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Some Properties of Controlled K-g-Frames in Hilbert C^*-Modules

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Abstract. This paper is devoted to studying the controlled $K-g-$frames in Hilbert C^*-modules, some useful results are presented. Also, the concept of controlled $K-g-$dual frames is given. Finally, we discuss the stability problem for controlled $K-g-$frames in Hilbert C^*-modules.

1. Introduction and Preliminaires

Frames for Hilbert spaces were introduced by Duffin and Schaefer [2] in 1952 to study some deep problems in nonharmonic Fourier series by abstracting the fundamental notion of Gabor [4] for signal processing.

Many generalizations of the concept of frame have been defined in Hilbert C^*-modules [3, 5, 6, 9, 11–15].

Controlled frames in Hilbert spaces have been introduced by P. Balazs [1] to improve the numerical efficiency of iterative algorithms for inverting the frame operator.

Rashidi and Rahimi [8] are introduced the concept of Controlled frames in Hilbert C^*-modules.

Let $\mathcal{A}$ be a unital C^*-algebra, let I be countable index set. Throughout this paper H and L are countably generated Hilbert $\mathcal{A}-$modules and $\{H_i\}_{i\in I}$ is a sequence of submodules of L. For each $i \in I$, $End^*_{\mathcal{A}}(H, H_i)$ is the collection of all adjointable $\mathcal{A}-$linear maps from H to H_i, and $End^*_{\mathcal{A}}(H, H)$

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is denoted by $End_{\mathcal{A}}^{*}(H)$. Also let $GL^{+}(H)$ be the set of all positive bounded linear invertible operators on H with bounded inverse.

Definition 1.1. *[10] Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-hilbert $\mathcal{A}$-Module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued in product $\langle .,. \rangle_A$:$\mathcal{H} \times \mathcal{H} \rightarrow \mathcal{A}$ such that is sesquilinear, positive definite and respects the module action. In the other words,*

(i) $\langle x, x\rangle_{\mathcal{A}} \geqslant 0$ *for all* $x \in \mathcal{H}$ *and* $\langle x, x\rangle_{\mathcal{A}} = 0$ *if and only if* $x = 0$.
(ii) $\langle ax + y, z\rangle_{\mathcal{A}} = a\langle x, z\rangle_{\mathcal{A}} + \langle y, z\rangle_{\mathcal{A}}$ *for all* $a \in \mathcal{A}$ *and* $x, y, z \in \mathcal{H}$.
(iii) $\langle x, y\rangle_{\mathcal{A}} = \langle y, x\rangle_{\mathcal{A}}^{*}$ *for all* $x, y \in \mathcal{H}$.

For $x \in \mathcal{H}$ *we define* $\|x\| = \|\langle x, x\rangle_{\mathcal{A}}\|^{\frac{1}{2}}$. *If* $\mathcal{H}$ *is complete with* $\|.\|$, *it is called a Hilbert* $\mathcal{A}$*-module or a Hilbert* C^**-module over* $\mathcal{A}$.
For every a in C^**-algebra* $\mathcal{A}$, *we have* $|a| = (a^*a)^{\frac{1}{2}}$ *and the* $\mathcal{A}$*-valued norm on* $\mathcal{H}$ *is defined by* $|x| = (x^*x)^{\frac{1}{2}}$ *for* $x \in \mathcal{H}$.
Let $\mathcal{H}$ *and* $\mathcal{K}$ *be tow Hilbert* $\mathcal{A}$ *modules, A map* $T : \mathcal{H} \rightarrow \mathcal{K}$ *is said to be adjointable if there exists a map* $T^* : \mathcal{K} \rightarrow \mathcal{H}$ *such that* $\langle Tx, y\rangle_{\mathcal{A}} = \langle x, T^*y\rangle_{\mathcal{A}}$ *for all* $y \in \mathcal{K}$ *and* $x \in \mathcal{H}$.

Lemma 1.1. *[18] Suppose that* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *two Hilbert* $\mathcal{A}$*-Modules* $\mathcal{H}$ *and* $L_1 \in End_{\mathcal{A}}^{*}(\mathcal{H}_1, \mathcal{H})$, $L_2 \in End_{\mathcal{A}}^{*}(\mathcal{H}_2, \mathcal{H})$. *Then the following assertions are equivalent:*

(i) $\mathcal{R}(L_1) \subseteq \mathcal{R}(L_2)$,
(ii) $L_1L_1^* \leq \lambda^2 L_2L_2^*$ *for some* $\lambda > 0$,
(iii) *There exists a mapping* $U \in End_{\mathcal{A}}^{*}(\mathcal{H}_1, \mathcal{H}_2)$ *such that* $L_1 = L_2U$.

Moreover, if above conditions are valid, then there exists a unique operator U such that

(i) $\|U\|^2 = \inf\{\alpha > 0 \ L_1L_1^* \leq \alpha L_2L_2^*\}$,
(ii) $\ker(L_1) = \ker(U)$,
(iii) $\mathcal{R}(U) \subseteq \overline{\mathcal{R}(L_2^*)}$.

If an operator U has a closed range, then there exists a right-inverse operator $U^\dagger$, *(pseudo-inverse of U) in the following sense.*

Lemma 1.2. *[17] Let* $U \in End_{\mathcal{A}}^{*}(\mathcal{H}_1, \mathcal{H}_2)$ *be a bounded operator with closed range* $\mathcal{R}(U)$. *Then there exists a bounded operator* $U^\dagger \in End_{\mathcal{A}}^{*}(\mathcal{H}_2, \mathcal{H}_1)$ *for which*

$$UU^\dagger x = x, \qquad x \in \mathcal{R}(U).$$

Lemma 1.3. *[10] Let* $\mathcal{H}$ *and* $\mathcal{K}$ *two Hilbert* $\mathcal{A}$*-module and* $T \in End_{\mathcal{A}}^{*}(\mathcal{H}, \mathcal{K})$. *Then, the following assertions are equivalent:*

(i) *The operator T is bounded and* $\mathcal{A}$*-linear,*
(ii) *There exist* $k > 0$ *such that* $\langle Tx, Tx\rangle_{\mathcal{A}} \leq k\langle x, x\rangle_{\mathcal{A}}$ *for all* $x \in \mathcal{H}$.

Definition 1.2. *[7] A family $\Lambda := \{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ is called a g-frame in Hilbert $\mathcal{A}$ module $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if there exist constants $0 < A \leq B < +\infty$ such that for each $f \in \mathcal{H}$,*

$$A\langle f, f\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle \Lambda_i f, \Lambda_i f\rangle_{\mathcal{A}} \leq B\langle f, f\rangle_{\mathcal{A}}.$$

Theorem 1.1. *[16] Let $\Lambda := \{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ be a g-frame in Hilbert $\mathcal{A}$ module $\mathcal{H}$ with respect to $\mathcal{H}\{_i \in I\}$ if and only if there exist constants $A, B > 0$*

$$A\left\|\langle f, f\rangle_{\mathcal{A}}\right\| \leq \left\|\sum_{i\in I}\langle \Lambda_i f, \Lambda_i f\rangle_{\mathcal{A}}\right\| \leq B\left\|\langle f, f\rangle_{\mathcal{A}}\right\|. \tag{1.1}$$

2. Some Properties of Controlled K-g-Frames

Now, we define controlled K-g-Frames in Hilbert C^*-modules.

Definition 2.1. *Let $C, C' \in \mathcal{GL}^+(\mathcal{H})$ and $K \in End^*_{\mathcal{A}}(\mathcal{H})$, we say that $\Lambda := \{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ is a (C, C')-controlled K-g-frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$ if there exist constants $0 < Acc' < Bcc' < +\infty$ such that for each $f \in \mathcal{H}$,*

$$Acc'\langle K^*f, K^*f\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle \Lambda_i C' f, \Lambda_i C f\rangle_{\mathcal{A}} \leq Bcc'\langle f, f\rangle_{\mathcal{A}}. \tag{2.1}$$

If the right hand of (2.1) holds, Λ is called a (C, C')-controlled $K-g-$Bessel sequence in Hilbert $\mathcal{A}$-module $\mathcal{H}$ with bound B_c.
We call Λ a Parseval $\mathcal{C}, \mathcal{C}'$-controlled K-g-frame if

$$\langle K^*f, K^*f\rangle_{\mathcal{A}} = \sum_{i\in I}\langle \Lambda_i C' f, \Lambda_i C f\rangle_{\mathcal{A}}.$$

If $K = I_{\mathcal{H}}$, then Λ is C, C'-controlled $g-$frame.
For simplicity, we will use a notation CC' instead of C, C'.
If Λ is a CC'-controlled g-frame on Hilbert $\mathcal{A}$-module $\mathcal{H}$, and $C^*\Lambda_i^*\Lambda_i C'$ is positive for all $i \in I$, then for each $f \in \mathcal{H}$,

$$Acc'\langle f, f\rangle_{\mathcal{A}} \leq \|(C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}} f\|^2 \leq Bcc'\langle f, f\rangle_{\mathcal{A}}.$$

Now, let

$$\mathcal{R} := \{(C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}} f : f \in \mathcal{H}\}_{i\in I} \subset (\sum_{i\in I} \oplus H)_{\ell^2}.$$

It is easy to check that $\mathcal{R}$ is a closed subspace of $(\sum_{i\in I} \oplus H)_{\ell^2}$.
Now, we can define the synthesis and analysis operators of the CC'-controlled g-frames as

$$T_{\mathcal{C}\mathcal{C}'} : \mathcal{R} \to \mathcal{H},$$

$$T_{\mathcal{C}\mathcal{C}'}((C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}} f)_{i\in I} = \sum_{i\in I}(C^*\Lambda_i^*\Lambda_i C' f),$$

and

$$T^*_{\mathcal{C}\mathcal{C}'} : \mathcal{H} \to \mathcal{R},$$

$$T^*_{\mathcal{C}\mathcal{C}'}(f) = ((C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}} f)_{i\in I}.$$

Thus, the CC'-controlled g-frame operator is given by

$$\begin{aligned} S_{\mathcal{C}\mathcal{C}'} f &= T_{\mathcal{C}\mathcal{C}'} T^*_{\mathcal{C}\mathcal{C}'} f \\ &= \sum_{i\in I} (C^*\Lambda_i^*\Lambda_i C' f). \end{aligned}$$

$S_{\mathcal{C}\mathcal{C}'}$ is positive, bounded, invertible and self-adjoint. Moreover

$$\langle S_{\mathcal{C}\mathcal{C}'} f, f\rangle = \sum_{i\in I} \langle \Lambda_i C' f, \Lambda_i C f\rangle$$

and

$$A_{\mathcal{C}\mathcal{C}'} I_{\mathcal{H}} < S_{\mathcal{C}\mathcal{C}'} < B_{\mathcal{C}\mathcal{C}'} I_{\mathcal{H}}.$$

Lemma 2.1. *Let $C, C' \in \mathcal{GL}^+(\mathcal{H})$. A sequence Λ is a CC'-controlled g-Bessel sequence in Hilbert $\mathcal{A}$-module with bound $B_{\mathcal{C}\mathcal{C}'}$ if and only if the operator*

$$T_{\mathcal{C}\mathcal{C}'} : \mathcal{R} \to \mathcal{H},$$
$$T_{\mathcal{C}\mathcal{C}'}((C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}} f)_{i\in I} = \sum_{i\in I} (C^*\Lambda_i^*\Lambda_i C' f)$$

is well defined and bounded with $\|T_{CC'}\| \leq \sqrt{B_{CC'}}$.

Proof. We only need to prove the sufficient condition. Let $T_{\mathcal{C}\mathcal{C}'}$ be a well-defined and bounded operator with $\|T_{CC'}\| \leq \sqrt{B_{CC'}}$. For each $f \in H$, we have

$$\begin{aligned} \sum_{i\in I} \langle \Lambda_i C' f, \Lambda_i C f\rangle_{\mathcal{A}} &= \sum_{i\in I} \langle C^*\Lambda_i^*\Lambda_i C' f, f\rangle_{\mathcal{A}} \\ &= \langle \sum_{i\in I} C^*\Lambda_i^*\Lambda_i C' f, f\rangle_{\mathcal{A}} \\ &= \langle T_{CC'}((C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}} f))_{i\in I}, f\rangle_{\mathcal{A}}. \end{aligned}$$

Hence,

$$\begin{aligned} \|\langle T_{CC'}((C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}} f))_{i\in I}, f\rangle_{\mathcal{A}}\| &\leq \|T_{CC'}((C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}} f))_{i\in I}\|\|f\| \\ &\leq \|T_{CC'}\|\|((C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}} f))_{i\in I}\|\|f\|. \end{aligned}$$

But

$$\|((C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}} f)\|^2 = \sum_{i\in I} \langle \Lambda_i C' f, \Lambda_i C f\rangle_{\mathcal{A}},$$
$$\|((C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}} f)\| \leq \|T_{CC'}\|\|f\|,$$
$$\|((C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}} f)\|^2 \leq \|T_{CC'}\|^2\|f\|^2.$$

It follows that

$$\sum_{i\in I} \langle \Lambda_i C' f, \Lambda_i C f\rangle_{\mathcal{A}} \leq Bcc' \|\langle f, f\rangle_{\mathcal{A}}\|.$$

and this means that Λ is a CC'-controlled g-Bessel sequence. □

Lemma 2.2. *Let $C, C' \in \mathcal{GL}^+(\mathcal{H})$. A sequence Λ is a CC'-controlled g-frame sequence in Hilbert $\mathcal{A}$-module if and only if the operator*

$$T_{\mathcal{CC}'} : \mathcal{R} \to \mathcal{H},$$

$$T_{\mathcal{CC}'}((C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}} f) = \sum_{i\in I} C^*\Lambda_i^*\Lambda_i C' f$$

is well defined, bounded and surjective.

Proof. Suppose that Λ is a CC'-controlled g-frame in Hilbert $\mathcal{A}$-module. Since, $S_{\mathcal{CC}'}$ is surjective operator, so $T_{\mathcal{CC}'}$. For the opposite implication, by Lemma 2.1; $T_{\mathcal{CC}'}$ is a well-defined and bounded operator. So Λ is a CC'-controlled g-Bessel sequence. Now, for each $f \in H$, we have $f = T_{CC'}T_{CC'}^{\dagger} f$. Hence

$$\begin{aligned}
\|f\|^4 &= \|\langle f, f\rangle\|^2 \\
&= \|\langle T_{CC'}T_{CC'}^{\dagger} f, f\rangle\|^2 \\
&= \|\langle T_{CC'}^{\dagger} f, T_{CC'}^{*} f\rangle\|^2 \\
&\leq \|\langle T_{CC'}^{\dagger} f, T_{CC'}^{\dagger} f\rangle\|^2 \|\langle T_{CC'}^{*} f, T_{CC'}^{*} f\rangle\|^2 \\
&\leq \|T_{CC'}^{\dagger} f\|^2 \|\|T_{CC'}^{*} f\|^2 \\
&\leq \|T_{CC'}^{\dagger}\|^2 \|f\|^2 \sum_{i\in I}\langle \Lambda_i C' f, \Lambda_i C f\rangle_{\mathcal{A}}.
\end{aligned}$$

We conclude that

$$(\|T_{CC'}^{\dagger}\|^2)^{-1} \|\langle f, f\rangle\| \leq \sum_{i\in I}\langle \Lambda_i C' f, \Lambda_i C f\rangle_{\mathcal{A}}.$$

□

Proposition 2.1. *Let Λ be a CC'-controlled K-g-frames in Hilbert $\mathcal{A}$-module $\mathcal{H}$ and K has a dense range. Suppose that $(C^*\Lambda_i^*\Lambda_i C')$ is positive and also $V_i = (C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}}$ for each $i \in I$. Then $(\bigcap_{i\in I} \ker V_i)^{\perp} = H$.*

Proof. Assume that Acc' and Bcc' are the frame bounds of Λ. Hence,

$$Acc'\langle K^* f, K^* f\rangle_{\mathcal{A}} \leq \|(C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}}\|^2 \leq Bcc'\langle f, f\rangle_{\mathcal{A}}. \tag{2.2}$$

Since $\ker K^* = (\mathcal{R}(K))^{\perp}$ and K has a dense range, K^* injective. Then from (2.2), for each $i \in I$, we get

$$\bigcap_{i\in I} \ker V_i \subseteq \ker K^* = \{0\}.$$

Remark 2.1. *Suppose that Λ is a CC'-controlled K-g-frame in Hilbert $\mathcal{A}$ with lower bound Acc'. Then, we have $S_{CC'} \geqslant Acc'KK^*$, so by Lemma 1.1, there exists an operator $U \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{R})$ such that*

$$T_{CC'}U = K. \tag{2.3}$$

Now, we can obtain optimal frame bounds of Λ by the operator U. Indeed, it is obvious that

$$B_{op} = \|S_{CC'}\| = \|T_{CC'}\|^2.$$

By Lemma 1.1, the equation (2.3) has a unique solution as U_0 such that

$$\begin{aligned}
\|U_0\|^2 &= \inf\{\alpha \geqslant 0 / KK^* \leq \alpha T_{CC'}T^*_{CC'}\} \\
&= \inf\{\alpha \geqslant 0 / \langle KK^*f, f\rangle \leq \alpha \langle T_{CC'}T^*_{CC'}f, f\rangle, f \in \mathcal{H}\} \\
&= \inf\{\alpha \geqslant 0 / \langle K^*f, K^*f\rangle \leq \alpha \langle T^*_{CC'}f, T^*_{CC'}f\rangle, f \in \mathcal{H}\} \\
&= \inf\{\alpha \geqslant 0 / \ \|\langle K^*f, K^*f\rangle\| \leq \alpha \|\langle T^*_{CC'}f, T^*_{CC'}f\rangle\|, f \in \mathcal{H}\} \\
&= \inf\{\alpha \geqslant 0 / \|K^*f\|^2 \leq \alpha \|T^*_{CC'}f\|^2, f \in \mathcal{H}\}.
\end{aligned}$$

Now, we have

$$\begin{aligned}
A_{op} &= \sup\{A > 0 \setminus A\|K^*f\|^2 \leq \|T^*_{CC'}f\|^2, f \in \mathcal{H}\} \\
&= (\inf\{\alpha \geqslant 0 \setminus \|K^*f\|^2 \leq \alpha \|T^*_{CC'}f\|^2, f \in \mathcal{H}\})^{-1} \\
&= U_0^{-2}.
\end{aligned}$$

□

In the following, we consider some proper relations between the operators $U, K \in End^*_{\mathcal{A}}(\mathcal{H})$ and $C, C' \in \mathcal{GL}^+(\mathcal{H})$ and investigate the cases that $\{\Lambda_i U\}_{i\in I}$, $\{\Lambda_i U^*\}_{i\in I}$ can also CC'-controlled K-g-frame. Next, by putting connections between the operators S_Λ, K, C and C', we reach to necessary and sufficient conditions that $\{\Lambda_i\}_{i\in I}$ can be a Parseval CC'-controlled K-g-frames.

Theorem 2.1. *Let Λ be a CC'-controlled K-g- frame in Hilbert $\mathcal{A}$ module $\mathcal{H}$. and $U \in End^*_{\mathcal{A}}(\mathcal{H})$ such that $\mathcal{R}(U) \subset \mathcal{R}(K)$. Then Λ is a CC'-controlled U-g-frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$.*

Proof. Suppose that $A_{CC'}$ is a lower frame bound of Λ. Using Lemma 1.1, there exists $\alpha > 0$ such that $UU^* \leq \alpha^2 KK^*$. Now, for each $f \in \mathcal{H}$. We have $\langle UU^*f, f\rangle_{\mathcal{A}} \leq \alpha^2 \langle KK^*f, f\rangle_{\mathcal{A}}$.
We have

$$\begin{aligned}
\frac{Acc'}{(\alpha^2)}\langle U^*f, U^*f\rangle_{\mathcal{A}} &\leq Acc'\langle K^*f, K^*f\rangle_{\mathcal{A}} \\
&\leq \sum_{i\in I}\langle \Lambda_i C'f, \Lambda_i Cf\rangle_{\mathcal{A}} \\
&\leq Bcc'\langle f, f\rangle_{\mathcal{A}}.
\end{aligned}$$

□

Theorem 2.2. *Let Λ be a CC'-controlled K-g- frame in Hilbert $\mathcal{A}$- module $\mathcal{H}$. Assume that K has a closed range and $U \in End^*_{\mathcal{A}}(\mathcal{H})$ such that $\mathcal{R}(U^*) \subset \mathcal{R}(K)$ Also suppose that U^* commutes with C and C'. Then $\{\Lambda_i U^*\}_{i\in I}$ is a CC'-controlled K-g- frame for $\mathcal{R}(U)$ if and only if there exists $\delta > 0$ such that for each $f \in \mathcal{R}(U)$,*

$$\|U^* f\| \geqslant \delta \|K^* f\|.$$

Proof. Suppose that $\{\Lambda_i U^*\}_{i\in I}$ is a CC'-controlled K-g-frame in Hilbert $\mathcal{A}$ module $\mathcal{H}$ with a lower frame bound $E_{CC'} > 0$. If $B_{CC'}$ is an upper frame bound of Λ then for each $f \in \mathcal{R}(U)$, we have

$$Ecc' \langle K^* f, K^* f\rangle_{\mathcal{A}} \leq \sum_{i\in I} \langle \Lambda_i U^* C' f, \Lambda_i U^* C f\rangle_{\mathcal{A}} = \sum_{i\in I} \langle \Lambda_i C' U^* f, \Lambda_i C U^* f\rangle_{\mathcal{A}},$$

thus

$$Ecc' \langle K^* f, K^* f\rangle_{\mathcal{A}} \leq \sum_{i\in I} \langle \Lambda_i C' U^* f, \Lambda_i C U^* f\rangle_{\mathcal{A}} \leq Bcc' \langle U^* f, U^* f\rangle_{\mathcal{A}},$$

Therefore

$$Ecc' \|\langle K^* f, K^* f\rangle_{\mathcal{A}}\| \leq \|\sum_{i\in I} \langle \Lambda_i C' U^* f, \Lambda_i C U^* f\rangle_{\mathcal{A}}\| \leq Bcc' \|\langle U^* f, U^* f\rangle_{\mathcal{A}}\|$$

thus $Ecc'\|K^* f\|^2 \leq Bcc'\|U^* f\|^2$. so $\sqrt{\frac{Ecc'}{Bcc'}}\|K^* f\| \leq \|U^* f\|$, for the opposite implication, for each $f \in H$, we have

$$\|U^* f\| = \|(K^\dagger)^* K^* U^* f\| \leq \|(K^\dagger)\| \|K^* U^* f\|.$$

Therefore, if $A_{CC'}$ is a lower frame bound of Λ, we have

$$\begin{aligned} A_{CC'} \delta^2 \|K^\dagger\|^{-2} \langle K^* f, K^* f\rangle &\leq A_{CC'} \|K^\dagger\|^{-2} \langle U^* f, U^* f\rangle \\ &\leq A_{CC'} \|K^* U^* f\|^2 \\ &\leq \sum_{i\in I} \langle \Lambda_i U^* C' f, \Lambda_i U^* C f\rangle_{\mathcal{A}}. \end{aligned}$$

For the upper bound, it is clear that

$$\sum_{i\in I} \langle \Lambda_i U^* C' f, \Lambda_i U^* C f\rangle_{\mathcal{A}} \leq Bcc' \langle U^* f, U^* f\rangle_{\mathcal{A}} \leq Bcc' \|U\|^2 \langle f, f\rangle_{\mathcal{A}}.$$

So, $(\Lambda_i U^*)_{i\in I}$ is a CC'-controlled K-g-frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$ with frame bounds $A_{CC'}\delta^2\|K^\dagger\|^{-2}$ and $Bcc'\|U\|^2$. □

Theorem 2.3. *Let Λ be a CC'-controlled K-g-frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$ and the operator K has a dense rang. Assume that $U \in End^*_{\mathcal{A}}(\mathcal{H})$ has a closed range and U and U^* commute with C and C'. If $\{\Lambda_i U^*\}_{i\in I}$ and $\{\Lambda_i U\}_{i\in I}$ are CC'-controlled K-g- frame in Hilbert $\mathcal{A}$- module H, then U is invertible.*

Proof. Suppose that $\{\Lambda_i U^*\}_{i\in I}$ is a CC'-controlled K-g-frame in Hilbert $\mathcal{A}$ module $\mathcal{H}$ with a lower frame bound A_1, and B_1. Then for each $f \in \mathcal{H}$,

$$A_1 \langle K^* f, K^* f\rangle_{\mathcal{A}} \leq \sum_{i\in I} \langle \Lambda_i U^* C' f, \Lambda_i U^* C f\rangle_{\mathcal{A}} \leq B_1 \langle f, f\rangle_{\mathcal{A}}.$$

We have

$$\|A_1\langle K^*f, K^*f\rangle_{\mathcal{A}}\|\leq \|\sum_{i\in I}\langle \Lambda_i U^*C'f, \Lambda_i U^*Cf\rangle_{\mathcal{A}}\|\leq \|B_1\langle f, f\rangle_{\mathcal{A}}\|, \tag{2.4}$$

hence,

$$A_1\|K^*f\|^2\leq \|\sum_{i\in I}\langle \Lambda_i U^*C'f, \Lambda_i U^*Cf\rangle\|\leq B_1\|f\|^2.$$

Since K has a dense range, K^* is injective. Moreover, $\mathcal{R}(U) = (\ker U^*)^{\perp} = H$ so U is surjective. Suppose that $\{\Lambda_i U^*\}_{i\in I}$ is a (CC')-controlled K-g-frame in Hilbert $\mathcal{A}$ module $\mathcal{H}$ with a lower frame bound A_2 and B_2. Then, for each $f \in \mathcal{H}$,

$$A_2\langle K^*f, K^*f\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle \Lambda_i U^*C'f, \Lambda_i U^*Cf\rangle_{\mathcal{A}} \leq B_2\langle f, f\rangle_{\mathcal{A}}$$

$$\|A_2\langle K^*f, K^*f\rangle_{\mathcal{A}}\|\leq \|\sum_{i\in I}\langle \Lambda_i U^*C'f, \Lambda_i U^*Cf\rangle_{\mathcal{A}}\|\leq \|B_2\langle f, f\rangle_{\mathcal{A}}\|$$

$$A_2\|K^*f\|^2\leq \|\sum_{i\in I}\langle \Lambda_i U^*C'f, \Lambda_i U^*Cf\rangle_{\mathcal{A}}\|\leq B_2\|f\|^2.$$

Therefore U is injective, since $\ker U \subseteq \ker K^*$. Thus, U is an invertible operator. □

Theorem 2.4. *Let Λ be a CC'-controlled K-g-frame in Hilbert $\mathcal{A}$- module $\mathcal{H}$ and $U \in End^*_{\mathcal{A}}(\mathcal{H})$ be a co-isometry (i.e. $UU^* = Id_H$) such that $UK = KU$ and U^* commutes with C and C'. Then $\{\Lambda_i U^*\}_{i\in I}$ is a CC'-controlled K-g-frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$.*

Proof. Suppose Λ be a CC'-controlled K-g- frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$ with a lower frame bound $A_{CC'}$. and $B_{CC'}$ for each $f \in \mathcal{H}$, we have

$$\sum_{i\in I}\langle \Lambda_i U^*C'f, \Lambda_i U^*Cf\rangle_{\mathcal{A}} = \sum_{i\in I}\langle \Lambda_i C'U^*f, \Lambda_i CU^*f\rangle_{\mathcal{A}} \leq B_{CC'}\langle U^*f, U^*f\rangle_{\mathcal{A}}$$

hence,

$$\sum_{i\in I}\langle \Lambda_i U^*C'f, \Lambda_i U^*Cf\rangle_{\mathcal{A}} \leq B_{CC'}\|U^*\|^2\langle f, f\rangle_{\mathcal{A}}.$$

So, $\{\Lambda_i U^*\}_{i\in I}$ is a CC'-controlled g-Bessel sequence. For the lower bound, we can write

$$\begin{aligned}\sum_{i\in I}\langle \Lambda_i U^*C'f, \Lambda_i U^*Cf\rangle_{\mathcal{A}} &= \sum_{i\in I}\langle \Lambda_i C'U^*f, \Lambda_i CU^*f\rangle_{\mathcal{A}}\\ &\geqslant A_{CC'}\langle K^*U^*f, K^*U^*f\rangle_{\mathcal{A}}\\ &= A_{CC'}\langle (UK)^*f, (UK)^*f\rangle_{\mathcal{A}}\\ &= A_{CC'}\langle (KU)^*f, (KU)^*f\rangle_{\mathcal{A}}\\ &= A_{CC'}\langle U^*K^*f, U^*K^*f\rangle_{\mathcal{A}}\\ &= A_{CC'}\langle UU^*K^*f, U^*K^*f\rangle_{\mathcal{A}}\\ &= A_{CC'}\langle K^*f, K^*f\rangle_{\mathcal{A}}.\end{aligned}$$

□

Theorem 2.5. *Let* $\Lambda := \{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ *and* $\circleddash := \{\circleddash_i \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ *be tow* CC'*-controlled* $K-g-$ *Bessel sequences in Hilbert* $\mathcal{A}$*- module* $\mathcal{H}$ *with bounds* B_Λ *and* $B_\circleddash$ *respectively. Suppose that* $T_{\Lambda,C,C'}$ *and* $T_{\circleddash,CC'}$ *are their synthesis operators such that* $T_{\circleddash,CC'}T^*_{\Lambda,C,C'} = K^*$*. Then* Λ *and* $\circleddash$ *are* CC'*-controlled* K *and* K^**-g-frames, respectively.*

Proof.

$$\begin{aligned}\|K^*f\|^4 &= \|\langle K^*f, K^*f\rangle_{\mathcal{A}}\|^2\\ &= \|\langle T_{\circleddash,CC'}T^*_{\Lambda,C,C'}f, K^*f\rangle_{\mathcal{A}}\|^2\\ &\leq \|T^*_{\Lambda,C,C'}f\|^2\|T^*_{\circleddash,CC'}K^*f\|^2\\ &= \sum_{i\in I}\langle \Lambda_iC'f, \Lambda_iCf\rangle_{\mathcal{A}}\sum_{i\in I}\langle \circleddash_iC'K^*f, \circleddash_iC'K^*f\rangle_{\mathcal{A}}\\ &\leq \sum_{i\in I}\langle \Lambda_iC'f, \Lambda_iCf\rangle_{\mathcal{A}}B_\circleddash\|\langle K^*f, K^*f\rangle_{\mathcal{A}}\|.\end{aligned}$$

So,

$$\|\langle K^*f, K^*f\rangle_{\mathcal{A}}\|\leq \sum_{i\in I}\langle \Lambda_iC'f, \Lambda_iCf\rangle_{\mathcal{A}}B_\circleddash$$

□

Thus

$$B_\circleddash^{-1}\|\langle K^*f, K^*f\rangle_{\mathcal{A}}\|\leq \sum_{i\in I}\langle \Lambda_iC'f, \Lambda_iCf\rangle_{\mathcal{A}}.$$

This that Λ is a CC'-controlled K-g-frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$ with frame operator S_Λ. For each $f \in \mathcal{A}$, we have $T_{\Lambda,C,C'}T^*_{\circleddash,CC'} = K$

$$\begin{aligned}\|Kf\|^4 &= \|\langle Kf, Kf\rangle_{\mathcal{A}}\|^2\\ &= \|\langle T_{\Lambda,C,C'}T^*_{\circleddash,CC'}f, Kf\rangle_{\mathcal{A}}\|^2\\ &\leq \|T^*_{\Lambda,C,C'}Kf\|^2\|T^*_{\circleddash,CC'}f\|^2\\ &= \sum_{i\in I}\langle \Lambda_iC'Kf, \Lambda_iCKf\rangle_{\mathcal{A}}\sum_{i\in I}\langle \circleddash_iC'f, \circleddash_iC'f\rangle_{\mathcal{A}}\\ &\leq \sum_{i\in I}\langle \circleddash_iC'f, \circleddash_iCf\rangle_{\mathcal{A}}B_\Lambda\|\langle Kf, Kf\rangle_{\mathcal{A}}\|.\end{aligned}$$

Thus

$$B_\Lambda^{-1}\|\langle Kf, Kf\rangle_{\mathcal{A}}\|\leq \sum_{i\in I}\langle \circleddash_iC'f, \circleddash_iCf\rangle_{\mathcal{A}}.$$

This that $\circleddash$ is a CC'-controlled K-g-frame in Hilbert $\mathcal{A}$- module $\mathcal{H}$.

Theorem 2.6. *Let* Λ *be a g-frame in Hilbert* $\mathcal{A}$*- module* $\mathcal{H}$ *with frame operator* S_Λ*. Also assume that* Λ *is a* CC'*-controlled g- Bessel sequence with frame operator* $S_{CC'}$*. Then* Λ *is a Parseval* CC'*-controlled* K*-g- frame in Hilbert* $\mathcal{A}$*-module* $\mathcal{H}$ *if and only if* $C = (S_\Lambda^{-p})^*\Phi$ *and* $C' = (S_\Lambda^{-q})\Psi$ *where* Φ*,* Ψ *are two operators in Hilbert* $\mathcal{A}$*- module* $\mathcal{H}$ *such that* $\Phi^*\Psi = KK^*$ *and* $p + q = 1$ *where* $p, q \in \mathbb{R}$*.*

Proof. Assume that Λ is a Parseval CC'-controlled K-g-frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$,

$$\begin{aligned}\sum_{i\in I}\langle \Lambda_i C' f, \Lambda_i C f\rangle_{\mathcal{A}} &= \langle K^* f, K^* f\rangle_{\mathcal{A}} \\ &= \sum_{i\in I}\langle f, C^*\Lambda_i^*\Lambda_i C' f\rangle_{\mathcal{A}} \\ &= \langle f, \sum_{i\in I} C^*\Lambda_i^*\Lambda_i C' f\rangle_{\mathcal{A}} \\ &= \langle f, S_{CC'} f\rangle_{\mathcal{A}} \\ &= \langle f, KK^* f\rangle_{\mathcal{A}}\end{aligned}$$

$$\begin{aligned}S_{CC'}(f) &= \sum_{i\in I} C^*\Lambda_i^*\Lambda_i C'(f) \\ &= C^*(\sum_{i\in I}\Lambda_i^*\Lambda_i C')(f) \\ &= C^* S_\Lambda C'(f).\end{aligned}$$

Hence $S_{CC'} = C^* S_\Lambda C'$ and $S_{CC'} = KK^*$. Therefore, for each $p, q \in \mathbb{R}$ such that $p+q=1$, we obtain

$$KK^* = C^* S_\Lambda^p S_\Lambda^q C'.$$

We define $\Phi = (S_\Lambda^p)^* C$ and $\Psi = (S_\Lambda^q)^* C'$ So

$$\Phi^*\Psi = C^* S_\Lambda^p S_\Lambda^q C' = KK^*.$$

Conversely, let Φ and Ψ be tow operators in Hilbert $\mathcal{A}$- module $\mathcal{H}$ such that $\Phi^*\Psi = KK^*$. Suppose that $C = (S_\Lambda^{-p})^*\Phi$ and $C' = (S_\Lambda^{-q})^*\Psi$ are tow operators on Hilbert $\mathcal{A}$- module $\mathcal{H}$ where$p, q \in \mathbb{R}$ and $p+q=1$, Since

$$KK^* = \Phi^*\Psi = C^* S_\Lambda^p S_\Lambda^q C' = C^* S_\Lambda C' = S_{CC'}.$$

So, for each $f \in \mathcal{H}$,

$$\langle KK^* f, f\rangle_{\mathcal{A}} = \langle K^* f, K^* f\rangle_{\mathcal{A}} = \langle \sum_{i\in I} C^*\Lambda_i^*\Lambda_i C' f, f\rangle_{\mathcal{A}}.$$

Thus Λ is Parseval CC'-controlled $k-g-$ frame on Hilbert $\mathcal{A}$- module $\mathcal{H}$. □

3. Duals of Controlled K-g-Frames

In this section, by the concept of K-g- dual pair, we present a bounded operator called dual operator and propose some known equalities and inequalities between dual operator CC'-controlled K-g-frame in Hilbert $\mathcal{A}$- module $\mathcal{H}$.

Definition 3.1. *Suppose that Λ is CC'-controlled k-g-frame on Hilbert $\mathcal{A}$- module $\mathcal{H}$ with synthesis operator $T_{\Lambda,C,C'}$ Then $\widetilde{\Lambda} := \{\widetilde{\Lambda}_i \in End^*_{\mathcal{A}}(\mathcal{H},\mathcal{H}_i)\}_{i\in I}$ is called a CC'-controlled $k-g-$ dual frame (or brevity$CC'-Kg-$ dual) for Λ if*

$$T_{\Lambda,C,C'}T^*_{\widetilde{\Lambda},C,C'} = K, \tag{3.1}$$

and $\widetilde{\Lambda}$ is a CC'-controlled $K-g-$ Bessel sequence. In this cas, $(\Lambda,\widetilde{\Lambda})$ is called a CC'-controlled $K-g-$ dual pair. The following results presents equivalent conditions of the CC'-K-g-dual.

Proposition 3.1. *Let $\widetilde{\Lambda}$ be a $CC'-K-g-$ dual for Λ. Then the following conditions are equivalent :*

(i) $T_{\Lambda,C,C'}T^*_{\widetilde{\Lambda},C,C'} = K$
(ii) $T_{\widetilde{\Lambda},C,C'}T^*_{\Lambda,C,C'} = K^*$
(iii) *for each $f, f' \in \mathcal{H}$, we have*

$$\langle Kf; f'\rangle = \langle T^*_{\widetilde{\Lambda},C,C'}f, T^*_{\widetilde{\Lambda},C,C'}f\rangle.$$

Theorem 3.1. *If $\widetilde{\Lambda}$ be a $CC'-K-g-$ dual for Λ, then $\widetilde{\Lambda}$ is a CC'-controlled K^*-g- frame in Hilbert $\mathcal{A}$- module $\mathcal{H}$.*

Proof. We have

$$\begin{aligned}
\|Kf\|^4 &= \|\langle Kf, Kf\rangle_{\mathcal{A}}\|^2 \\
&= \|\langle T_{\Lambda,C,C'}T^*_{\widetilde{\Lambda},C,C'}f, Kf\rangle\|^2 \\
&= \|\langle T^*_{\widetilde{\Lambda},C,C'}f, T^*_{\Lambda,C,C'}\rangle\|^2 \\
&\leq \|T^*_{\widetilde{\Lambda},C,C'}f\|^2\|T^*_{\Lambda,C,C'}Kf\|^2 \\
&\leq (\sum_{i\in I}\langle\widetilde{\Lambda}_iC'f, \widetilde{\Lambda}_iCf\rangle_{\mathcal{A}})(\sum_{i\in I}\langle\Lambda_iC'Kf, \Lambda_iCKf\rangle_{\mathcal{A}}) \\
&\leq B_C\|Kf\|^2(\sum_{i\in I}\langle\widetilde{\Lambda}_iC'f, \widetilde{\Lambda}_iCf\rangle_{\mathcal{A}}),
\end{aligned}$$

It follows that

$$B_C^{-1}A_{CC'}\|\langle Kf, Kf\rangle_{\mathcal{A}}\| \leq \sum_{i\in I}\langle\widetilde{\Lambda}_iC'f, \widetilde{\Lambda}_iCf\rangle_{\mathcal{A}} \leq B_{CC'}\|\langle f, f\rangle_{\mathcal{A}}\|.$$

Therefore, $\widetilde{\Lambda}$ is a CC'-controlled K^*-g- frame in Hilbert $\mathcal{A}$- module $\mathcal{H}$. □

Theorem 3.2. *Assume that C_{OP} and D_{OP} are the optimal bounds of $\widetilde{\Lambda}$, respectively. Then*

$$C_{OP} \geqslant B^{-1}_{op}, \qquad D_{op} \geqslant A^{-1}_{op},$$

for which A_{op} and B_{op} are the optimal bounds of Λ, respectively. Assume $(\Lambda,\widetilde{\Lambda})$ is called a CC'-controlled $K-g-$ dual pair and $\mathcal{J} \subset \mathcal{I}$. We define

$$S_{\mathcal{J}}f := \sum_{i\in\mathcal{J}}(C^*\Lambda^*_i\Lambda_iC')^{\frac{1}{2}}(C^*\widetilde{\Lambda^*_i}\widetilde{\Lambda}_iC')^{\frac{1}{2}}f, f \in \mathcal{H},$$

and we call it a dual operator.

It is clear that $S_{\mathcal{J}} \in End_{\mathcal{A}}^{*}(\mathcal{H})$ *and* $S_{\mathcal{J}} + S_{\mathcal{J}^c} = K$ *where* $\mathcal{J}^c$ *is the complement of* $\mathcal{J}$. *If* B_1*and* B_2 *are the bounds of* Λ *and* $\widetilde{\Lambda}$ *respectively, then, we have*

$$\begin{aligned}\|S_{\mathcal{J}}f\|^2 &= (\sup_{\|g\|=1}\|\langle S_{\mathcal{J}}f, g\|\rangle)^2\\ &\leq (\sup_{\|g\|=1}(\sum_{i\in\mathcal{J}}\|\langle(C^*\Lambda_i^*\Lambda_iC')^{\frac{1}{2}}(C^*\widetilde{\Lambda_i^*}\widetilde{\Lambda}_iC')^{\frac{1}{2}}f\rangle\|)^2\\ &\leq (\sum_{i\in\mathcal{I}}\|(C^*\Lambda_i^*\Lambda_iC')^{\frac{1}{2}}f\|^2)((\sup_{\|g\|=1}(\sum_{i\in\mathcal{J}}\|C^*\widetilde{\Lambda_i^*}\widetilde{\Lambda}_iC')^{\frac{1}{2}}\|)^2\\ &\leq B_1B_2\|f\|^2.\end{aligned}$$

So $S_{\mathcal{J}}$ *is bounded. Now, by that operator* $S_{\mathcal{J}}$ *we extend some well known equalities and inequalities for controlled* K*-g- frames in the following theorems.*

Theorem 3.3. *If* $f \in \mathcal{H}$ *then* $(\sum_{i\in\mathcal{J}}\langle(C^*\widetilde{\Lambda_i^*}\widetilde{\Lambda}_iC')^{\frac{1}{2}}f, (C^*\Lambda_i^*\Lambda_iC')^{\frac{1}{2}}Kf\rangle - \|S_{\mathcal{J}}f\|^2 = (\sum_{i\in\mathcal{J}^c}\overline{\langle C^*\widetilde{\Lambda_i^*}\widetilde{\Lambda}_iC')^{1/2}f, (C^*\Lambda_i^*\Lambda_iC')^{1/2}Kf\rangle} - \|S_{\mathcal{J}^c}f\|^2$.

Proof. Let $f \in \mathcal{H}$. We can write

$$\begin{aligned}(\sum_{i\in\mathcal{J}}\langle(C^*\widetilde{\Lambda_i^*}\widetilde{\Lambda}_iC')^{\frac{1}{2}}f, (C^*\Lambda_i^*\Lambda_iC')^{\frac{1}{2}}Kf\rangle - \|S_{\mathcal{J}}f\|^2 &= \langle K^*S_{\mathcal{J}}f, f\rangle - \|S_{\mathcal{J}}f\|^2\\ &= \langle K^*S_{\mathcal{J}}f, f\rangle - \langle S_{\mathcal{J}}^*S_{\mathcal{J}}f, f\rangle\\ &= \langle (K - S_{\mathcal{J}})^*S_{\mathcal{J}}f, f\rangle\\ &= \langle S_{\mathcal{J}^c}^*(K - S_{\mathcal{J}}), f\rangle\\ &= \langle S_{\mathcal{J}^c}^*Kf, f\rangle - \langle S_{\mathcal{J}^c}^*S_{\mathcal{J}^c}f, f\rangle\\ &= \langle Kf, S_{\mathcal{J}^c}f\rangle - \langle S_{\mathcal{J}^c}f, S_{\mathcal{J}^c}f\rangle\\ &= \overline{\langle S_{\mathcal{J}^c}f, Kf\rangle} - \|S_{\mathcal{J}^c}f\|^2\\ &= (\sum_{i\in\mathcal{J}^c}\overline{\langle(C^*\widetilde{\Lambda_i^*}\widetilde{\Lambda}_iC')^{\frac{1}{2}}f, (C^*\Lambda_i^*\Lambda_iC')^{\frac{1}{2}}Kf\rangle}\\ &- \|S_{\mathcal{J}^c}f\|^2.\end{aligned}$$

□

Theorem 3.4. *Let* Λ *be a Parseval* CC'*-controlled* K*-g-frame in Hilbert* $\mathcal{A}$*-module* $\mathcal{H}$ *if* $J \subseteq I$ *and* $E \subseteq J^c$, *then for each* $f \in \mathcal{H}$,

$$\begin{aligned}&\|\sum_{i\in J\cup E}(C^*\Lambda_i^*\Lambda_iC')f\|^2 - \|\sum_{i\in J^c\setminus E}(C^*\Lambda_i^*\Lambda_iC')f\|^2\\ &= \|\sum_{i\in J}(C^*\Lambda_i^*\Lambda_iC')f\|^2 - \|\sum_{i\in J^c}(C^*\Lambda_i^*\Lambda_iC')f\|^2 + 2Re(\sum_{i\in E}\langle\Lambda_iC'f, \Lambda_iC^*KK^*f\rangle).\end{aligned}$$

Proof. Let

$$S_{\Lambda,J}f=\sum_{i\in J}(C^*\Lambda_i^*\Lambda_iC')f,$$

therefore, $S_{\Lambda,I}+S_{\Lambda,I^c}=KK^*$.
Hence

$$\begin{aligned}S_{\Lambda,J}^2-S_{\Lambda,J^c}^2&=S_{\Lambda,J}^2-(KK^*-S_{\Lambda,J})^2\\&=KK^*S_{\Lambda,J}+S_{\Lambda,J}KK^*-(KK^*)^2\\&=KK^*S_{\Lambda,J}-S_{\Lambda,J^c}KK^*.\end{aligned}$$

Now, for each $f\in H$, we obtain

$$\|S_{\Lambda,J}^2\|^2-\|S_{\Lambda,J^c}^2\|^2=\langle KK^*S_{\Lambda,J}f,f\rangle-\langle S_{\Lambda,J^c}KK^*f,f\rangle,$$

consequently, for $J\cup E$ instead of J:

$$\begin{aligned}&\|\sum_{i\in J\cup E}(C^*\Lambda_i^*\Lambda_iC')f\|^2-\|\sum_{i\in J^c\setminus E}(C^*\Lambda_i^*\Lambda_iC')f\|^2\\&=(\sum_{i\in J\cup E}\langle\Lambda_iC'f,\Lambda_iC^*KK^*f\rangle)-\sum_{i\in J^c\setminus E}\overline{\langle\Lambda_iC'f,\Lambda_iC^*KK^*f\rangle}\\&=(\sum_{i\in J}\langle\Lambda_iC'f,\Lambda_iC^*KK^*f\rangle)-\sum_{i\in J^c}\overline{\langle\Lambda_iC'f,\Lambda_iC^*KK^*f\rangle}+2Re(\sum_{i\in E}\langle\Lambda_iC'f,\Lambda_iC^*KK^*f\rangle)\\&=\sum_{i\in J}(C^*\Lambda_i^*\Lambda_iC')f\|^2-\|\sum_{i\in J^c}(C^*\Lambda_i^*\Lambda_iC')f\|^2+2Re(\sum_{i\in E}\langle\Lambda_iC'f,\Lambda_iC^*KK^*f\rangle).\end{aligned}$$

□

Theorem 3.5.

Let Λ be a Parseval CC'-controlled K-g-frame in Hilbert $\mathcal{A}$- module $\mathcal{H}$ if $J\subseteq I$, then for each $f\in\mathcal{H}$,

$$\begin{aligned}&\|\sum_{i\in J}(C^*\Lambda_i^*\Lambda_iC')f\|^2+Re\left(\sum_{i\in J^c}\langle\Lambda_iC'f,\Lambda_iC^*KK^*f\rangle\right)\\&=\|\sum_{i\in J^c}(C^*\Lambda_i^*\Lambda_iC')f\|^2+Re\left(\sum_{i\in J}\langle\Lambda_iC'f,\Lambda_iC^*KK^*f\rangle\right)\geqslant\frac{3}{4}\|KK^*f\|^2.\end{aligned}$$

Proof. using the the proof of Theorem 3.4, we have

$$S_{\Lambda,J}^2-S_{\Lambda,J^c}^2=KK^*S_{\Lambda,J}-S_{\Lambda,J^c}KK^*.$$

Therefore

$$S_{\Lambda,J}^2+S_{\Lambda,J^c}^2=2\left(\frac{KK^*}{2}-S_{\Lambda,J}\right)^2+\frac{(KK^*)^2}{2}\geqslant\frac{(KK^*)^2}{2}.$$

Thus

$$KK^*S_{\Lambda,J}+S^2_{\Lambda,J^c}+(KK^*S_{\Lambda,J}+S^2_{\Lambda,J^c})^*=KK^*S_{\Lambda,J}+S^2_{\Lambda,J^c}+S_{\Lambda,J}KK^*+S^2_{\Lambda,J^c}$$
$$=KK^*(S_{\Lambda,J}+S_{\Lambda,J^c})+S^2_{\Lambda,J}+S^2_{\Lambda,J^c}\geqslant\frac{3}{4}(KK^*)^2.$$

Now, for each $f\in H$, we obtain

$$\|\sum_{i\in J}(C^*\Lambda_i^*\Lambda_iC')f\|^2+Re(\sum_{i\in J}\langle\Lambda_iC'f,\Lambda_iC^*KK^*f\rangle)$$
$$=\langle KK^*S_{\Lambda,J}f,f\rangle+\langle S^2_{\Lambda,J^c}f,f\rangle+\langle KK^*+S^2_{\Lambda,J^c}f,f\rangle+\langle f,S^2_{\Lambda,J^c}f\rangle\geqslant\frac{3}{4}(KK^*)^2.$$

□

4. The stability problem of controlled $K-g-$frames

Stability of the wavelet and Gabor frames under perturbation is one of the important problems in frame theory. At first this problem was studied by Paley and Wienes for bases and then extended to frames.But the most important results are obtained by Casazza and Christensen. Here we study the perturbation of CC'-controlled K-g-frames.in Hilbert $\mathcal{A}$-module $\mathcal{H}$.

Theorem 4.1. *Let Λ be a CC'-controlled K-g- frame on Hilbert $\mathcal{A}$- module $\mathcal{H}$ with bounds $A_{CC'}$ and $A_{CC'}$. Assume that $\ominus:=\{\ominus_i\in End^*_{\mathcal{A}}(\mathcal{H},\mathcal{H}_i)_{i\in I}\}$ is a sequence of operators such that for each $f\in H$ and $i\in I$,*

$$\|(C^*\Lambda_i^*\Lambda_iC'-C^*\ominus_i^*\ominus_iC')^{1/2}f\|$$
$$\leq\lambda_1\|(C^*\Lambda_i^*\Lambda_iC')^{1/2}f\|+\lambda_2\|C^*\ominus_i^*\ominus_iC')^{1/2}f\|+c_i\langle K^*f,K^*f\rangle^{\frac{1}{2}}$$

where $\{c_i\}_{i\in I}$ is a sequence of positive numbers such that $\eta:=\sum_{i\in I}c_i^2<\infty$ and $0\leq\lambda_1,\ \lambda_2\leq 1$. Then $\ominus$ is a CC'-controlled $k-g-$ frame on Hilbert $\mathcal{A}$-module $\mathcal{H}$ with bounds:

$$\left(\frac{(1-\lambda_1)\sqrt{A_{CC'}}-\eta}{1+\lambda_2}\right)^2,\left(\frac{(1+\lambda_1)\sqrt{B_{CC'}}+\eta\|K\|}{1-\lambda_2}\right)^2.$$

Proof. For each $f\in H$, we have

$$\|C^*\ominus_i^*\ominus_iC')^{1/2}f\|=\|(C^*\ominus_i^*\ominus_iC'-C^*\Lambda_i^*\Lambda_iC')^{1/2}f+(C^*\Lambda_i^*\Lambda_iC')^{1/2}f\|$$

$$\leq\|(C^*\ominus_i^*\ominus_iC'-C^*\Lambda_i^*\Lambda_iC')^{1/2}f\|+(C^*\Lambda_i^*\Lambda_iC')^{1/2}f\|$$
$$\leq\lambda_1\|(C^*\Lambda_i^*\Lambda_iC')^{1/2}f\|+\lambda_2\|C^*\ominus_i^*\ominus_iC')^{1/2}f\|+c_i\langle K^*f,K^*f\rangle^{\frac{1}{2}}+\|+(C^*\Lambda_i^*\Lambda_iC')^{1/2}f\|.$$

Hence

$$(1-\lambda_2)\|(C^*\ominus_i^*\ominus_iC')^{1/2}f\|\leq(1+\lambda_1)\|(C^*\Lambda_i^*\Lambda_iC')^{1/2}f\|+c_i\langle K^*f,K^*f\rangle^{\frac{1}{2}}$$

Since Λ is a CC'-controlled K-g-frame, so

$$\begin{aligned}\|T^*_{CC'}\|^2 &= \|(C^*\Lambda_i^*\Lambda_i C')^{1/2}f\|^2\\ &= \sum_{i\in I}\langle \Lambda_i C' f, \Lambda_i C f\rangle_{\mathcal{A}}\\ &\leq Bcc'\langle f, f\rangle_{\mathcal{A}}.\end{aligned}$$

Therefore

$$\|(C^*\ominus_i^*\ominus_i C')^{1/2}f\| \leq \frac{(1+\lambda_1)\|(C^*\Lambda_i^*\Lambda_i C')^{1/2}f\| + c_i\langle K^*f, K^*f\rangle^{\frac{1}{2}}}{1-\lambda_2},$$

$$\|((C^*\ominus_i^*\ominus_i C')^{1/2}f\|^2 \leq (\frac{(1+\lambda_1)\sqrt{Bcc'}+\eta\|K\|}{1-\lambda_2}))^2\langle f, f\rangle_{\mathcal{A}}.$$

Now, for the lower bound we get

$$\begin{aligned}\|(C^*\ominus_i^*\ominus_i C')^{1/2}\| &= \|C^*\Lambda_i^*\Lambda_i C')^{1/2}f - (C^*\Lambda_i^*\Lambda_i C' - C^*\ominus_i^*\ominus_i C')^{1/2}f\|\\ &\geqslant \|C^*\Lambda_i^*\Lambda_i C')^{1/2}f\| - \|(C^*\Lambda_i^*\Lambda_i C' - C^*\ominus_i^*\ominus_i C')^{1/2}f\|\\ &\geqslant \|C^*\Lambda_i^*\Lambda_i C')^{1/2}f\| - \lambda_1\|(C^*\Lambda_i^*\Lambda_i C')^{1/2}f\|\\ &\qquad - \lambda_2\|C^*\ominus_i^*\ominus_i C')^{1/2}f\| - c_i\langle K^*f, K^*f\rangle^{\frac{1}{2}}.\end{aligned}$$

Therefore

$$(1+\lambda_2)\|(C^*\ominus_i^*\ominus_i C')^{1/2}f\| \geqslant (1-\lambda_1)\|(C^*\Lambda_i^*\Lambda_i C')^{1/2}f\| - c_i\langle K^*f, K^*f\rangle^{\frac{1}{2}}$$

or

$$\|(C^*\ominus_i^*\ominus_i C')^{1/2}f\| \geqslant \frac{(1-\lambda_1)\|(C^*\Lambda_i^*\Lambda_i C')^{1/2}f\| - c_i\langle K^*f, K^*f\rangle^{\frac{1}{2}}}{(1+\lambda_2)}.$$

Since,

$$\|T^*_{CC'}\|^2 = \|(C^*\Lambda_i^*\Lambda_i C')^{1/2}f\|^2 = \sum_{i\in I}\langle \Lambda_i C' f, \Lambda_i C f\rangle_{\mathcal{A}} \geqslant Acc'\langle K^*f, K^*f\rangle_{\mathcal{A}}.$$

Thus

$$\|(C^*\Lambda_i^*\Lambda_i C')^{1/2}f\|^2 \geqslant (\frac{(1-\lambda_1)\sqrt{Acc'}-\eta}{(1+\lambda_2)})^2\langle K^*f, K^*f\rangle_{\mathcal{A}}.$$

□

Proposition 4.1. *Let Λ be a CC'-controlled $k-g-$ frame on Hilbert $\mathcal{A}$- module $\mathcal{H}$ with bounds $A_{CC'}$ and $B_{CC'}$. Assume that $\ominus := \{\ominus_i \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_i)_{i\in I}\}$ is a sequence of operators such that for each $f \in H$ and $i \in I$,*

$$\|(C^*\Lambda_i^*\Lambda_i C' - C^*\ominus_i^*\ominus_i C')^{1/2}f\| \leq c_i\langle K^*f, K^*f\rangle^{\frac{1}{2}}.$$

where $\{c_i\}_{i\in I}$ is a sequence of positive numbers such that $\eta := \sum_{i\in I} c_i^2 < \infty$. Then $\ominus$ is a CC'-controlled $k-g-$ frame on Hilbert $\mathcal{A}$- module $\mathcal{H}$ with bounds :

$$(\sqrt{A_{CC'}}-\eta)^2, (\sqrt{B_{CC'}}+\eta\|K\|)^2.$$

Proof. For each $f \in H$, we have

$$\begin{aligned}\|(C^* \ominus_i^* \ominus_i C')^{1/2} f\| &= \|C^*\Lambda_i^*\Lambda_i C')^{1/2} f - (C^*\Lambda_i^*\Lambda_i C' - C^* \ominus_i^* \ominus_i C')^{1/2} f\| \\ &\geqslant \|C^*\Lambda_i^*\Lambda_i C')^{1/2} f\| - \|(C^*\Lambda_i^*\Lambda_i C' - C^* \ominus_i^* \ominus_i C')^{1/2} f\| \\ &\geqslant \sqrt{A_{CC'}} \langle K^* f, K^* f\rangle^{\frac{1}{2}} - \eta \langle K^* f, K^* f\rangle^{\frac{1}{2}} \\ &\geqslant (\sqrt{A_{CC'}} - \eta)\langle K^* f, K^* f\rangle^{\frac{1}{2}}.\end{aligned}$$

Thus

$$\|(C^* \ominus_i^* \ominus_i C')^{1/2} f\|^2 \geqslant (\sqrt{A_{CC'}} - \eta)^2 \langle K^* f, K^* f\rangle_{\mathcal{A}}.$$

On the other hand

$$\begin{aligned}\|C^* \ominus_i^* \ominus_i C')^{1/2} f\| &= \|(C^* \ominus_i^* \ominus_i C' - C^*\Lambda_i^*\Lambda_i C')^{1/2} f + (C^*\Lambda_i^*\Lambda_i C')^{1/2} f\| \\ &\leq \|(C^* \ominus_i^* \ominus_i C' - C^*\Lambda_i^*\Lambda_i C')^{1/2} f\| + \|(C^*\Lambda_i^*\Lambda_i C')^{1/2} f\| \\ &\leq \sqrt{B_{CC'}} \langle f, f\rangle^{\frac{1}{2}} + \eta \langle K^* f, K^* f\rangle_{\mathcal{A}}^{\frac{1}{2}} \\ &\leq (\sqrt{B_{CC'}} + \eta\|K\|)\langle f, f\rangle_{\mathcal{A}}^{\frac{1}{2}}.\end{aligned}$$

Thus

$$\|(C^* \ominus_i^* \ominus_i C')^{1/2} f\|^2 \leq (\sqrt{B_{CC'}} + \eta\|K\|)^2 \langle f, f\rangle_{\mathcal{A}}.$$

□

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