, and so $Q(x*z)\leq\sup\{Q((x*y)*z),Q(y*z)\}$ for all $x,y,z\in X$. Therefore, Q is a doubt m-polar fuzzy positive implicative ideal of X. □

Corollary 5.8 If Q is a doubt m-polar fuzzy positive implicative ideal of a BCK-algebra X, then $Q^s_{[\sigma]} \neq \phi$ is a positive implicative ideal of x for all $\sigma \in [0,1]^m$.

Proof. straightforward. □

Theorem 5.9 Let ω be an element of a BCK-algebra X. If Q is a doubt m-polar fuzzy positive implicative ideal of X, then X_ω is a positive implicative ideal of X.

Proof. Let Q is a doubt m-polar fuzzy positive implicative ideal of BCK-algebra X, then it is a doubt m- polar fuzzy ideal of X and so X_ω is an ideal. Thus $0 \in X_\omega$. Now, assume that $(x * y) * z \in X_\omega$ and $y * z \in X_\omega$ for any $x, y, z \in X$. Then

$$Q(((x * y) * z) \leq Q(\omega) \text{ and } Q(y * z) \leq Q(\omega).$$

It follow from *Definition (5.1)* that

$$Q(x * z) \leq \sup\{Q((x * y) * z, Q(y * z)\} \leq Q(\omega).$$

Thus $Q(x*z) \leq Q(\omega)$. Hence , $x*z \in X_\omega$ and therefore X_ω and therefore X_ω is a positive implicative ideal of X. □

6. **Doubt m-polar fuzzy commutative ideals.**

In this section, we introduce doubt m-polar fuzzy commutative ideals in BCK- algebra, several fundamental properties and theorems related to this concept are and theorems related to this concept are also studied and investigated.

Definition 6.1 An m-polar fuzzy set Q in X is called a doubt m-polar fuzzy Commutative ideal If it satisfies the following conditions for all $x, y, z \in X$:

(1) $Q(0) \leq Q(x)$.

(2)$Q(x * (y \wedge x)) \leq \sup\{Q((x * y) * z), Q(z)\}$.

That is, $(\forall \ x, y \ \in \ X)$ $(p_i \circ Q(x * (y \wedge x)) \leq \sup\{p_i \circ Q((x * y) * z), p_i \circ Q(z)\})$ for each $i = 1, 2, ..., m$.

Example 6.2 Consider a BCK- algebra $x = \{0, a, b, c\}$ which is given in *Example (3.2)* define an m-polar fuzzy set $Q : X \to [0,1]^m$ by:

$$Q(x) = \left\{ \begin{array}{c} \alpha = (\alpha_1, \alpha_2, \ldots\ldots, \alpha_m), \text{if } x = 0, \\ \beta = (\beta_1, \beta_2, \ldots\ldots, \beta_m), \text{if } x = a, \\ \gamma = (\gamma_1, \gamma_2, \ldots\ldots, \gamma_m), \text{if } x = b, c. \end{array} \right\}$$

where $\alpha, \beta, \gamma \in [0,1]^m$ and $\alpha < \beta < \gamma$.

By routine calculation, we know that Q is a doubt m-polar fuzzy commutative ideal of X.

Theorem 6.3 Every doubt m-polar fuzzy commutative ideal of BCK-algebra X is a doubt m- polar fuzzy ideal of X.

Proof. Let Q be a doubt m- polar fuzzy commutative ideal of BCK-algebra X. Then $Q(0) \leq Q(x)$. Now, for any $x, y, z \in X$, we have

$$\begin{aligned} Q(x) &= Q(x * (0 \wedge x)) \\ &\leq \sup \{Q(x * 0) * z), Q(z)\} \\ &= \sup\{Q(x * z), Q(z)\}. \end{aligned}$$

Hence, Q is a doubt m-polar fuzzy commutative ideal of X. □

The converse of *Theorem (6.3)* is not true in general as seen in the following example.

Example 6.4 Consider a BCK- algebra $x = \{0, 1, 2, 3, 4\}$ with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

Defined an m-polar fuzzy set $Q : X \to [0, 1]^m$ by:

$$Q(x) = \left\{ \begin{array}{c} \alpha = (\alpha_1, \alpha_2, \ldots\ldots, \alpha_m), \text{if } x = 0, \\ \beta = (\beta_1, \beta_2, \ldots\ldots, \beta_m), \text{if } x = 1, \\ \gamma = (\gamma_1, \gamma_2, \ldots\ldots, \gamma_m), \text{if } x = 2, 3, 4. \end{array} \right\}$$

Where $\alpha, \beta, \gamma \in [0, 1]^m$ and $\alpha < \beta < \gamma$.

By routine calculation, we know that Q is a doubt m-polar fuzzy ideal of X.But it is not a doubt m-polar fuzzy commutative ideal of X. Since:

$$Q(2 * (3 \wedge 2)) \nleq \sup \{Q(2 * 3) * 0), Q(0)\}.$$

Now we give the condition for a doubt m-polar fuzzy ideal to be a doubt m-polar fuzzy commutative ideal of X.

Theorem 6.5 Let Q be a doubt m-polar fuzzy ideal of a BCK-algebra X. Then Q is a doubt m-polar fuzzy commutative ideal of X if and only if the following condition is valid for all $x, y \in X$

$$Q(x * (y \wedge x)) \leq Q(x * y). \tag{6.1}$$

Proof. Assume that Q is a doubt m-polar fuzzy Commutative ideal of a BCK-algebra X. We have:

$$Q(x * (y \wedge x)) \leq \sup\{Q(x * y) * z), Q(z)\}.$$

By taking $z = 0$; then we get Q

$$(x * (y \wedge x)) \leq Q(x * y).$$

Conversely, suppose that a doubt m-polar fuzzy ideal Q of a BCK- algebra X satisfies the condition (6.1). Then

$$Q(x * y) \leq \sup\left\{Q\left((x * y) * z\right), Q\left(z\right)\right\}. \tag{6.2}$$

For all $x, y \in X$. Using (6.1) and (6.2), we have

$$Q(x * (y \wedge x)) \leq \sup\{Q(x * y) * z), Q(z)\}.$$

Therefore, Q is a doubt m-polar fuzzy commutative ideal of X. □

Theorem 6.6 In commutative BCK-algebra X, every doubt m-polar fuzzy ideal is a doubt m- polar fuzzy commutative ideal.

Proof. Let Q be a doubt m-polar fuzzy ideal of a commutative BCK-algebra X, we have

$$\begin{aligned} ((x * (y \wedge x)) * ((x * y) * z)) * z &= ((x * (y \wedge x)) * z) * ((x * y) * z) \\ &\leq (x * (y \wedge x)) * (x * y) \\ &= (x \wedge y) * (y \wedge x) \\ &= 0. \end{aligned}$$

And so $((x * (y \wedge x)) * ((x * y) * z)) * z = 0$, i.e.,$((x * (y \wedge x)) * ((x * y) * z)) \leq z$ for all $x, y, z \in X$. Since Q is a doubt m-polar fuzzy ideal, it follow from *Proposition (4.4)*

$$Q(x * (y \wedge x)) \leq \sup\{Q((x * y) * z), Q(z)\}.$$

Hence, Q is a doubt m-polar fuzzy commutative ideal of X. □

Theorem 6.7 Let Q be an m-polar fuzzy set of a BCK- algebra X. Then Q is a doubt m-polar fuzzy commutative ideal of X if and only if it satisfies

$$(\forall \sigma \in [0,1]^m)(Q_{[\sigma]} \neq \phi \Rightarrow Q_{[\sigma]} \text{ is a commutative ideal of } X \text{ for all } \sigma \in [0,1]^m). \tag{6.3}$$

Proof. Let Q is a doubt m-polar fuzzy commutative ideal of X. Then Q is a doubt m-polar fuzzy ideal of X and so every σ-level cut set $Q_{[\sigma]}$ of Q is an ideal of X. Let $x, y, z \in X$ be such that $(x * y) * z \in Q_{[\sigma]}$ and $z \in Q_{[\sigma]}$. Then $Q((x * y) * z) \leq \sigma$ and $Q(z) \leq \sigma$. It follow that :

$$Q(x * (y \wedge x)) \leq \sup\{Q((x * y) * z), Q(z)\} \leq \sigma.$$

So that $x * (y \wedge x) \in Q_{[\sigma]}$. Hence $Q_{[\sigma]}$ is a commutative ideal of X.

Conversely, suppose that (6.3) is valid, $Q(0) \leq Q(h)$ for all $x \in X$. Let $Q((x * y) * z) = \alpha = (\alpha_1, \alpha_2, ..., \alpha_m)$ and $Q(z) = \beta = (\beta_1, \beta_2, ..., \beta_m)$ for all $x, y, z \in X$. Then $(x * y) * z \in Q_{[\beta]}$ and $z \in Q_{[\beta]}$. Without loss of generality, we may assume that $\beta \leq \alpha$. Then $Q_{[\alpha]} \subseteq Q_{[\sigma]}$, and so $z \in Q_{[\sigma]}$. Since $Q_{[\sigma]}$ is a commutative ideal of by hypothesis, we obtain that $(x * (y \wedge x)) \in Q_{[\alpha]}$, and so $Q(x * (y \wedge x)) \leq \alpha = \sup\{Q((x * y) * z), Q(z)\}$. Therefore, Q is a doubt m-polar fuzzy commutative ideal of X. □

Corollary 6.8 If Q is a doubt m-polar fuzzy commutative ideal of a BCK-algebra X, then $Q^s_{[\sigma]} \neq \phi$ is a commutative ideal of X for all $\sigma \in [0, 1]^m$.

Proof. straightforward. □

Theorem 6.9 Let ω be an element of a BCK-algebra X. If Q is a doubt m-polar fuzzy commutative ideal of X, then X_ω is a commutative ideal of X.

Proof. If Q is a doubt m-polar fuzzy commutative ideal of BCK-algebra X, then it is a doubt m- polar fuzzy ideal of X and so X_ω is an ideal. Thus $0 \in X_\omega$.Now Let $x * y \in X_\omega$ for any $x, y \in X$. Then $Q(x * y) \leq Q(\omega)$. It follows from *Theorem (6.5)* that

$$Q(x * (y \wedge x)) \leq Q(x * y) \leq Q(\omega).$$

Thus, $x * (y \wedge x) \in X_\omega$ and therefore $X\ _\omega$. Hence, X_ω is a commutative ideal of X by *Lemma (2.5)*. □

7. Conclusions

We discussed the notion of doubt m-polar fuzzy subalgebras and ideals of BCK-algebras.Also, we introduced the notion of Doubt m-polar fuzzy positive implicative (commutative)ideal of BCK -algebras. Our defintions probably can be applied in other kinds of doubt m-polar fuzzy ideals of BCK-algebras.

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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