Int. J. Anal. Appl. (2022), 20:26

International Journal of Analysis and Applications

Fractional Order Riemann Curvature Tensor in Differential Geometry

Wedad Saleh*

Department of Mathematics, Taibah University, Al- Medina, Saudi Arabia

**Corresponding author:* wed_10_777@hotmail.com

Abstract. This study discussed some interesting aspects and features of fractional curvature in the differential manifold. In particular, Riemannian fractional curvature tensor, Livi-Civita fractional connection and Bianchi fractional identity are presented.

1. Introduction

In mathematics, several special functions appear in many applications such as the Gamma function that plays some significant roles in the theory of integral differential equations in particular fractional calculus. Thus, we begin with some definitions, for the details we refer to ([1], [15], [8]).

The Gamma function of a positive integer η is again a positive integer, while the gamma function $\Gamma(-\eta)$ of a negative integer changes to infinities. The Gamma function any positive η value is defined as follows:

$$\Gamma(\eta)=\int_0^\infty t^{\eta-1}e^{-t}dt.$$

The Gamma function $\Gamma(\eta)$ is considered as a generalization of the factorial and $\Gamma(\eta)$ is defined for $\eta>0$ by the integral

$$\Gamma(\eta)=\int_0^\infty t^{\eta-1}e^{-t}\,dt.$$

In the classical sense since $\Gamma(0)=\dfrac{\Gamma(1)}{0}$, then it follows that $\Gamma(\eta)$ is not defined for integers $\eta\leq 0$. However, the extension formula gives finite values for $\Gamma(\eta)$, for $\Re(\eta)\leq 0$ since $\Gamma(\eta)$ is analytic

Received: Mar. 7, 2022.

2010 *Mathematics Subject Classification.* 26A33, 58A05, 58D17.

Key words and phrases. fractional geometrical object; fractional manifold; revised Riemann-Liouville fractional calculus on manifolds.

https://doi.org/10.28924/2291-8639-20-2022-26
ISSN: 2291-8639

everywhere except at $\eta = 0, -1, -2, ...$, and the residue at $\eta = k$ is given by

$$\mathrm{Res}_{\eta=k}\Gamma(\eta) = \frac{(-1)^k}{k!}.$$

Now, if $\eta > 0$, then

$$\Gamma(\eta+1) = \eta\Gamma(\eta). \tag{1.1}$$

Equation (1.1) can be used to define $\Gamma(\eta)$ for $\eta < 0$ and $\eta \neq -1, -2, \ldots$ and further, this is one of the most important formulas that were satisfied by the Gamma function.

Even though the Gamma function is defined as a locally summable function on the real line by [17]

$$\Gamma(\eta) = \int_0^\infty t^{\eta-1} e^{-t} dt, \qquad \eta > 0. \tag{1.2}$$

In the classical sense, $\Gamma(\eta)$ function was not defined for the negative integer thus, there was an open problem to give a satisfactory definition. However, by using the neutral limit, it has been shown in [21] that the Gamma function (1.2) is defined as follows:

$$\Gamma(\eta) = N - \lim_{\varepsilon \to 0} \int_\varepsilon^\infty t^{\eta-1} e^{-t} dt$$

for $\eta \neq 0, -1, -2, ...$, and this function can be defined by neutral limit such as

$$\begin{aligned} \Gamma(-n) &= N - \lim_{\varepsilon \to 0} \int_\varepsilon^\infty t^{-n-1} e^{-t} dt \\ &= \int_1^\infty t^{-n-1} e^{-t} dt \\ &\quad + \int_0^1 t^{-n-1} \left[e^{-t} - \sum_{i=0}^{n} \frac{(-1)^i}{i!} t^i \right] dt - \sum_{i=0}^{n-1} \frac{(-1)^i}{i!(n-i)}, n \in \mathbb{N}. \end{aligned}$$

It was also proven in [20] the existence of r the derivative of the Gamma function and defined it by equation

$$\begin{aligned} \Gamma^{(r)}(0) &= N - \lim_{\varepsilon \to 0} \int_\varepsilon^\infty t^{-1} \ln^r t e^{-t} dt \\ &= \int_1^\infty t^{-1} \ln^r t e^{-t} dt + \int_0^1 t^{-1} \ln^r t \left[e^{-t} - 1 \right] dt \\ \Gamma^{(r)}(-n) &= N - \lim_{\varepsilon \to 0} \int_\varepsilon^\infty t^{-n-1} \ln^r t e^{-t} dt \\ &= \int_1^\infty t^{-n-1} \ln^r t e^{-t} dt \\ &\quad + \int_0^1 t^{-n-1} \ln^r t \left[e^{-t} - \sum_{i=0}^{n} \frac{(-1)^i}{i!} t^i \right] dt \\ &\quad - \sum_{i=0}^{n-1} \frac{(-1)^i}{i!} r!(n-i)^{-r-1} \end{aligned}$$

for $r \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Also,

$$\Gamma(-r) = \frac{(-1)^r}{r!}\psi(r) - \frac{(-1)^r}{r!}\gamma$$

for $r = 1, 2, \dots,$ where

$$\psi(r) = \sum_{i=1}^{r} \frac{1}{i}.$$

Thus, the definition can be extended to the whole real line where,

$$\Gamma(0) = \Gamma^{'}(1) = -\gamma,$$

where γ denotes Euler's constant, see [22].

For a function $f : V \subset \mathbb{R} \to \mathbb{R}$ with $0 \in V$, the fractional derivative of order α is defined by:

$$\frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(-\alpha)} \int_0^t \frac{f(s) - f(0)}{(t-s)^{1+\alpha}} ds, \;\; \alpha < 0 \tag{1.3}$$

$$\frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s) - f(0)}{(t-s)^{\alpha-n+1}} ds, \;\; \alpha > 0 \tag{1.4}$$

where n is the first integer greater than or equal to α.
The relation (1.3) gives a fractional integral and (1.4) gives a fractional derivative.

We express some of the operators of fractional derivatives, see for example, [4, 7, 9, 10, 12, 16].

(1) $\frac{d^\alpha}{dt^\alpha} t^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)} t^{\gamma-\alpha}$, $\alpha \in \mathbb{R}$ or $(\alpha \in \mathbb{C})$ and $1+\gamma \neq 0, -1, ..., -n$,
(2) $\frac{d^n}{dt^n} \frac{d^\alpha}{dt^\alpha} f(t) = \frac{d^{n+\alpha}}{dt^{n+\alpha}} f(t), \;\; n \in \mathbb{N}$,
(3) $\frac{d^\alpha}{dt^\alpha}(f_1(t) + f_2(t)) = \frac{d^\alpha}{dt^\alpha} f_1(t) + \frac{d^\alpha}{dt^\alpha} f_2(t)$,
(4) $\frac{d^\alpha}{dt^\alpha}(Cf(t)) = C\frac{d^\alpha}{dt^\alpha} f(t)$, where C is a constant,
(5) $\frac{d^\alpha}{dt^\alpha} f(\beta t) = \beta^\alpha \frac{d^\alpha}{[d(\beta t)]^\alpha} f(\beta t)$.

It is well known that fractional calculus is an essential and advantageous branch of mathematics, having a broad range of applications at almost every department of sciences. Techniques of fractional calculus have been employed in the modeling of many different phenomena in engineering, physics , and mathematics.The problem in fractional calculus is not only essential but also quite challenging ,which usually involves complicated mathematical solution techniques. However, a general solution theory for almost every issue in this area has yet to be established. Each application has developed its approaches and implementations. Consequently, a single standard method for the problems in fractional calculus has not emerged yet. Therefore, funding reliable and efficient solution techniques along with fast implementation methods are significantly essential and still active research areas.

Further, it is also realized that the operators of fractional integration and derivation have physical and geometric interpretations, which streamline along with their utilization for related issues in various fields of science(see [2], [8], [10], [11], [12], [14], [18], [19]). Moreover, the fractional differential calculus on a differential manifold is studied in([2], [3], [4], [6], [13]). Even though fractional calculus is a handy and important topic, however, the research on geometric interpretation and applications are limited ,and not many in current literature. Thus, in this study, we focus on the Riemannian curvature tensor, Livi-Civita connection and Bianchi's identity on fractional differentiable manifolds and discuss some related properties. We also give some examples.

2. Fractional Differential Calculus on Manifolds

Assume that N be an m-dimensional differential manifold (V, x_i) a local coordinate system on N and $V_0 = \{x \in V : 0 \leq x_i \leq b_i, i = 1, 2, ..., m\}$ [5].

For a function $f : V_0 \to \mathbb{R}$, the fractional derivative with respect to x_i:

$$\partial_i^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\partial_{x_i}^n \int_0^{x_i} \frac{f(x_1, ..., x_{i-1}, s, x_{i+1}, ..., x_m) - f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_m)}{(x_i - s)^{\alpha-n+1}} ds,$$

where $\partial_{x_i}^n = \frac{\partial}{\partial x_i} \circ \frac{\partial}{\partial x_i} \circ \ldots \circ \frac{\partial}{\partial x_i}$ (n times, i is fixed, $\alpha \geq 0$).

For $\alpha \in (0, 1), \gamma > -1$,

$$\partial_i^\alpha (x_i)^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} \quad ; \partial_i^\alpha = \delta_i^j.$$

A fractional vector field $V \subset N$ is an object of the form $X^\alpha = X_i^\alpha \partial_i^\alpha$, where $X_i^\alpha \in \Im_V(N) \quad i = 1, ..., m$.

The fractional vector fields on V and χ_V^α is generated by the operators $\partial_i^\alpha, i = 1, 2, ..., m$ are denoted by χ_V^α . If $c : x = x(t), t \in I$ is a parameterized curve in U then the fractional tangent vector field of c is given by

$$x^\alpha(t) = \frac{1}{\Gamma(1+\alpha)} \partial_t^\alpha x_i(t) \partial_i^\alpha.$$

A fractional covariant derivative is given by

$$\nabla_{X^\alpha}^\alpha Y^\alpha = X_i^\alpha(\partial_i^\alpha Y_j^\alpha + \tilde{\Gamma}_{ik}^j Y_k^\alpha)\partial_j^\alpha$$

where $X^\alpha, Y^\alpha \in \chi_U^\alpha$ and $\tilde{\Gamma}_{ik}^j$ the functions defining the coefficients of a fractional linear connection on N. They are determined by the relations

$$\nabla_{\partial_i^\alpha}^\alpha \partial_k^\alpha = \tilde{\Gamma}_{ik}^j \partial_j^\alpha.$$

Since it is essential to study fractional vector fields on a differentiable manifold N. For $\mathbb{R}^n$, there is an obvious way to do this. Recall that $\chi^\alpha(\mathbb{R}^n)$ denotes the space of fractional differentiable vector fields defined on $\mathbb{R}$. Examples are the fractional vector fields $\frac{\partial^\alpha}{\partial u_1^\alpha}, ..., \frac{\partial^\alpha}{\partial u_n^\alpha}$ determined by the natural coordinate functions $u_1, ..., u_n$.

Definition 2.1. *Fractional Riemannian metric F on m-dimensional manifold N defines for every point $p \in N$, the scalar product of fractional tangent vectors in the fractional tangent space $T_p^\alpha N$ depending on the point p.*

Let $A^\alpha = A_i^\alpha \partial_i^\alpha$ and $B^\alpha = B_j^\alpha \partial_j^\alpha$ any two fractional vectors tangent to the manifold N at the point p with coordinates $x = (x_1, ..., x_m)$ $(A^\alpha, B^\alpha \in T_p^\alpha N)$ the scalar product is equal to

$$\begin{aligned}\langle A^\alpha, B^\alpha\rangle_F \mid_p &= A_i^\alpha(x)\tilde{g}_{ij}(x)B_j^\alpha(x)\\ &= (A_1^\alpha, ..., A_n^\alpha)\begin{pmatrix}\tilde{g}_{11} & \dots & \tilde{g}_{1n}\\ . & \dots & .\\ . & \dots & .\\ . & \dots & .\\ \tilde{g}_{n1} & \dots & \tilde{g}_{nn}\end{pmatrix}\begin{pmatrix}B_1^\alpha\\ .\\ .\\ .\\ B_n^\alpha\end{pmatrix}\end{aligned}$$

where

(1) $F(A^\alpha, B^\alpha) = F(B^\alpha, A^\alpha)$, i.e., $\tilde{g}_{ij} = \tilde{g}_{ji}$ (symmetricity condition).
(2) $F(A^\alpha, A^\alpha) > 0$ if $A^\alpha \neq 0$, i.e. $\tilde{g}_{ij}u_i^\alpha u_j^\alpha \geq 0$, $\tilde{g}_{ij}u_i^\alpha u_j^\alpha = 0$ iff $u_1^\alpha = ... = u_n^\alpha = 0$ (positive definiteness).
(3) $F(A^\alpha, B^\alpha) \mid_{p=x}$, i.e. $\tilde{g}_{ij}(x)$ are smooth function where $0 < \alpha < 1$.

Components of tensor field F in coordinate system are matrix valued functions $\tilde{g}_{ij}(x)$

$$F = \tilde{g}_{ij}(x)d^\alpha x_i \otimes d^\alpha x_j.$$

Rule of Transformation for Entries of the Matrix $\tilde{g}_{ij}(x)$

$\tilde{g}_{ij}(x)$- entries of the matrix $\parallel \tilde{g}_{ij} \parallel$ are components of tensor field F in a given coordinate system. How do these components transform under transformation of coordinates $\{x_i\} \to \{x_{i'}\}$?

$$\begin{aligned}F &= \tilde{g}_{ij}dx_i^\alpha \otimes dx_j^\alpha\\ &= \tilde{g}_{ij}\left(\frac{\partial x_i^\alpha}{\partial x_{i'}^\alpha}dx_{i'}^\alpha\right) \otimes \left(\frac{\partial x_j^\alpha}{\partial x_{j'}^\alpha}dx_{j'}^\alpha\right)\\ &= \frac{\partial x_i^\alpha}{\partial x_{i'}^\alpha}\tilde{g}_{ij}\frac{\partial x_j^\alpha}{\partial x_{j'}^\alpha}dx_{i'}^\alpha \otimes dx_{j'}^\alpha\\ &= \tilde{g}_{i'j'}dx_{i'}^\alpha \otimes dx_{j'}^\alpha.\end{aligned}$$

Hence,

$$\tilde{g}_{i'j'} = \frac{\partial x_i^\alpha}{\partial x_{i'}^\alpha}\tilde{g}_{ij}\frac{\partial x_j^\alpha}{\partial x_{j'}^\alpha}.$$

Example 2.1. Consider $\mathbb{R}^2$ with fractional polar coordinates in the domain $y > 0$, $X = (r^\alpha cos^\alpha\varphi, r^\alpha sin^\alpha\varphi)$, then

$$\frac{\partial^\alpha X}{\partial r^\alpha} = (\alpha!\cos^\alpha\varphi, \alpha!\sin^\alpha\varphi).$$

$$\frac{\partial^\alpha X}{\partial \varphi^\alpha} = \left(\alpha!e^{i\alpha\pi}r^\alpha\sin^\alpha\varphi, \alpha!r^\alpha\cos^\alpha\varphi\right).$$

$$\tilde{g}_{ij} = \begin{pmatrix}(\alpha!)^2\left[\cos^{2\alpha}\varphi + \sin^{2\alpha}\varphi\right] & (\alpha!)^2r^\alpha\left[e^{2\alpha\pi} + 1\right]\sin^\alpha\varphi\cos^\alpha\varphi\\ (\alpha!)^2r^\alpha\left[e^{\alpha\pi i} + 1\right]\sin^\alpha\varphi\cos^\alpha\varphi & (\alpha!)^2r^{2\alpha}\left[e^{2\alpha\pi i}\sin^{2\alpha}\varphi + \cos^{2\alpha}\varphi\right]\end{pmatrix}$$

We have that

$$\begin{aligned}F = (\alpha!)^2\left[\cos^{2\alpha}\varphi + \sin^{2\alpha}\varphi\right](dr^\alpha)^2\\ +2(\alpha!)^2r^\alpha\left[\left[e^{i\alpha\pi} + 1\right]\sin^\alpha\varphi\cos^\alpha\varphi\right]dr^\alpha d\varphi^\alpha\\ +(\alpha!)^2r^{2\alpha}\left[e^{2i\alpha\pi}\sin^{2\alpha}\varphi + \cos^{2\alpha}\varphi\right](d\varphi^\alpha)^2\}.\end{aligned}$$

Notice that , as expected, when $\alpha = 1$, one recovers the classical formula.

$$F = (dr)^2 + r^2 (d\varphi)^2 .$$

$\alpha =$	0.1	0.2
$\tilde{g}_{rr}$	$0.905[C^{0.2} + S^{0.2}]$	$0.843[C^{0.4} + S^{0.4}]$
$\tilde{g}_{r\varphi} = \tilde{g}_{\varphi r}$	$0.905r^{0.1}[i+1]S^{0.2}C^{0.2}$	0
$\tilde{g}\varphi\varphi$	$0.905r^{0.2}[-S^{0.2} - C^{0.2}]$	$0.843r^{0.4}[-S^{0.4} + C^{0.4}]$
F	$0.905[C^{0.2} + S^{0.2}]dr^{0.2}$ $+1.8101r^{0.1}[i+1]S^{0.2}C^{0.2}(dr)^{.1}(d\varphi)^{.1}$ $+0.905r^{0.2}[-S^{0.2} - C^{0.2}](d\varphi)^{.2}$	$0.843[C^{0.4} + S^{0.4}]dr^{0.4}$ $+$ $0.843r^{0.4}[-S^{0.4} + C^{0.4}](d\varphi)^{0.4}$

Table 1. $C = \cos\varphi, S = \sin\varphi$

$\alpha =$	0.3	0.4
$\tilde{g}_{rr}$	$0.805[C^{0.6} + S^{0.6}]$	$0.787[C^{0.8} + S^{0.8}]$
$\tilde{g}_{r\varphi} = \tilde{g}_{\varphi r}$	$0.805r^{0.3}[[i+1]S^{0.3}C^{0.3}]$	$1.574r^{0.4}S^{0.3}C^{0.3}$
$\tilde{g}\varphi\varphi$	$0.805r^{0.6}[-S^{0.6} + C^{0.6}]$	$0.787r^{0.8}[S^{0.8} + C^{0.8}]$
F	$0.805[C^{0.6} + S^{0.6}](dr)^{0.6}$ $+1.61r^{0.3}[[i+1]S^{0.3}C^{0.3}](dr)^{.3}(d\varphi)^{.3}$ $+0.805r^{0.6}[-S^{0.6} + C^{0.6}](d\varphi)^{.6}$	$0.787[C^{0.8} + S^{0.8}](dr)^{0.8}$ $+3.148r^{0.4}S^{0.4}C^{0.4}dr^{.4}d\varphi^{.4}$ $+0.787r^{0.8}[S^{0.8} + C^{0.8}](d\varphi)^{.8}$

Table 2. $C = \cos\varphi, S = \sin\varphi$

$\alpha =$	0.5	0.6
$\tilde{g}_{rr}$	$0.785[C + S]$	$0.798[C^{1.2} + S^{1.2}]$
$\tilde{g}_{r\varphi} = \tilde{g}_{\varphi r}$	$0.785r^{0.5}[i+1]S^{0.5}C^{0.5}$	0
$\tilde{g}\varphi\varphi$	$0.785r[-S + C]$	$0.798r^{1.2}[S^{1.2} + C^{1.2}]$
F	$0.785[C + S]r(dr)$ $+1.57r^{0.5}[i+1]S^{0.5}C^{0.5}(dr)^{0.5}(d\varphi)^{0.5}$ $0.785r[-S + C]d\varphi$	$0.798[C^{1.2} + S^{1.2}](dr)^{1.2}$ $+$ $+0.798r^{1.2}[S^{1.2} + C^{1.2}](d\varphi)^{1.2}$

Table 3. $C = \cos\varphi, S = \sin\varphi$

$\alpha =$	0.7	0.8
$\tilde{g}_{rr}$	$0.826[C^{1.4}+S^{1.4}]$	$0.867[C^{1.6}+S^{1.6}]$
$\tilde{g}_{r\varphi}=\tilde{g}_{\varphi r}$	$0.826r^{0.7}[i+1]S^{0.7}C^{0.7}$	$1.734r^{0.8}S^{0.8}C^{0.8}$
$\tilde{g}\varphi\varphi$	$0.826r^{1.4}[-S^{1.4}+C^{1.4}]$	$0.867r^{1.6}[S^{1.6}+C^{1.6}]$
F	$0.826[C^{1.4}+S^{1.4}](dr)^{1.4}$ $+1.652r^{0.7}[i+1]S^{0.7}C^{0.7}(dr)^{0.7}(d\varphi)^{0.7}$ $+0.826r^{1.4}[-S^{1.4}+C^{1.4}](d\varphi)^{1.4}$	$0.867[C^{1.6}+S^{1.6}](dr)^{1.6}$ $+3.468r^{0.8}S^{0.8}C^{0.8}dr^{0.8}d\varphi^{0.8}$ $+0.867r^{1.6}[S^{1.6}+C^{1.6}](d\varphi)^{1.6}$

Table 4. $C=\cos\varphi, S=\sin\varphi$

$\alpha =$	0.9	1
$\tilde{g}_{rr}$	$0.925[C^{1.8}+S^{1.8}]$	1
$\tilde{g}_{r\varphi}=\tilde{g}_{\varphi r}$	$0.925r^{0.9}[i+1]S^{0.9}C^{0.9}$	0
$\tilde{g}\varphi\varphi$	$0.925r^{1.8}[-S^{1.8}+C^{1.8}]$	r^2
F	$0.925[C^{1.8}+S^{1.8}](dr)^{1.8}$ $+1.85r^{0.9}[i+1]S^{0.9}C^{0.9}(dr)^{0.9}(d\varphi)^{0.9}$ $+0.925r^{1.8}[-S^{1.8}+C^{1.8}](d\varphi)^{1.8}$	$(dr)^2$ $+$ $r^2\,(d\varphi)^2$

Table 5. $C=\cos\varphi, S=\sin\varphi$

Remark 2.1. *Let N is an m-dimensional Riemannian manifold with fractional metric tensor $\tilde{g}$,then we shall denote the fractional derivatives of the elements of tensor $\tilde{g}$ as follows:*

$$\tilde{g}_{ij,k}=\frac{\partial^{\alpha}}{\partial x_k^{\alpha}}\tilde{g}_{ij},$$

and

$$\begin{aligned}\tilde{g}_{ij,kl} &= \frac{\partial^{\alpha}}{\partial x_l^{\alpha}}\frac{\partial^{\alpha}}{\partial x_k^{\alpha}}\tilde{g}_{ij}\\ &= \frac{\partial^{2\alpha}}{\partial x_l^{\alpha}\partial x_k^{\alpha}}\tilde{g}_{ij},\ i,j,k,l=1,...,n.\end{aligned}$$

Definition 2.2. *Asymmetric fractional connection is called Levi-Civita fractional connection if it is compatible with metric, i.e., if it preserves the scalar product:*

$$\partial^{\alpha}_{X^{\alpha}}\langle Y^{\alpha},Z^{\alpha}\rangle=\langle\nabla^{\alpha}_{X^{\alpha}}Y^{\alpha},Z^{\alpha}\rangle+\langle Y^{\alpha},\nabla^{\alpha}_{X^{\alpha}}Z^{\alpha}\rangle$$

for arbitrary fractional vector fields X^{α}, Y^{α}, and Z^{α}.

In local coordinates Christoffel symbols of Levi-Civita fractional connection are given by:

$$\tilde{\Gamma}^k_{ij}=\frac{1}{2}\tilde{g}^{kl}(\partial^{\alpha}_j\tilde{g}_{il}+\partial^{\alpha}_i\tilde{g}_{lj}-\partial^{\alpha}_l\tilde{g}_{ij}).$$

Proof. Since

$$\partial^{\alpha}_j e_i=\tilde{\Gamma}^m_{ij}e_m \tag{2.1}$$

$$\tilde{\Gamma}^m_{ij}e_me_l=(\partial^{\alpha}_j e_i)e_l \tag{2.2}$$

$$\begin{aligned}\tilde{\Gamma}^m_{ij} g_{ml} &= \partial^\alpha_j (e_i . e_l) - e_i(\partial^\alpha_j e_l) \\ &= \partial^\alpha_j g_{il} - \tilde{\Gamma}^m_{lj} e_m e_i \\ &= \partial^\alpha_j g_{il} - \tilde{\Gamma}^m_{lj} g_{mi}, \end{aligned} \tag{2.3}$$

then

$$\tilde{\Gamma}^m_{ij} g_{ml} + \tilde{\Gamma}^m_{lj} g_{mi} = \partial^\alpha_j g_{il}, \tag{2.4}$$

which implies that

$$\tilde{\Gamma}^m_{ij} \tilde{g}_{ml} + \tilde{\Gamma}^m_{lj} \tilde{g}_{mi} = \partial^\alpha_j \tilde{g}_{il}. \tag{2.5}$$

In this equation, the index m is a dummy, so only the indices i,j ,and l are specified. We can cyclically permute these indices to generate two more equations:

$$\tilde{\Gamma}^m_{jl} \tilde{g}_{mi} + \tilde{\Gamma}^m_{il} \tilde{g}_{mj} = \partial^\alpha_l \tilde{g}_{ji} \tag{2.6}$$

$$\tilde{\Gamma}^m_{li} \tilde{g}_{mj} + \tilde{\Gamma}^m_{ji} \tilde{g}_{ml} = \partial^\alpha_i \tilde{g}_{lj} \tag{2.7}$$

since $\tilde{\Gamma}^m_{ij} = \tilde{\Gamma}^m_{ji}$, then

$$\tilde{\Gamma}^m_{lj} \tilde{g}_{mi} + \tilde{\Gamma}^m_{il} \tilde{g}_{mj} = \partial^\alpha_l \tilde{g}_{ij} \tag{2.8}$$

$$\tilde{\Gamma}^m_{il} \tilde{g}_{mj} + \tilde{\Gamma}^m_{ij} \tilde{g}_{ml} = \partial^\alpha_i \tilde{g}_{lj}. \tag{2.9}$$

We can now add (2.5)to (2.9) and subtract (2.8)to get

$$2\tilde{\Gamma}^m_{ij} \tilde{g}_{ml} = \partial^\alpha_j \tilde{g}_{il} + \partial^\alpha_i \tilde{g}_{lj} - \partial^\alpha_l \tilde{g}_{ij}$$

$$2\tilde{\Gamma}^m_{ij} \tilde{g}_{ml} \tilde{g}^{kl} = \tilde{g}^{kl}(\partial^\alpha_j \tilde{g}_{il} + \partial^\alpha_i \tilde{g}_{lj} - \partial^\alpha_l \tilde{g}_{ij})$$

since $\tilde{g}_{ml}\tilde{g}^{kl} = \delta^k_m$, then

$$\tilde{\Gamma}^k_{ij} = \frac{1}{2}\tilde{g}^{kl}(\partial^\alpha_j \tilde{g}_{il} + \partial^\alpha_i \tilde{g}_{lj} - \partial^\alpha_l \tilde{g}_{ij}),$$

we can write

$$\tilde{\Gamma}^k_{ij} = \frac{1}{2}\tilde{g}^{kl}(\tilde{g}_{il,j} + \tilde{g}_{lj,i} - \tilde{g}_{ij,l}).$$

□

Example 2.2. *For 2-dimential polar coordinates $X = (r^\alpha cos^\alpha\varphi, r^\alpha sin^\alpha\varphi)$.*

The metric tensor and its inverse here are:

$$\tilde{g}_{ij} = \begin{pmatrix} (\alpha!)^2\left[\cos^{2\alpha}\varphi + \sin^{2\alpha}\varphi\right] & (\alpha!)^2 r^\alpha\left[e^{2\alpha\pi}+1\right]\sin^\alpha\varphi\cos^\alpha\varphi \\ (\alpha!)^2 r^\alpha\left[e^{\alpha\pi i}+1\right]\sin^\alpha\varphi\cos^\alpha\varphi & (\alpha!)^2 r^{2\alpha}\left[e^{2\alpha\pi i}\sin^{2\alpha}\varphi + \cos^{2\alpha}\varphi\right]\end{pmatrix}$$

$$\tilde{g}^{ij} = \begin{pmatrix} A^{-1}(\alpha!)^2 r^{2\alpha}\left[e^{2\alpha\pi i}\sin^{2\alpha}\varphi + \cos^{2\alpha}\varphi\right] & -A^{-1}(\alpha!)^2 r^\alpha\left[e^{2\alpha\pi}+1\right]\sin^\alpha\varphi\cos^\alpha\varphi \\ -A^{-1}(\alpha!)^2 r^\alpha\left[e^{2\alpha\pi}+1\right]\sin^\alpha\varphi\cos^\alpha\varphi & A^{-1}(\alpha!)^2\left[\cos^{2\alpha}\varphi + \sin^{2\alpha}\varphi\right]\end{pmatrix}.$$

where

$$A = (\alpha!)^4 r^{2\alpha}\left\{\left[\cos^{2\alpha}\varphi + \sin^{2\alpha}\varphi\right]\left[e^{2\alpha\pi i}\sin^{2\alpha}\varphi + \cos^{2\alpha}\varphi\right] - \left[e^{\alpha\pi i}+1\right]\sin^{2\alpha}\varphi\cos^{2\alpha}\varphi\right\}$$

Therefore,

$$\partial^\alpha_r \tilde{g}_{ij} = \begin{pmatrix} 0 & (\alpha!)^3\left[e^{\alpha\pi i}+1\right]\sin^\alpha\varphi\cos^\alpha\varphi \\ (\alpha!)^3\left[e^{\alpha\pi i}+1\right]\sin^\alpha\varphi\cos^\alpha\varphi & (\alpha!)(2\alpha)! r^\alpha\left[e^{2\alpha\pi i}\sin^{2\alpha}\varphi + \cos^{2\alpha}\varphi\right]\end{pmatrix}$$

$$\partial^\alpha_\varphi \tilde{g}_{ij} = \begin{pmatrix} (\alpha!)(2\alpha)!\left[e^{\alpha\pi i}+1\right] cos^\alpha\varphi sin^\alpha\varphi & (\alpha!)^3 r^\alpha\left[e^{\alpha\pi i}+1\right]\left[e^{\alpha\pi i}\sin^{2\alpha}\varphi + \cos^{2\alpha}\varphi\right] \\ (\alpha!)^3 r^\alpha\left[e^{\alpha\pi i}+1\right]\left[e^{\alpha\pi i}\sin^{2\alpha}\varphi + \cos^{2\alpha}\varphi\right] & (\alpha!)(2\alpha)! r^{2\alpha}\left[e^{\alpha\pi i}+1\right]\sin^\alpha\varphi\cos^\alpha\varphi\end{pmatrix}.$$

Then ,

$$\begin{aligned}\tilde{\Gamma}^{r}_{rr} &= \frac{1}{2}\tilde{g}^{rl}(\partial^{\alpha}_{r}\tilde{g}_{rl} + \partial^{\alpha}_{r}\tilde{g}_{lr} - \partial^{\alpha}_{l}\tilde{g}_{rr}) \\ &= -A^{-1}(\alpha!)^{5}r^{\alpha}\left[e^{\alpha\pi i}+1\right]^{2}\sin^{2\alpha}\varphi\cos^{2\alpha}\varphi,\end{aligned}$$

$$\begin{aligned}\tilde{\Gamma}^{r}_{r\varphi} &= \frac{1}{2}\tilde{g}^{rl}(\partial^{\alpha}_{\varphi}\tilde{g}_{rl} + \partial^{\alpha}_{r}\tilde{g}_{l\varphi} - \partial^{\alpha}_{l}\tilde{g}_{r\varphi}) \\ &= 0 \\ &= \tilde{\Gamma}^{r}_{\varphi r},\end{aligned}$$

$$\begin{aligned}\tilde{\Gamma}^{r}_{\varphi\varphi} &= \frac{1}{2}\tilde{g}^{rl}(\partial^{\alpha}_{\varphi}\tilde{g}_{\varphi l} + \partial^{\alpha}_{\varphi}\tilde{g}_{l\varphi} - \partial^{\alpha}_{l}\tilde{g}_{\varphi\varphi}) \\ &= (\alpha!)^{2}r^{\alpha}(2A)^{-1}\left\{\left[e^{2\alpha\pi i}sin^{2\alpha}\varphi+\cos^{2\alpha}\varphi\right]\left[2(\alpha!)^{3}r^{\alpha}\left[e^{\alpha\pi i}+1\right]\left[e^{\alpha\pi i}\sin^{2\alpha}\varphi+\cos^{2\alpha}\varphi\right]\right.\right. \\ &\quad \left.\left.-\ (\alpha!)(2\alpha)!r^{\alpha}\left[e^{2\alpha\pi i}\sin^{2\alpha}\varphi+\sin^{2\alpha}\varphi\right]\right]-(\alpha!)(2\alpha)!r^{\alpha}\left[e^{\alpha\pi i}+1\right]\sin^{2\alpha}\varphi\cos^{2\alpha}\varphi\right\},\end{aligned}$$

$$\begin{aligned}\tilde{\Gamma}^{\varphi}_{rr} &= \frac{1}{2}\tilde{g}^{\varphi l}(\partial^{\alpha}_{\varphi}\tilde{g}_{rl} + \partial^{\alpha}_{r}\tilde{g}_{lr} - \partial^{\alpha}_{l}\tilde{g}_{rr}) \\ &= -(\alpha!)^{2}(2A)^{-1}\left\{(\alpha!)(2\alpha)!r^{\alpha}\left[e^{\alpha\pi i}+1\right]^{2}\sin^{2\alpha}\varphi\cos^{2\alpha}\varphi+\left[\cos^{2\alpha}\varphi+\sin^{2\alpha}\varphi\right]\left[e^{\alpha\pi i}+1\right]\right. \\ &\quad \left.\times\left[(\alpha!)^{3}r^{\alpha}\left[e^{\alpha\pi i}\sin^{2\alpha}\varphi+\cos^{2\alpha}\varphi\right]+(\alpha!)^{3}\sin^{\alpha}\varphi\cos^{\alpha}\varphi-(\alpha!)(2\alpha)!\cos^{\alpha}\varphi\sin^{\alpha}\varphi\right]\right\},\end{aligned}$$

$$\begin{aligned}\tilde{\Gamma}^{\varphi}_{r\varphi} &= \frac{1}{2}\tilde{g}^{\varphi l}(\partial^{\alpha}_{\varphi}\tilde{g}_{rl} + \partial^{\alpha}_{r}\tilde{g}_{l\varphi} - \partial^{\alpha}_{l}\tilde{g}_{r\varphi}) \\ &= (\alpha!)^{2}(2A)^{-1}\left\{-(\alpha!)(2\alpha)!^{2}r^{\alpha}\left[e^{\alpha\pi i}+1\right]\cos^{2\alpha}\varphi\sin^{2\alpha}\varphi\right. \\ &\quad \left.+(\alpha!)(2\alpha)!r^{\alpha}\left[\cos^{2\alpha}\varphi+\sin^{2\alpha}\varphi\right]\left[e^{2\alpha\pi i}\sin^{2\alpha}\varphi+\cos^{2\alpha}\varphi\right]\right\} \\ &= \tilde{\Gamma}^{\varphi}_{\varphi r},\end{aligned}$$

$$\begin{aligned}\tilde{\Gamma}^{\varphi}_{\varphi\varphi} &= \frac{1}{2}\tilde{g}^{\varphi l}(\partial^{\alpha}_{\varphi}\tilde{g}_{\varphi l} + \partial^{\alpha}_{\varphi}\tilde{g}_{l\varphi} - \partial^{\alpha}_{l}\tilde{g}_{\varphi\varphi}) \\ &= (\alpha!)^{2}r^{2\alpha}(2A)^{-1}\left\{-\left[e^{\alpha\pi i}+1\right]\sin^{\alpha}\varphi\cos^{\alpha}\varphi\left[2(\alpha!)^{3}\left[e^{\alpha\pi i}+1\right]\left[e^{\alpha\pi i}\sin^{2\alpha}\varphi+\cos^{2\alpha}\varphi\right]\right.\right. \\ &\quad \left.\left.-(\alpha!)(2\alpha)!\left[e^{2\alpha\pi i}\sin^{2\alpha}\varphi+\cos^{2\alpha}\varphi\right]\right]+(\alpha!)(2\alpha)!\left[e^{\alpha\pi i}+1\right]\left[\sin^{2\alpha}\varphi+\cos^{2\alpha}\varphi\right]\sin^{\alpha}\varphi\cos^{\alpha}\varphi\right\}.\end{aligned}$$

3. Fractional Curvature

Definition 3.1. *The fractional curvature $\tilde{R}$ of order α of a Riemannian manifold N is a correspondence that associates to every pair $X^{\alpha}, Y^{\alpha} \in \chi^{\alpha}$ a mapping $\tilde{R}(X^{\alpha}, Y^{\alpha})\colon \chi^{\alpha}(N)\times\chi^{\alpha}(N)\to\chi^{\alpha}(N)$ given by*

$$\tilde{R}(X^{\alpha},Y^{\alpha})Z^{\alpha} = \triangledown^{\alpha}_{X^{\alpha}}\triangledown^{\alpha}_{Y^{\alpha}}Z^{\alpha} - \triangledown^{\alpha}_{Y^{\alpha}}\triangledown^{\alpha}_{X^{\alpha}}Z^{\alpha} - \triangledown^{\alpha}_{[X^{\alpha},Y^{\alpha}]}Z^{\alpha},$$

where $Z^{\alpha}\in\chi^{\alpha}$ and $\triangledown^{\alpha}$ is the fractional Riemannian connection.

Remark 3.1.

$$\begin{aligned}\tilde{R}(X^{\alpha},Y^{\alpha})Z^{\alpha} &= \triangledown^{\alpha}_{X^{\alpha}}\triangledown^{\alpha}_{Y^{\alpha}}Z^{\alpha} - \triangledown^{\alpha}_{Y^{\alpha}}\triangledown^{\alpha}_{X^{\alpha}}Z^{\alpha} - \triangledown^{\alpha}_{[X^{\alpha},Y^{\alpha}]}Z^{\alpha} \\ &= -(\triangledown^{\alpha}_{Y^{\alpha}}\triangledown^{\alpha}_{X^{\alpha}}Z^{\alpha} - \triangledown^{\alpha}_{X^{\alpha}}\triangledown^{\alpha}_{Y^{\alpha}}Z^{\alpha} - \triangledown^{\alpha}_{[Y^{\alpha},X^{\alpha}]}Z^{\alpha}) \\ &= -\tilde{R}(Y^{\alpha},X^{\alpha})Z^{\alpha}.\end{aligned}$$

Proposition 3.1. *The fractional curvature $\tilde{R}$ of a Riemannian manifold has the following properties:*

(1) *$\tilde{R}$ is bilinear in $\chi^\alpha(N)\times\chi^\alpha(N)$, that is,*

$$\tilde{R}(fX^\alpha+gY^\alpha,Z^\alpha)W^\alpha=f\tilde{R}(X^\alpha,Z^\alpha)W^\alpha+g\tilde{R}(Y^\alpha,Z^\alpha)W^\alpha,$$

$$\tilde{R}(X^\alpha,fY^\alpha+gZ^\alpha)W^\alpha=f\tilde{R}(X^\alpha,Y^\alpha)W^\alpha+g\tilde{R}(X^\alpha,Z^\alpha)W^\alpha,$$

where $f,g\in\Im(M),X^\alpha,Y^\alpha,Z^\alpha,W^\alpha\in\chi^\alpha(N)$

(2) *For any $X^\alpha,Y^\alpha\in\chi^\alpha(N),\tilde{R}(X^\alpha,Y^\alpha)$ is linear*

$$\tilde{R}(X^\alpha,Y^\alpha)(Z^\alpha+W^\alpha)=\tilde{R}(X^\alpha,Y^\alpha)Z^\alpha+\tilde{R}(X^\alpha,Y^\alpha)W^\alpha,$$

$$\tilde{R}(X^\alpha,Y^\alpha)(fZ^\alpha)=f\tilde{R}(X^\alpha,Y^\alpha)Z^\alpha,$$

where $Z^\alpha,W^\alpha\in\chi^\alpha(N)$

Proof. (1)

$$\begin{aligned}
&\tilde{R}(fX^\alpha+gY^\alpha,Z^\alpha)W^\alpha\\
&=\nabla^\alpha_{fX^\alpha+gY^\alpha}\nabla^\alpha_{Z^\alpha}W^\alpha-\nabla^\alpha_{Z^\alpha}\nabla^\alpha_{fX^\alpha+gY^\alpha}W^\alpha-\nabla^\alpha_{[fX^\alpha+gY^\alpha,Z^\alpha]}W^\alpha\\
&=(f\nabla^\alpha_{X^\alpha}+g\nabla^\alpha_{Y^\alpha})\nabla^\alpha_{Z^\alpha}W^\alpha-\nabla^\alpha_{Z^\alpha}(f\nabla^\alpha_{X^\alpha}W^\alpha+g\nabla^\alpha_{Y^\alpha}W^\alpha)\\
&\quad-\nabla^\alpha_{f[X^\alpha,Z^\alpha]+g[Y^\alpha,Z^\alpha]-(Z^\alpha f)X^\alpha-(Z^\alpha g)Y^\alpha}W^\alpha\\
&=f\nabla^\alpha_{X^\alpha}\nabla^\alpha_{Z^\alpha}W^\alpha+g\nabla^\alpha_{Y^\alpha}\nabla^\alpha_{Z^\alpha}W^\alpha-(Z^\alpha f)\nabla^\alpha_{X^\alpha}W^\alpha\\
&\quad-f\nabla^\alpha_{Z^\alpha}\nabla^\alpha_{X^\alpha}W^\alpha-(Z^\alpha g)\nabla^\alpha_{Y^\alpha}W^\alpha-g\nabla^\alpha_{Z^\alpha}\nabla^\alpha_{Y^\alpha}W^\alpha-f\nabla^\alpha_{[X^\alpha,Z^\alpha]}W^\alpha\\
&\qquad-g\nabla^\alpha_{[Y^\alpha,Z^\alpha]}W^\alpha+(Z^\alpha f)\nabla^\alpha_{X^\alpha}W^\alpha+(Z^\alpha g)\nabla^\alpha_{Y^\alpha}W^\alpha\\
&=f(\nabla^\alpha_{X^\alpha}\nabla^\alpha_{Z^\alpha}W^\alpha-\nabla^\alpha_{Z^\alpha}\nabla^\alpha_{X^\alpha}W^\alpha-\nabla^\alpha_{[X^\alpha,Z^\alpha]}W^\alpha)\\
&\quad+g(\nabla^\alpha_{Y^\alpha}\nabla^\alpha_{Z^\alpha}W^\alpha-\nabla^\alpha_{Z^\alpha}\nabla^\alpha_{Y^\alpha}W^\alpha-\nabla^\alpha_{[Y^\alpha,Z^\alpha]}W^\alpha)\\
&=f\tilde{R}(X^\alpha,Z^\alpha)W^\alpha+gf\tilde{R}(Y^\alpha,Z^\alpha)W^\alpha.
\end{aligned}$$

Also,

$$\begin{aligned}
\tilde{R}(X^\alpha,fY^\alpha+gZ^\alpha)W^\alpha &= -\tilde{R}(fY^\alpha+gZ^\alpha,X^\alpha)\\
&= -f\tilde{R}(Y^\alpha,X^\alpha)W^\alpha-g\tilde{R}(Z^\alpha,X^\alpha)W^\alpha\\
&= f\tilde{R}(X^\alpha,Y^\alpha)W^\alpha+g\tilde{R}(X^\alpha,Z^\alpha)W^\alpha.
\end{aligned}$$

(2)

$$\begin{aligned}
&\tilde{R}(X^\alpha,Y^\alpha)(Z^\alpha+W^\alpha)\\
&=\nabla^\alpha_{X^\alpha}\nabla^\alpha_{Y^\alpha}(Z^\alpha+W^\alpha)-\nabla^\alpha_{Y^\alpha}\nabla^\alpha_{X^\alpha}(Z^\alpha+W^\alpha)-\nabla^\alpha_{[X^\alpha,Y^\alpha]}(Z^\alpha+W^\alpha)\\
&=\nabla^\alpha_{X^\alpha}\nabla^\alpha_{Y^\alpha}Z^\alpha+\nabla^\alpha_{X^\alpha}\nabla^\alpha_{Y^\alpha}W^\alpha-\nabla^\alpha_{Y^\alpha}\nabla^\alpha_{X^\alpha}Z^\alpha\\
&\quad-\nabla^\alpha_{Y^\alpha}\nabla^\alpha_{X^\alpha}W^\alpha-\nabla^\alpha_{[X^\alpha,Y^\alpha]}Z^\alpha-\nabla^\alpha_{[X^\alpha,Y^\alpha]}W^\alpha\\
&=(\nabla^\alpha_{X^\alpha}\nabla^\alpha_{Y^\alpha}Z^\alpha-\nabla^\alpha_{Y^\alpha}\nabla^\alpha_{X^\alpha}Z^\alpha-\nabla^\alpha_{[X^\alpha,Y^\alpha]}Z^\alpha)\\
&\quad+(\nabla^\alpha_{X^\alpha}\nabla^\alpha_{Y^\alpha}W^\alpha-\nabla^\alpha_{Y^\alpha}\nabla^\alpha_{X^\alpha}W^\alpha-\nabla^\alpha_{[X^\alpha,Y^\alpha]}W^\alpha)\\
&=\tilde{R}(X^\alpha,Y^\alpha)Z^\alpha+\tilde{R}(X^\alpha,Y^\alpha)W^\alpha.
\end{aligned}$$

Also,

$$\begin{aligned}
&\tilde{R}(X^\alpha, Y^\alpha)(fZ^\alpha)\\
&\quad= \bigtriangledown^\alpha_{X^\alpha} \bigtriangledown^\alpha_{Y^\alpha} (fZ^\alpha) - \bigtriangledown^\alpha_{Y^\alpha} \bigtriangledown^\alpha_{X^\alpha} (fZ^\alpha) - \bigtriangledown^\alpha_{[X^\alpha,Y^\alpha]}(fZ^\alpha)\\
&\quad= \bigtriangledown^\alpha_{X^\alpha}((Y^\alpha f)Z^\alpha + f \bigtriangledown^\alpha_{Y^\alpha} Z^\alpha) - \bigtriangledown^\alpha_{Y^\alpha}((X^\alpha f)Z^\alpha + f \bigtriangledown^\alpha_{X^\alpha} Z^\alpha)\\
&\qquad -(([X^\alpha, Y^\alpha]f)Z^\alpha + f \bigtriangledown^\alpha_{[X^\alpha,Y^\alpha]} Z^\alpha)\\
&\quad= X^\alpha(Y^\alpha f)Z^\alpha + (Y^\alpha f) \bigtriangledown^\alpha_{X^\alpha} Z^\alpha + (X^\alpha f) \bigtriangledown^\alpha_{Y^\alpha} Z^\alpha + f \bigtriangledown^\alpha_{X^\alpha} \bigtriangledown^\alpha_{Y^\alpha} Z^\alpha\\
&\qquad -Y^\alpha(X^\alpha f)Z^\alpha + (X^\alpha f) \bigtriangledown^\alpha_{Y^\alpha} Z^\alpha + (Y^\alpha f) \bigtriangledown^\alpha_{X^\alpha} Z^\alpha + f \bigtriangledown^\alpha_{Y^\alpha} \bigtriangledown^\alpha_{X^\alpha} Z^\alpha\\
&\qquad\quad -([X^\alpha, Y^\alpha]f)Z^\alpha - f \bigtriangledown^\alpha_{[X^\alpha,Y^\alpha]} Z^\alpha\\
&\quad= ([X^\alpha, Y^\alpha]f)Z^\alpha + f(\bigtriangledown^\alpha_{X^\alpha} \bigtriangledown^\alpha_{Y^\alpha} Z^\alpha - \bigtriangledown^\alpha_{Y^\alpha} \bigtriangledown^\alpha_{X^\alpha} Z^\alpha - \bigtriangledown^\alpha_{[X^\alpha,Y^\alpha]} Z^\alpha) - ([X^\alpha, Y^\alpha]f)Z^\alpha\\
&\quad= f\tilde{R}(X^\alpha, Y^\alpha)Z^\alpha.
\end{aligned}$$

□

Proposition 3.2 (Bianchi Fractional Identity)**.**

$$\tilde{R}(X^\alpha, Y^\alpha)Z^\alpha + \tilde{R}(Y^\alpha, Z^\alpha)X^\alpha + \tilde{R}(Z^\alpha, X^\alpha)Y^\alpha = 0.$$

Proof.

$$\begin{aligned}
&\tilde{R}(X^\alpha, Y^\alpha)Z^\alpha + \tilde{R}(Y^\alpha, Z^\alpha)X^\alpha + \tilde{R}(Z^\alpha, X^\alpha)Y^\alpha\\
&\quad= \bigtriangledown^\alpha_{X^\alpha} \bigtriangledown^\alpha_{Y^\alpha} Z^\alpha - \bigtriangledown^\alpha_{Y^\alpha} \bigtriangledown^\alpha_{X^\alpha} Z^\alpha - \bigtriangledown^\alpha_{[X^\alpha,Y^\alpha]} Z^\alpha\\
&\qquad + \bigtriangledown^\alpha_{Y^\alpha} \bigtriangledown^\alpha_{Z^\alpha} X^\alpha - \bigtriangledown^\alpha_{Z^\alpha} \bigtriangledown^\alpha_{Y^\alpha} X^\alpha - \bigtriangledown^\alpha_{[Y^\alpha,Z^\alpha]} X^\alpha\\
&\qquad\quad + \bigtriangledown^\alpha_{Z^\alpha} \bigtriangledown^\alpha_{X^\alpha} Y^\alpha - \bigtriangledown^\alpha_{X^\alpha} \bigtriangledown^\alpha_{Z^\alpha} Y^\alpha - \bigtriangledown^\alpha_{[Z^\alpha,X^\alpha]} Y^\alpha\\
&\quad= \bigtriangledown^\alpha_{X^\alpha}[Y^\alpha, Z^\alpha] + \bigtriangledown^\alpha_{Y^\alpha}[Z^\alpha, X^\alpha] + \bigtriangledown^\alpha_{Z^\alpha}[X^\alpha, Y^\alpha]\\
&\qquad - \bigtriangledown^\alpha_{[X^\alpha,Y^\alpha]} Z^\alpha - \bigtriangledown^\alpha_{[Y^\alpha,Z^\alpha]} X^\alpha - \bigtriangledown^\alpha_{[Z^\alpha,X^\alpha]} Y^\alpha\\
&\quad= [X^\alpha, [Y^\alpha, Z^\alpha]] + [Y^\alpha, [Z^\alpha, X^\alpha]] + [Z^\alpha, [X^\alpha, Y^\alpha]] = 0.
\end{aligned}$$

□

In local coordinates

$$\tilde{R}(\partial^\alpha_i, \partial^\alpha_j)\partial^\alpha_k = \tilde{R}^l_{ijk}\partial^\alpha_l,$$

and

$$\begin{aligned}
\tilde{R}_{ijkm} &= \langle \tilde{R}(\partial^\alpha_i, \partial^\alpha_j)\partial^\alpha_k, \partial^\alpha_m \rangle\\
&= \left\langle \tilde{R}^l_{ijk}\partial^\alpha_l, \partial^\alpha_m \right\rangle\\
&= \tilde{R}^l_{ijk} \langle \partial^\alpha_l, \partial^\alpha_m \rangle\\
&= \tilde{R}^l_{ijk}\tilde{g}_{ml}.
\end{aligned}$$

The fractional Riemannian curvature tensor acts on fractional vector fields as follows:

$$\tilde{R}(X^\alpha, Y^\alpha, Z^\alpha, W^\alpha) = \langle \tilde{R}(X^\alpha, Y^\alpha)Z^\alpha, W^\alpha \rangle.$$

Proposition 3.3. (1) $\tilde{R}_{ijkl} + \tilde{R}_{jkil} + \tilde{R}_{kijl} = 0$.

(2) $\tilde{R}_{ijkl} = -\tilde{R}_{jikl}$.

(3) $\tilde{R}_{ijkl} = -\tilde{R}_{ijlk}$.

(4) $\tilde{R}_{ijkl} = \tilde{R}_{klij}$.

Proof. (1) is just the Bianchi fractional identity again.

(2)

$$\begin{aligned}\tilde{R}_{ijkl} &= \left\langle \tilde{R}(\partial_i^\alpha, \partial_j^\alpha)\partial_k^\alpha, \partial_l^\alpha \right\rangle \\ &= \left\langle -\tilde{R}(\partial_j^\alpha, \partial_i^\alpha)\partial_k^\alpha, \partial_l^\alpha \right\rangle \\ &= -\left\langle \tilde{R}(\partial_j^\alpha, \partial_i^\alpha)\partial_k^\alpha, \partial_l^\alpha \right\rangle \\ &= -\tilde{R}_{jikl}.\end{aligned}$$

(3) is equivalent to $\tilde{R}_{ijkk} = 0$, whose proof follows:

$$\begin{aligned}\tilde{R}_{ijkk} &= \left\langle \tilde{R}(\partial_i^\alpha, \partial_j^\alpha)\partial_k^\alpha, \partial_k^\alpha \right\rangle \\ &= \left\langle \nabla^\alpha_{\partial_i^\alpha} \nabla^\alpha_{\partial_j^\alpha} \partial_k^\alpha - \nabla^\alpha_{\partial_j^\alpha} \nabla^\alpha_{\partial_i^\alpha} \partial_k^\alpha - \nabla^\alpha_{[\partial_i^\alpha, \partial_j^\alpha]}\partial_k^\alpha, \partial_k^\alpha \right\rangle,\end{aligned}$$

but

$$\left\langle \nabla^\alpha_{\partial_j^\alpha} \nabla^\alpha_{\partial_i^\alpha} \partial_k^\alpha, \partial_k^\alpha \right\rangle = \partial_j^\alpha \left\langle \nabla^\alpha_{\partial_i^\alpha}\partial_k^\alpha, \partial_k^\alpha \right\rangle - \left\langle \nabla^\alpha_{\partial_i^\alpha}\partial_k^\alpha, \nabla^\alpha_{\partial_j^\alpha}\partial_k^\alpha \right\rangle,$$

and

$$\left\langle \nabla^\alpha_{[\partial_i^\alpha, \partial_j^\alpha]}\partial_k^\alpha, \partial_k^\alpha \right\rangle = \frac{1}{2}\left[\partial_i^\alpha, \partial_j^\alpha\right] \langle \partial_k^\alpha, \partial_k^\alpha \rangle,$$

then

$$\begin{aligned}\tilde{R}_{ijkk} &= \partial_j^\alpha \left\langle \nabla^\alpha_{\partial_i^\alpha}\partial_k^\alpha, \partial_k^\alpha \right\rangle - \partial_i^\alpha \left\langle \nabla^\alpha_{\partial_j^\alpha}\partial_k^\alpha, \partial_k^\alpha \right\rangle + \frac{1}{2}\left[\partial_i^\alpha, \partial_j^\alpha\right] \langle \partial_k^\alpha, \partial_k^\alpha \rangle \\ &= \frac{1}{2}\partial_j^\alpha(\partial_i^\alpha \langle \partial_k^\alpha, \partial_k^\alpha \rangle) - \frac{1}{2}\partial_i^\alpha\left(\partial_j^\alpha \langle \partial_k^\alpha, \partial_k^\alpha \rangle\right) + \frac{1}{2}\left[\partial_i^\alpha, \partial_j^\alpha\right] \langle \partial_k^\alpha, \partial_k^\alpha \rangle \\ &= -\frac{1}{2}\left[\partial_i^\alpha, \partial_j^\alpha\right] \langle \partial_k^\alpha, \partial_k^\alpha \rangle + \frac{1}{2}\left[\partial_i^\alpha, \partial_j^\alpha\right] \langle \partial_k^\alpha, \partial_k^\alpha \rangle = 0.\end{aligned}$$

(4) By Bianchi fractional identity we have

$$\begin{aligned}\tilde{R}_{ijkl} + \tilde{R}_{jkil} + \tilde{R}_{kijl} &= 0 \\ \tilde{R}_{jkli} + \tilde{R}_{klji} + \tilde{R}_{ljki} &= 0 \\ \tilde{R}_{klij} + \tilde{R}_{likj} + \tilde{R}_{iklj} &= 0 \\ \tilde{R}_{lijk} + \tilde{R}_{ijlk} + \tilde{R}_{jlik} &= 0\end{aligned}$$

summing the equations above, we obtain

$$2\tilde{R}_{kijl} + 2\tilde{R}_{ljki} = 0,$$

then

$$\tilde{R}_{kijl} = -\tilde{R}_{ljki} = \tilde{R}_{jlki}.$$

□

Proposition 3.4. *The following expression holds*

$$2\tilde{R}_{ijkm} = \tilde{g}_{jm,ki} + \tilde{g}_{km,ji} - \tilde{g}_{jk,mi} - \tilde{g}_{im,kj} - \tilde{g}_{km,ij} + \tilde{g}_{ik,mj} - 2\tilde{\Gamma}^r_{jk}\tilde{\Gamma}^s_{im}\tilde{g}_{rs} + 2\tilde{\Gamma}^r_{ik}\tilde{\Gamma}^s_{jm}\tilde{g}_{rs}.$$

Proof. From the definition of the Christoffel symbols, $\nabla^\alpha_{\partial_i^\alpha}\partial_j^\alpha = \tilde{\Gamma}^k_{ij}\partial_k^\alpha$,

$$\begin{aligned}2\left\langle \nabla^\alpha_{\partial_i^\alpha}\partial_j^\alpha, \partial_m^\alpha \right\rangle &= 2\left\langle \tilde{\Gamma}^k_{ij}\partial_k^\alpha, \partial_m^\alpha \right\rangle \\ &= 2\tilde{\Gamma}^k_{ij}\tilde{g}_{mk} \\ &= \tilde{g}_{im,j} + \tilde{g}_{jm,i} - \tilde{g}_{ij,m},\end{aligned}$$

an appropriate rearrangement of the indices yields the following expression:

$$\begin{aligned} 2\left\langle \nabla^{\alpha}_{\partial^{\alpha}_j}\partial^{\alpha}_k, \partial^{\alpha}_m \right\rangle &= 2\left\langle \tilde{\Gamma}^i_{jk}\partial^{\alpha}_i, \partial^{\alpha}_m \right\rangle \\ &= 2\tilde{\Gamma}^i_{jk}\tilde{g}_{im} \\ &= \tilde{g}_{jm,k} + \tilde{g}_{km,j} - \tilde{g}_{jk,m}. \end{aligned} \tag{3.1}$$

$$\partial^{\alpha}_i < \nabla^{\alpha}_{\partial^{\alpha}_j}\partial^{\alpha}_k, \partial^{\alpha}_m >=< \nabla^{\alpha}_{\partial^{\alpha}_i} \nabla^{\alpha}_{\partial^{\alpha}_j} \partial^{\alpha}_k, \partial^{\alpha}_m > + < \nabla^{\alpha}_{\partial^{\alpha}_j}\partial^{\alpha}_k, \nabla^{\alpha}_{\partial^{\alpha}_i}\partial^{\alpha}_m >$$

whence, by (3.1)

$$\begin{aligned} 2\left\langle \nabla^{\alpha}_{\partial^{\alpha}_i} \nabla^{\alpha}_{\partial^{\alpha}_j} \partial^{\alpha}_k, \partial^{\alpha}_m \right\rangle + 2\left\langle \nabla^{\alpha}_{\partial^{\alpha}_j}\partial^{\alpha}_k, \nabla^{\alpha}_{\partial^{\alpha}_i}\partial^{\alpha}_m \right\rangle &= 2\partial^{\alpha}_i \left\langle \nabla^{\alpha}_{\partial^{\alpha}_j}\partial^{\alpha}_k, \partial^{\alpha}_m \right\rangle \\ &= \partial^{\alpha}_i \left(\tilde{g}_{im} g^{il} (\partial^{\alpha}_k g_{jl} + \partial^{\alpha}_j g_{kl} - \partial^{\alpha}_l g_{jk}) \right) \\ &= \tilde{g}_{jm,ki} + \tilde{g}_{km,ji} - \tilde{g}_{jk,mi}. \end{aligned} \tag{3.2}$$

By switching i and j we also have that

$$2\left\langle \nabla^{\alpha}_{\partial^{\alpha}_j} \nabla^{\alpha}_{\partial^{\alpha}_i} \partial^{\alpha}_k, \partial^{\alpha}_m \right\rangle + 2\left\langle \nabla^{\alpha}_{\partial^{\alpha}_i}\partial^{\alpha}_k, \nabla^{\alpha}_{\partial^{\alpha}_j}\partial^{\alpha}_m \right\rangle = \tilde{g}_{im,kj} + \tilde{g}_{km,ij} - \tilde{g}_{ik,mj}. \tag{3.3}$$

Combining (3.2) and (3.3) yields

$$\begin{aligned} &2\left\langle \nabla^{\alpha}_{\partial^{\alpha}_i} \nabla^{\alpha}_{\partial^{\alpha}_j} \partial^{\alpha}_k, \partial^{\alpha}_m \right\rangle - 2\left\langle \nabla^{\alpha}_{\partial^{\alpha}_j} \nabla^{\alpha}_{\partial^{\alpha}_i} \partial^{\alpha}_k, \partial^{\alpha}_m \right\rangle = \\ &\quad \tilde{g}_{jm,ki} + \tilde{g}_{km,ji} - \tilde{g}_{jk,mi} - \tilde{g}_{im,kj} - \tilde{g}_{km,ij} + \tilde{g}_{ik,mj} \\ &\quad -2\left\langle \nabla^{\alpha}_{\partial^{\alpha}_j}\partial^{\alpha}_k, \nabla^{\alpha}_{\partial^{\alpha}_i}\partial^{\alpha}_m \right\rangle + 2\left\langle \nabla^{\alpha}_{\partial^{\alpha}_i}\partial^{\alpha}_k, \nabla^{\alpha}_{\partial^{\alpha}_j}\partial^{\alpha}_m \right\rangle. \end{aligned}$$

By definition

$$\tilde{R}\left(\partial^{\alpha}_i, \partial^{\alpha}_j\right)\partial^{\alpha}_k = \nabla^{\alpha}_{\partial^{\alpha}_i} \nabla^{\alpha}_{\partial^{\alpha}_j} \partial^{\alpha}_k - \nabla^{\alpha}_{\partial^{\alpha}_j} \nabla^{\alpha}_{\partial^{\alpha}_i} \partial^{\alpha}_k$$

whence

$$\begin{aligned} 2\tilde{R}_{ijkm} &= 2\left\langle \tilde{R}(\partial^{\alpha}_i, \partial^{\alpha}_j)\partial^{\alpha}_k, \partial^{\alpha}_m \right\rangle \\ &= 2\left\langle \nabla^{\alpha}_{\partial^{\alpha}_i} \nabla^{\alpha}_{\partial^{\alpha}_j} \partial^{\alpha}_k, \partial^{\alpha}_m \right\rangle - 2\left\langle \nabla^{\alpha}_{\partial^{\alpha}_j} \nabla^{\alpha}_{\partial^{\alpha}_i} \partial^{\alpha}_k, \partial^{\alpha}_m \right\rangle, \end{aligned}$$

so we have proven that

$$\begin{aligned} &2\tilde{R}_{ijkm} = \\ &\tilde{g}_{jm,ki} + \tilde{g}_{km,ji} - \tilde{g}_{jk,mi} - \tilde{g}_{im,kj} - \tilde{g}_{km,ij} + \tilde{g}_{ik,mj} \\ &\quad -2\left\langle \nabla^{\alpha}_{\partial^{\alpha}_j}\partial^{\alpha}_k, \nabla^{\alpha}_{\partial^{\alpha}_i}\partial^{\alpha}_m \right\rangle + 2\left\langle \nabla^{\alpha}_{\partial^{\alpha}_i}\partial^{\alpha}_k, \nabla^{\alpha}_{\partial^{\alpha}_j}\partial^{\alpha}_m \right\rangle. \end{aligned}$$

By the definition of the Christoffels,

$$\begin{aligned} \left\langle \nabla^{\alpha}_{\partial^{\alpha}_i}\partial^{\alpha}_k, \nabla^{\alpha}_{\partial^{\alpha}_j}\partial^{\alpha}_m \right\rangle &= \left\langle \tilde{\Gamma}^r_{ik}\partial^{\alpha}_r, \tilde{\Gamma}^s_{jm}\partial^{\alpha}_s \right\rangle \\ &= \tilde{\Gamma}^r_{ik}\tilde{\Gamma}^s_{jm}\left\langle \partial^{\alpha}_r, \partial^{\alpha}_s \right\rangle \\ &= \tilde{\Gamma}^r_{ik}\tilde{\Gamma}^s_{jm}\tilde{g}_{rs}, \end{aligned}$$

$$\begin{aligned} \left\langle \nabla^{\alpha}_{\partial^{\alpha}_j}\partial^{\alpha}_k, \nabla^{\alpha}_{\partial^{\alpha}_i}\partial^{\alpha}_m \right\rangle &= \left\langle \tilde{\Gamma}^r_{jk}\partial^{\alpha}_r, \tilde{\Gamma}^s_{im}\partial^{\alpha}_s \right\rangle \\ &= \tilde{\Gamma}^r_{jk}\tilde{\Gamma}^s_{im}\left\langle \partial^{\alpha}_r, \partial^{\alpha}_s \right\rangle \\ &= \tilde{\Gamma}^r_{jk}\tilde{\Gamma}^s_{im}\tilde{g}_{rs}, \end{aligned}$$

then

$$2\tilde{R}_{ijkm}=\tilde{g}_{jm,ki}+\tilde{g}_{km,ji}-\tilde{g}_{jk,mi}-\tilde{g}_{im,kj}-\tilde{g}_{km,ij}+\tilde{g}_{ik,mj}-2\tilde{\Gamma}^r_{jk}\tilde{\Gamma}^s_{im}\tilde{g}_{rs}+2\tilde{\Gamma}^r_{ik}\tilde{\Gamma}^s_{jm}\tilde{g}_{rs}.$$

□

Remark 3.2. *If $\alpha=1$, then*

$$2R_{ijkm}=g_{jm,ki}+g_{km,ji}-g_{jk,mi}-g_{im,kj}-g_{km,ij}+g_{ik,mj}-2\Gamma^r_{jk}\Gamma^s_{im}g_{rs}+2\Gamma^r_{ik}\Gamma^s_{jm}g_{rs}.$$

Since $g_{km,ji}=g_{km,ij}$, then

$$2R_{ijkm}=g_{jm,ki}-g_{jk,mi}-g_{im,kj}+g_{ik,mj}-2\Gamma^r_{jk}\Gamma^s_{im}g_{rs}+2\Gamma^r_{ik}\Gamma^s_{jm}g_{rs},$$

then

$$2R_{ijkm}=\tilde{g}_{jm,ki}+\tilde{g}_{km,ji}-\tilde{g}_{jk,mi}-\tilde{g}_{im,kj}-\tilde{g}_{km,ij}+\tilde{g}_{ik,mj}-2\Gamma^r_{jk}\Gamma^s_{im}g_{rs}+2\Gamma^r_{ik}\Gamma^s_{jm}g_{rs}.$$

Remark 3.3. *For any pair of fractional tangent vectors $X^\alpha, Y^\alpha \in T^\alpha_p N$ we shall denote with $\tilde{\Gamma}(X^\alpha, Y^\alpha)$ the following fractional vector in $T^\alpha_p N$:*

$$\tilde{\Gamma}(X^\alpha,Y^\alpha)=\tilde{\Gamma}^k_{ij}X^\alpha_i Y^\alpha_j \partial^\alpha_k.$$

Proposition 3.5. *The following expressions hold for any pair $X^\alpha, Y^\alpha \in T^\alpha_p N$:*

$$\begin{aligned}
&2\tilde{R}\left(X^\alpha,Y^\alpha,Y^\alpha,X^\alpha\right)\\
&\quad=\partial^\alpha_i\left(\tilde{g}_{im}g^{il}\right)\left(\partial^\alpha_k g_{jl}+\partial^\alpha_j g_{kl}-\partial^\alpha_l g_{jk}\right)+\tilde{g}_{im}g^{il}\left(\partial^\alpha_i\partial^\alpha_k g_{jl}+\partial^\alpha_i\partial^\alpha_j g_{kl}-\partial^\alpha_i\partial^\alpha_l g_{jk}\right)\\
&\quad-\partial^\alpha_j\left(\tilde{g}_{jm}g^{jl}\right)\left(\partial^\alpha_k g_{il}+\partial^\alpha_i g_{kl}-\partial^\alpha_l g_{ik}\right)-\tilde{g}_{jm}g^{jl}\left(\partial^\alpha_j\partial^\alpha_k g_{il}+\partial^\alpha_j\partial^\alpha_i g_{kl}-\partial^\alpha_j\partial^\alpha_l g_{ik}\right)\\
&\quad+2\parallel\tilde{\Gamma}\left(X^\alpha,Y^\alpha\right)\parallel^2-2\left\langle\tilde{\Gamma}\left(X^\alpha,X^\alpha\right),\tilde{\Gamma}\left(Y^\alpha,Y^\alpha\right)\right\rangle.
\end{aligned}$$

Proof. Since

$$\begin{aligned}
\tilde{g}_{rs}X^\alpha_i Y^\alpha_j Y^\alpha_k X^\alpha_m\tilde{\Gamma}^r_{ik}\tilde{\Gamma}^s_{jm} &= \left\langle X^\alpha_i Y^\alpha_k\tilde{\Gamma}^r_{ik}\partial^\alpha_r, Y^\alpha_j X^\alpha_m\tilde{\Gamma}^s_{jm}\partial^\alpha_s\right\rangle\\
&= \left\langle\tilde{\Gamma}(X^\alpha,Y^\alpha),\tilde{\Gamma}(X^\alpha,Y^\alpha)\right\rangle=\parallel\tilde{\Gamma}(X^\alpha,Y^\alpha)\parallel^2,
\end{aligned}$$

and

$$\begin{aligned}
\tilde{g}_{rs}X^\alpha_i Y^\alpha_j Y^\alpha_k X^\alpha_m\tilde{\Gamma}^r_{jk}\tilde{\Gamma}^s_{im} &= \left\langle X^\alpha_i X^\alpha_m\tilde{\Gamma}^s_{im}\partial^\alpha_s, Y^\alpha_j Y^\alpha_k\tilde{\Gamma}^r_{jk}\partial^\alpha_r\right\rangle\\
&= \left\langle\tilde{\Gamma}(X^\alpha,X^\alpha),\tilde{\Gamma}(Y^\alpha,Y^\alpha)\right\rangle,
\end{aligned}$$

This completes the proof. □

Remark 3.4. *If $\alpha=1$, then*

$$\begin{aligned}
&2R(X,Y,Y,X)=g_{jm,ki}-g_{jk,mi}-g_{im,kj}+g_{ik,mj}\\
&+2\parallel\Gamma(X,Y)\parallel^2-2\left\langle\Gamma(X,X),\Gamma(Y,Y)\right\rangle.
\end{aligned}$$

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] I.D. Albu, M. Neamtu, D. Opris, The Geometry of Fractional Osculator Bundle of Higher Order and Applications, in: Processing of International Conference on Differential Geometry Lagrange and Hamilton Spaces, Dedicated to Acad. Radu Miron at eighty, September 3-8, 2007, Iasi, Romania https://doi.org/10.48550/arXiv.0709.2000.

[2] M. Axtell, M.E. Bise, Fractional Calculus Application in Control Systems, in: IEEE Conference on Aerospace and Electronics, IEEE, Dayton, OH, USA, 1990: pp. 563–566. https://doi.org/10.1109/NAECON.1990.112826.

[3] Y. Ding, X. Wu, Fractional Integrals on Product Manifolds, Potential Anal. 30 (2009), 371–383. https://doi.org/10.1007/s11118-009-9120-1.

[4] G. Jumarie, Riemann-Christoffel Tensor in Differential Geometry of Fractional Order Application to Fractal Space-Time, Fractals. 21 (2013), 1350004. https://doi.org/10.1142/S0218348X13500047.

[5] I. D. Albu and D. Opris, The Geometry of Fractional Tangent Bundle and Applications, in: BSG Proceedings 16. The Int. Conf. of Diff. Geom. and Dynamical Systems (DGDS-2008) and The V-th Int. Colloq. of Mathematics in Engineering and Numerical Physics (MENP-5) - math. sections, August 29 - September 2, 2008, Mangalia, Romania, pp. 1-11. http://repository.utm.md/handle/5014/17678.

[6] A. Kilicman, W. Saleh, Note on the Fractional Mittag-Leffler Functions by Applying the Modified Riemann-Liouville Derivatives, Bol. Soc. Paranaense Mat. 40 (2022), 1–16. https://doi.org/10.5269/bspm.44103.

[7] K.M. Kolwankar, Studies of Fractal Structures and Processes Using Methods of Fractional Calculus, ArXiv:Chao-Dyn/9811008. (1998). http://arxiv.org/abs/chao-dyn/9811008.

[8] M. Delkhosh, Introduction of Derivatives and Integrals of Fractional Order and Its Applications, Appl. Math. Phys. 1 (2013), 103–119 .

[9] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley and Sons, Inc. New York, (1993).

[10] K.B. Oldham, J. Spanier, The Fractional Calculus, Academic Press, New York, (1974).

[11] I. Podlubny, Fractional-Order Systems and Fractional-Order Controllers, Technical Report UEF-03-94, Slovak Acad. Sci. (1994).

[12] B. Ross, ed., Fractional Calculus and Its Applications: Proceedings of the International Conference Held at the University of New Haven, June 1974, Springer Berlin Heidelberg, Berlin, Heidelberg, 1975. https://doi.org/10.1007/BFb0067095.

[13] W. Saleh, A. Kılıçman, Some Remarks on Geometry of Fractional Calculus, Appl. Math. Inform. Sci. 9 (2015), 1265–1275.

[14] W. Saleh and A. Kılıçman, Some Inequalities for Generalized s-Convex Functions, JP J. Geom. Topol. 17 (2015), 63–82.

[15] I.N. Sneddon, Special Functions of Mathematical Physics and Chemistry, Oliver & Boyd, Edinburgh, 1956.

[16] M.H. Tavassoli, A. Tavassoli, M.R. Ostad Rahimi, The Geometric and Physical Interpretation of Fractional Order Derivatives of Polynomial Functions, Differ. Geom. Dyn. Syst. 15 (2013), 93–104.

[17] A. Kılıçman and B. Fisher, Some Results on the Composition of Singular Distribution, Appl. Math. Inform. Sci. 5 (2011), 597–608.

[18] A. Kılıçman, W. Saleh, NOtions of Generalized s-Convex Functions on Fractal Sets, J. Inequal. Appl. 2015 (2015), 312. https://doi.org/10.1186/s13660-015-0826-x.

[19] A. Kılıçman, W. Saleh, Some Generalized Hermite-Hadamard Type Integral Inequalities for Generalized S-Convex Functions on Fractal Sets, Adv. Differ. Equ. 2015 (2015), 301. https://doi.org/10.1186/s13662-015-0639-8.

[20] B. Fisher, B. Jolevsaka-Tuneska, A. KiliÇman, On Defining the Incomplete Gamma Function, Integral Transforms Spec. Funct. 14 (2003), 293–299. https://doi.org/10.1080/1065246031000081667.

[21] B. Fisher, A. Kilicman, D. Nicholas, On the Beta Function and the Neutrix Product of Distributions, Integral Transforms Spec. Funct. 7 (1998), 35–42. https://doi.org/10.1080/10652469808819184.

[22] B. Fisher, A. Kılıçman, Results on the Gamma Function, Appl. Math. Inform. Sci. 6 (2012), 173–176.