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COMMON FIXED POINT OF FOUR MAPS IN S_m-METRIC SPACE

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Abstract. In this paper, first, we deal with new metric space S_m-metric space that combines multiplicative metric space and S-metric space. We generate a common fixed point theorem in a S_m-metric space using the notions of reciprocally continuous mappings, faintly compatible mappings and occasionally weakly compatible mappings (OWC). We are also studying the well-posedness of S_m metric space. Further, some examples are presented to support our outcome.

1. Introduction

The idea of Multiplicative metric space(MMS for short) was first introduced by Bashirove [1] in 2008.Ozaksar and Cevical [2] investigated and proved the properties of MMS. Following that, several theorems like [3] and [4] in this area of MMS were developed. Sedhi.S et al. [5] introduced a new structure of S-metric space and developed some fixed point theorems. Pant et al. [6] used the concept of reciprocally continuous mappings which is weaker than continuous mappings. In this article, we use the multiplicative metric space and S metric space and generated a new S_m-metric space [7]. We used the concept of occasionally weakly compatible (OWC for shot) [9]mappings, reciprocally continuous and faintly compatible mappings [10] to generate a common fixed point theorem in S_m-metric space. We also discuss the well-posedness property [11] in S_m-metric space. Furthermore, some examples are provided to support our new findings.

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2. Mathematical Preliminaries:

Definition 2.1. *[1] " Let $X \neq \phi$. An operator $\delta : X^2 \to \mathbb{R}_+$ be a multiplicative metric space (MMS) holding in the conditions below:*

(M1) $\delta(\alpha, \beta) \geq 1$, *and* $\delta(\alpha, \beta) = 1 \iff \alpha = \beta$

(M2) $\delta(\alpha, \beta) = \delta(\beta, \alpha)$

(M3) $\delta(\alpha, \beta) \leq \delta(\alpha, \gamma)\delta(\gamma, \beta), \forall \alpha, \beta, \gamma \in X$.

Mapping δ together with X, (X, δ) is called a MMS".

A three-dimensional metric space was proposed by Sedghi et al . [5], and it is called S-metric space.

Definition 2.2. *[5] " Let $X \neq \phi$ defined on a function $S : X^3 \to [0, \infty)$ satisfying:*

(S1) $S(\alpha, \beta, \gamma) \geq 0$;

(S2) $S(\alpha, \beta, \gamma) = 0; \iff \alpha = \beta = \gamma$,

(S3) $S(\alpha, \beta, \gamma) \leq S(\alpha, \alpha, \omega) + S(\beta, \beta, \omega) + S(\gamma, \gamma, \omega), \forall \alpha, \beta, \gamma, \omega \in X$.

The pair (X, S) is known as S-metric space on X ".

We now present the concept of S_m-metric space which is consolidation of multiplicative metric space defined by Bashirov [1] and S-metric space defined by Sedgi [5] by as follows

Definition 2.3. *[7] " Let $X \neq \phi$. A function $S_m : X^3 \to \mathbb{R}_+$ holding the conditions below:*

(MS1) $S_m(\alpha, \beta, \gamma) \geq 1$

(MS2) $S_m(\alpha, \beta, \gamma) = 1 \iff \alpha = \beta = \gamma$

(MS3) $S_m(\alpha, \beta, \gamma) \leq S_m(\alpha, \alpha, \omega)S_m(\beta, \beta, \omega)S_m(\gamma, \gamma, \omega), \forall \alpha, \beta, \gamma, \omega \in X$.

Mapping S_m together with X, (X, S_m) is known as S_m-metric space."

Example 2.1. *" Let $X \neq \phi, S_m : X^3 \to [0, \infty)$ by*
$S_m(\alpha, \beta, \gamma) = a^{|\alpha-\gamma|+|\beta-\gamma|}$, *where $\alpha, \beta, \gamma, a \in X$, then (X, S_m) is a S_m -metric space on X."*

Example 2.2. *Let X=$\mathbb{R}_+$,define $S_m : X^3 \to [0, \infty)$ by*
$S_m(\alpha, \beta, \gamma) = a^{|\beta+\gamma-2\alpha|+|\beta-\gamma|}$, *where $\alpha, \beta, \gamma \in X$, then (X, S_m) is a S_m -metric space on X.*

Now we present some definitions in S_m -metric space.

Definition 2.4. *[7] Suppose (X, S_m) is a S_m-metric space, a sequence $\{\alpha_k\} \in X$ is called*

(2.4.1) *cauchy sequence* $\iff S_m(\alpha_k, \alpha_k, \alpha_l) \to 1$, *for all* $k, l \to \infty$;

(2.4.2) *convergent* $\iff \exists\alpha \in X$ *such that* $S_m(\alpha_k, \alpha_k, \alpha) \to 1$ *as* $k \to \infty$;

(2.4.3) *is complete if every cauchy sequence is convergent.*

Definition 2.5. *[8] " The mappings G and I be compatible mappings in S_m-metric space if $S_m(GI\alpha_k, GI\alpha_k, IG\alpha_k) = 1$, whenever a sequence $\{\alpha_k\}$ in X such that $lim_{k\to\infty}G\alpha_k = lim_{k\to\infty}I\alpha_k = \eta$ for some $\eta \in X$. "*

Definition 2.6. *[8] "Let G and I be weakly compatible mappings in S_m-metric space if for all $\eta \in X$, $G\eta = I\eta \implies GI\eta = IG\eta$".*

Definition 2.7. *[9] " Suppose G and I are mappings in S_m-metric are said to be occasionally weakly compatible (OWC for shot) iff $\exists\eta \in X$ such that $G\eta = I\eta \implies GI\eta = IG\eta$. "*

Example 2.3. *Let $X = [0, \infty)$ is a S_m-metric space on X ,*
$S_m(\alpha, \beta, \gamma) = a^{|\alpha-\beta|+|\beta-\gamma|+|\gamma-\alpha|}$, *for every* $\alpha, \beta, \gamma \in X$.
Construct two self maps G and I as
$G(\alpha) = 3\alpha - 2$ *and* $I(\alpha) = \alpha^2$.
Consider a sequence $\{\alpha_k\}$ *given by* $\alpha_k = 2 + \frac{1}{k}$ *for* $k \geq 0$.
$G(\alpha_k) = 3(2 + \frac{1}{k}) - 2 = 4$ *and* $I(\alpha_k) = (2 + \frac{1}{k})^2 = 4$ *as* $k \to \infty$
Therefore $G\alpha_k = I\alpha_k = 4 \neq \phi$.
Moreover, $GI(\beta_k) = GI(2 + \frac{1}{k}) = G(2 + \frac{1}{k})^2 = G(4 + 4\frac{1}{k} + \frac{1}{k^2}) = 3(4 + 4\frac{1}{k} + \frac{1}{k^2}) - 2$ *=10*
and $IG(\beta_k) = IG(2 + \frac{1}{k}) = I(3(2 + \frac{1}{k}) - 2) = I(4 + \frac{3}{k}) = (4 + \frac{3}{k})^2 = 16$ *as* $k \to \infty$.
This gives $S_m(GI\alpha_k, GI\alpha_k, IG\alpha_k) = S_m(10, 10, 16) \neq 1$.
Hence, (G,I) is not compatible.
Now G(1)= I(1)=1 also GI(1)= IG(1)=1 $\implies$ *GI(1)= IG(1).*
G(2)=4, I(2)=4 also GI(2)=10 , IG(2)=16 $\implies GI(2) \neq IG(2)$.
As a result G and I have OWC, but not weakly compatible.

Definition 2.8. *[10] " Two self maps G and I in S_m-metric space as conditionally- compatible if there exists a sequence $\{\alpha_k\} \in X$ such that $G\alpha_k = I\alpha_k \neq \phi, \exists$ a sequence $\{\beta_k\} \in X$ such that $G\beta_k = I\beta_k \to \eta$ for some $\eta \in X$ and $S_m(GI\beta_k, GI\beta_k, IG\eta) = 1$ as $k \to \infty$. "*

Definition 2.9. *[10] " Two self maps G and I in S_m-metric space are called as faintly compatible iff (G , I)is conditionally- compatible and G and I commute on a non -empty subset of the set of coincidence points if the collection of coincidence points is non- empty."*

Definition 2.10. *[6] "A reciprocally continuous mappings G and I of a S_m-metric space is defined as $S_m(GI\alpha_k, GI\alpha_k, I\eta) = 1 and S_m(IG\alpha_k, IG\alpha_k, G\eta) = 1$ letting $k \to \infty$ if there exists a sequence $\{\alpha_k\} \in X$ such that $lim_{k\to\infty} G\alpha_k = lim_{k\to\infty} I\alpha_k = \eta$ as $\eta \in X$. "*

Example 2.4. *Let $X = [0, \infty)$ is a S_m-metric space on X ,*
$S_m(\alpha, \beta, \gamma) = a^{|\alpha-\beta|+|\beta-\gamma|+|\gamma-\alpha|}$, for every $\alpha, \beta, \gamma \in X$.
Construct two self maps G and I as
$G(\alpha) = \alpha^2 - 3\alpha + 2$ and $I(\alpha) = 3\alpha^2 - 7\alpha + 2$.
Consider a sequence $\{\alpha_k\}$ given by $\alpha_k = 2 + \frac{1}{k}$ for $k \geq 0$.
then $G(\alpha_k) = (2 + \frac{1}{k})^2 - 3(2 + \frac{1}{k}) + 2 = 0$ and $I(\alpha_k) = 3(2 + \frac{1}{k})^2 - 7(2 + \frac{1}{k}) + 2 = 0$ as $k \to \infty$ therefore $lim_{k\to\infty} G\alpha_k = lim_{k\to\infty} I\alpha_k = 0 \neq \phi$.
Moreover, $GI(\alpha_k) = GI(2 + \frac{1}{k}) = G[3(2 + \frac{1}{k})^2 - 7(2 + \frac{1}{k}) + 2] = G(\frac{3}{k^2} + \frac{5}{k}) = (\frac{3}{k^2} + \frac{5}{k})^2 - 3(\frac{3}{k^2} + \frac{5}{k}) + 2 = 2$ and $IG(\alpha_k) = IG(2+\frac{1}{k}) = I[(2+\frac{1}{k})^2 - 3(2+\frac{1}{k}) + 2] = I(\frac{1}{k^2} + \frac{1}{k}) = 3(\frac{1}{k^2} + \frac{1}{k})^2 - 7(\frac{1}{k^2} + \frac{1}{k}) + 2 = 2$ as $k \to \infty$.
$\implies S_m(GI\alpha_k, GI\alpha_k, IG\alpha_k) = S_m(2, 2, 2) = 1$. Hence, (G,I) is compatible.
Consider another sequence $\{\beta_k\}$ given by $\beta_k = \frac{1}{k}$ for $k \geq 0$.
$G(\beta_k) = (\frac{1}{k^2} - \frac{3}{k} + 2) = 2$ and $I(\beta_k) = (\frac{3}{k^2} - \frac{7}{k} + 2) = 2$ as $k \to \infty$
therefore $lim_{k\to\infty} G\beta_k = lim_{k\to\infty} I\beta_k = 2$.
Further $GI(\beta_k) = GI(\frac{1}{k}) = G(\frac{3}{k^2} - \frac{7}{k} + 2) = (\frac{3}{k^2} - \frac{7}{k} + 2)^2 - 3(\frac{3}{k^2} - \frac{7}{k} + 2) + 2 = 0$ and $IG(\beta_k) = IG(\frac{1}{k}) = I(\frac{1}{k^2} - \frac{3}{k} + 2) = 3(\frac{1}{k^2} - \frac{3}{k} + 2) - 7(\frac{1}{k^2} - \frac{3}{k} + 2) + 2 = -6$ as $k \to \infty$.
This gives $S_m(GI\alpha_k, IG\alpha_k, \eta) = S_m(0, 0, -6) \neq 1$.
the pair (G,I) is not compatible
Hence (G,I) is conditionally compatible.
Compatibility is distinct from the concept of conditional compatibility,
Now G(2)=0,I(2)=0 and GI(2)=2,IG(2)=2.
Also G(0)=I(0)=2 and GI(0)=IG(0)=0.
Hence the pair (G,I) is faintly compatible.
As a result, the mappings G and I have faintly compatible, but they are not compatible.

Definition 2.11. *[11] "The mappings G and I of a S_m-metric space are called **well-posed** if*

- *G and I have a unique common fixed point η in X*
- *If $\alpha_k \in X$ such that $S_m(G\alpha_k, G\alpha_k, \alpha_k) = 1$ and $S_m(I\alpha_k, I\alpha_k, \alpha_k) = 1$ as $k \to \infty$ we have $S_m(\alpha_k, \alpha_k, \eta) = 1$ as $k \to \infty$."*

3. MAIN THEOREM

Theorem:

Suppose G, H, I and J are self-mapping in a complete S_m-metric space X, suppose that there exist $\lambda \in (0, \frac{1}{2})$ such that the conditions

(3.1.1) $G(X) \subseteq J(X)$ and $H(X) \subseteq I(X)$

(3.1.2)

$$S_m(G\alpha, G\alpha, H\beta) \leq \Big\{ max[S_m(G\alpha, G\alpha, I\alpha)S_m(H\beta, H\beta, J\beta), S_m(G\alpha, G\alpha, J\beta)S_m(I\alpha, I\alpha, H\beta), S_m(G\alpha, G\alpha, J\beta)S_m(H\beta, H\beta, J\beta), S_m(G\alpha, G\alpha, I\alpha)S_m(H\beta, H\beta, I\alpha)] \Big\}^{\lambda}$$

(3.1.3) the pair (H,J) is OWC

(3.1.4) and the pair (G,I) is reciprocally continuous and faintly compatible.

Then the common fixed point problem of G, H, I and J is Well-posed.

Proof:

We begin with using **(3.1.1)**, then there is a point $\alpha_0 \in X$, such that $G\alpha_0 = J\alpha_1 = \beta_0$. For this point $\alpha_1$then there $\exists \alpha_2 \in X$ such that $H\alpha_1 = I\alpha_2 = \beta_1$.
In general, by induction choose α_{k+1} so that

$$\beta_{2k} = G\alpha_{2k} = J\alpha_{2k+1} \text{ and } \beta_{2k+1} = H\alpha_{2k+1} = I\alpha_{2k+2} \text{ for } k \geq 0.$$

We show that $\{\beta_k\}$ is a cauchy sequence in S_m - metric space .
Indeed, it follows that $S_m(\beta_{2k}, \beta_{2k}, \beta_{2k+1}) =$

$$S_m(G\alpha_{2k}, G\alpha_{2k}, H\alpha_{2k+1}) \leq \max \Big\{ S_m(G\alpha_{2k}, G\alpha_{2k}, I\alpha_{2k})S_m(H\alpha_{2k+1}, H\alpha_{2k+1}, J\alpha_{2k+1}), S_m(G\alpha_{2k}, G\alpha_{2k}, J\alpha_{2k+1})S_m(H\alpha_{2k+1}, H\alpha_{2k+1}, I\alpha_{2k}), S_m(G\alpha_{2k}, G\alpha_{2k}, J\alpha_{2k+1})S_m(H\alpha_{2k+1}, H\alpha_{2k+1}, J\alpha_{2k+1}), S_m(G\alpha_{2k}, G\alpha_{2k}, I\alpha_{2k})S_m(H\alpha_{2k+1}, H\alpha_{2k+1}, I\alpha_{2k}) \Big\}^{\lambda}$$

$$S_m(\beta_{2k},\beta_{2k},\beta_{2k+1}) \leq \max \begin{cases} S_m(\beta_{2k},\beta_{2k},\beta_{2k-1})S_m(\beta_{2k+1},\beta_{2k+1},\beta_{2k}), \\ S_m(\beta_{2k},\beta_{2k},\beta_{2k})S_m(\beta_{2k+1},\beta_{2k+1},\beta_{2k-1}), \\ S_m(\beta_{2k},\beta_{2k},\beta_{2k})S_m(\beta_{2k+1},\beta_{2k+1},\beta_{2k}), \\ S_m(\beta_{2k},\beta_{2k},\beta_{2k-1})S_m(\beta_{2k+1},\beta_{2k+1},\beta_{2k-1}) \end{cases}^{\lambda}$$

on simplification

$$S_m(\beta_{2k},\beta_{2k},\beta_{2k+1}) \leq S_m(\beta_{2k-1},\beta_{2k-1},\beta_{2k+1})^{\lambda}.$$

$$S_m(\beta_{2k},\beta_{2k},\beta_{2k+1}) \leq \{S_m(\beta_{2k-1},\beta_{2k-1},\beta_{2k})S_m(\beta_{2k},\beta_{2k},\beta_{2k+1})\}^{\lambda}.$$

$$S_m^{1-\lambda}(\beta_{2k},\beta_{2k},\beta_{2k+1}) \leq S_m^{\lambda}(\beta_{2k-1},\beta_{2k-1},\beta_{2k}).$$

$$S_m(\beta_{2k},\beta_{2k},\beta_{2k+1}) \leq S_m^{\frac{\lambda}{1-\lambda}}(\beta_{2k-1},\beta_{2k-1},\beta_{2k}).$$

$$S_m(\beta_{2k},\beta_{2k},\beta_{2k+1}) \leq S_m^{p}(\beta_{2k-1},\beta_{2k-1},\beta_{2k}). \text{ where } p = \frac{\lambda}{1-\lambda}$$

Now this gives

$$S_m(\beta_k,\beta_k,\beta_{k+1}) \leq S_m^{p}(\beta_{k-1},\beta_{k-1},\beta_k) \leq S_m^{p^2}(\beta_{k-2},\beta_{k-2},\beta_{k-1}) \leq \cdots S_m^{p^n}(\beta_0,\beta_0,\beta_n)$$

By using triangular inequality,

$$S_m(\beta_k,\beta_k,\beta_n) \leq S_m^{p^k}(\beta_0,\beta_0,\beta_l) \leq S_m^{p^{k+1}}(\beta_0,\beta_0,\beta_n) \leq \cdots S_m^{p^{n-1}}(\beta_0,\beta_0,\beta_n)$$

Hence $\{\beta_k\}$ is a cauchy sequence in S_m-metric space.

Now X being complete in S_m-metric space $\exists \eta \in X$ such that $lim_{k\to\infty}\beta_k \to \eta$.

Consequently, the sub sequences $\{G\alpha_{2k}\}$, $\{I\alpha_{2k}\}$, $\{J\alpha_{2k+1}\}$ and $\{H\alpha_{2k+1}\}$ of $\{\beta_k\}$ also converges to the point $\eta \in X$.

Since the pair (G,I) is faintly compatible mappings, so that $\exists$ another sequence $\nu_k \in X$ such that $lim_{k\to\infty}G\nu_k = lim_{k\to\infty}I\nu_k = \omega$ for $\omega \in X$ satisfying

$$lim_{k\to\infty}S(GI\nu_k, GI\nu_k, IG\nu_k) = 1 \text{ and the pair (G,I) is reciprocally continuous}$$

$$S_m(GI\nu_k, GI\nu_k, I\omega) = 1, \text{and } S_m(IG\nu_k, IG\nu_k, G\omega) = 1. \text{ as } k \to \infty.$$

$$G\omega = I\omega \tag{3.1}$$

On putting $\alpha = \omega$ and $\beta = \alpha_{2k+1}$ in (3.1.2) we get

$$S_m(G\omega, G\omega, H\alpha_{2k+1}) \leq \Big\{ max[S_m(G\omega, G\omega, I\omega)S_m(H\alpha_{2k+1}, H\alpha_{2k+1}, J\alpha_{2k+1}),$$
$$S_m(G\omega, G\omega, J\alpha_{2k+1})S_m(I\omega, I\eta, H\alpha_{2k+1}),$$
$$S_m(G\omega, G\omega, J\alpha_{2k+1})S_m(H\alpha_{2k+1}, H\alpha_{2k+1}, J\alpha_{2k+1}),$$
$$S_m(G\omega, G\omega, I\omega)S_m(H\alpha_{2k+1}, H\alpha_{2k+1}, I\omega)] \Big\}^{\lambda}$$

and

$$S_m(G\omega, G\omega, \eta) \leq \Big\{ max[S_m(G\omega, G\omega, I\omega)S_m(\eta, \eta, \eta), S_m(G\omega, G\omega, \eta)S_m(I\omega, I\omega, \eta),$$
$$S_m(G\omega, G\omega, \eta)S^*(\eta, \eta, \eta), S_m(G\omega, G\omega, I\omega)S_m(\eta, \eta, I\omega)] \Big\}^{\lambda}$$

which gives

$$S_m(G\omega, G\omega, \eta) \leq \Big\{ max[S_m(G\omega, G\omega, G\omega)S_m(\eta, \eta, \eta), S_m(G\omega, G\omega, \eta)S_m(G\omega, G\omega, \eta),$$
$$S_m(G\omega, G\omega, \eta)S_m(\eta, \eta, \eta), S_m(G\omega, G\omega, G\omega)S_m(\eta, \eta, G\omega)] \Big\}^{\lambda}$$

implies

$$S_m(G\omega, G\omega, \eta) \leq \Big\{ max[1, S_m^2(G\omega, G\omega, \eta), S_m(G\omega, G\omega, \eta), S_m(G\omega, G\omega, \eta)] \Big\}^{\lambda}$$

this gives

$$S_m(G\omega, G\omega, \eta) \leq \Big\{ S_m^{2\lambda}(G\omega, G\omega, \eta) \Big\}$$

this implies $G\omega = \eta$.

$$\text{therefore } G\omega = I\omega = \eta. \tag{3.2}$$

Since the pair (G,I) is faintly compatible, so that $G\omega = I\omega$ this gives $GI\omega = IG\omega$

this implies $G\eta = I\eta$.

By using the inequality (3.1.2) on putting $\alpha = \eta$ and $\beta = \alpha_{2k+1}$ we get

$$S_m(G\eta, G\eta, H\alpha_{2k+1}) \leq \Big\{ max[S_m(G\eta, G\eta, I\eta)S_m(H\alpha_{2k+1}, H\alpha_{2k+1}, J\alpha_{2k+1}),$$
$$S_m(G\eta, G\eta, J\alpha_{2k+1})S_m(I\eta, I\eta, H\alpha_{2k+1}),$$
$$S_m(G\eta, G\eta, J\alpha_{2k+1})S_m(H\alpha_{2k+1}, H\alpha_{2k+1}, J\alpha_{2k+1}),$$
$$S_m(G\eta, G\eta, I\eta)S_m(H\alpha_{2k+1}, H\alpha_{2k+1}, I\eta)] \Big\}^{\lambda}$$

and

$$S_m(G\eta, G\eta, \eta) \leq \Big\{max[S_m(G\eta, G\eta, I\eta)S_m(\eta,\eta,\eta), S_m(G\eta, G\eta, \eta)S_m(I\eta, I\eta, \eta),$$
$$S_m(G\eta, G\eta, \eta)S_m(\eta,\eta,\eta), S_m(G\eta, G\eta, I\eta)S_m\eta, \eta, I\eta)]\Big\}^{\lambda}$$

which gives

$$S_m(G\eta, G\eta, \eta) \leq \Big\{max[S_m(G\eta, G\eta, G\eta)S_m(\eta,\eta,\eta), S_m(G\eta, G\eta, \eta)S_m(G\eta, G\eta, \eta),$$
$$S_m(G\eta, G\eta, \eta)S_m(\eta,\eta,\eta), S_m(G\eta, G\eta, G\eta)S_m(\eta, \eta, G\eta)]\Big\}^{\lambda}$$

implies

$$S_m(G\eta, G\eta, \eta) \leq \Big\{max[1, {S_m}^2(G\eta, G\eta, \eta), S_m(G\eta, G\eta, \eta), S_m(G\eta, G\eta, \eta)]\Big\}^{\lambda}$$

which implies

$$S_m(G\eta, G\eta, \eta) \leq \Big\{{S_m}^{2\lambda}(G\eta, G\eta, \eta)\Big\}$$

$$\implies G\eta = \eta. \tag{3.3}$$

$$G\eta = I\eta = \eta. \tag{3.4}$$

$\implies \eta = G\eta \in G(X) \subseteq J(X) \implies G\eta = Jv$ for some $v \in X$.

$$G\eta = I\eta = Jv = \eta. \tag{3.5}$$

Using the inequality (3.1.2) on putting $\alpha = \eta$ and $\beta = v$ we have

$$S_m(G\eta, G\eta, Hv) \leq \Big\{max[S_m(G\eta, G\eta, I\eta)S_m(Hv, Hv, Jv), S_m(G\eta, G\eta, Jv)S_m(I\eta, I\eta, Hv),$$
$$S_m(G\eta, G\eta, Jv)S_m(Hv, Hv, Jv), S_m(G\eta, G\eta, I\eta)S_m(Hv, Hv, I\eta)]\Big\}^{\lambda}$$

this implies

$$S_m(\eta, \eta, Hv) \leq \Big\{max[S_m(\eta,\eta,\eta)S_m(Hv, Hv, \eta), S_m(\eta,\eta,\eta)S_m(\eta, \eta, Hv),$$
$$S_m(\eta,\eta,\eta)S_m(Hv, Hv, \eta), S_m(\eta,\eta,\eta)S_m(Hv, Hv, \eta)]\Big\}^{\lambda}$$

which implies

$$S_m(\eta, \eta, Hv) \leq \Big\{max[S_m(Hv, Hv, \eta), S_m(\eta, \eta, Hv), \qquad S_m(Hv, Hv, \eta), S_m(Hv, Hv, \eta)]\Big\}^{\lambda}$$

this gives

$$S_m(\eta,\eta,Hv) \leq \left\{ S_m(Hv,Hv,\eta) \right\}^{\lambda}$$

which gives $Hv = \eta$.

$$G\eta = I\eta = Jv = Hv = \eta. \tag{3.6}$$

Again (H,J) is OWC with $v \in X$ so that $Hv = Jv \implies HJv = JHv$

which implies that $H\eta = J\eta$.

Using the inequality (3.1.2) and take $\alpha = \eta$ and $\beta = \eta$ we get

$$S_m(G\eta,G\eta,H\eta) \leq \Big\{ max[S_m(G\eta,G\eta,I\eta)S_m(H\eta,H\eta,J\eta), S_m(G\eta,G\eta,J\eta)S_m(I\eta,I\eta,H\eta), \\ S_m(G\eta,G\eta,J\eta)S_m(H\eta,H\eta,J\eta), S_m(G\eta,G\eta,I\eta)S_m(H\eta,H\eta,I\eta)] \Big\}^{\lambda}$$

this implies

$$S_m(\eta,\eta,H\eta) \leq \Big\{ max[S_m(\eta,\eta,\eta)S_m(H\eta,H\eta,\eta), S_m(\eta,\eta,\eta)S_m(\eta,\eta,H\eta), \\ S_m(\eta,\eta,\eta)S_m(H\eta,H\eta,\eta), S_m(\eta,\eta,\eta)S_m(H\eta,H\eta,\eta)] \Big\}^{\lambda}$$

where

$$S_m(\eta,\eta,H\eta) \leq \left\{ max[S_m(H\eta,H\eta,\eta), S_m(\eta,\eta,H\eta), S_m(H\eta,H\eta,\eta), S_m(H\eta,H\eta,\eta)] \right\}^{\lambda}$$

this gives

$$S_m(\eta,\eta,H\eta) \leq \left\{ S_m(H\eta,H\eta,\eta) \right\}^{\lambda}$$

this gives $H\eta = \eta$.

$$H\eta = J\eta = \eta. \tag{3.7}$$

From(3.4) and (3.7)

$$G\eta = I\eta = J\eta = H\eta = \eta. \tag{3.8}$$

$\implies$ η is a common fixed point for the mappings G,H,I and J.

For the proof of well-posed property

Suppose $\rho(\rho \neq \eta)$ is one more fixed point of G,I,H and J i.e $G\rho = I\rho = H\rho = J\rho = \rho$.

Using the inequality **(3.1.2)** take $\alpha = \rho$ and $\beta = \eta$ we have

$$S_m(G\rho, G\rho, H\eta) \leq \Big\{ \max[S_m(G\rho, G\rho, I\eta)S_m(H\eta, H\eta, J\eta), S_m(G\rho, G\rho, J\eta)S_m(I\rho, I\rho, H\eta), \\ S_m(G\rho, G\rho, J\eta)S_m(H\eta, H\eta, J\eta), S_m(G\rho, G\rho, I\rho)S_m(H\eta, H\eta, I\rho)] \Big\}^{\lambda}$$

this gives

$$S_m(\rho, \rho, \eta) \leq \Big\{ \max[S_m(\rho, \rho, \eta)S_m(\eta, \eta, \eta), S_m(\rho, \rho, \eta)S_m(\rho, \rho, \eta), \\ S_m(\rho, \rho, \eta)S_m(\eta, \eta, \eta), S_m(\rho, \rho, \rho)S_m(\eta, \eta, \rho)] \Big\}^{\lambda}$$

this gives

$$S_m(\rho, \rho, \eta) \leq \Big\{ \max[1, S_m(\rho, \rho, \eta), 1, 1] \Big\}^{\lambda}$$

which gives

$$\therefore S_m(\rho, \rho, \eta) \leq S_m(\rho, \rho, \eta)^{\lambda}$$

This gives $\rho = \eta$.

Hence η is the unique common fixed point of G,H,I and J

Suppose $\{\alpha_k\}$ be a sequence in X such that

$S_m(G\alpha_k, G\alpha_k, \alpha_k) = S_m(I\alpha_k, I\alpha_k, \alpha_k) = 1$

and $S_m(H\alpha_k, H\alpha_k, \alpha_k) = S_m(J\alpha_k, J\alpha_k, \alpha_k) = 1$ as $k \to \infty$.

We have to show that $S_m(\alpha_k, \alpha_k, \eta) = 1$, $S_m(\alpha_k, \alpha_k, \eta) \leq S_m(G\alpha_k, G\alpha_k, \eta)S_m(G\alpha_k, G\alpha_k, \alpha_k)$

$$S_m(\alpha_k, \alpha_k, \eta) \leq \\ \Big\{ \max[S_m(G\alpha_k, G\alpha_k, I\alpha_k)S_m(H\eta, H\eta, J\eta), S_m(G\alpha_k, G\alpha_k, J\eta)S_m(I\alpha_k, I\alpha_k, H\eta), \\ S_m(G\alpha_k, G\alpha_k, J\eta)S_m(H\eta, H\eta, J\eta), S_m(G\alpha_k, G\alpha_k, I\alpha_k)S_m(H\eta, H\eta, I\alpha_k)] \Big\}^{\lambda} S_m(G\alpha_k, G\alpha_k, \alpha_k)$$

this gives

$$S_m(\alpha_k, \alpha_k, \eta) \leq \\ \Big\{ \max[S_m(G\alpha_k, G\alpha_k, G\eta)S_m(I\alpha_k, I\alpha_k, I\eta)S_m(H\eta, H\eta, J\eta), S_m(G\alpha_k, G\alpha_k, J\eta)S_m(I\alpha_k, I\alpha_k, H\eta), \\ S_m(G\alpha_k, G\alpha_k, J\eta)S_m(H\eta, H\eta, J\eta), S_m(G\alpha_k, G\alpha_k, I\alpha_k)S_m(H\eta, H\eta, I\alpha_k)] \Big\}^{\lambda} S_m(G\alpha_k, G\alpha_k, \alpha_k)$$

which gives

$$S_m(\alpha_k,\alpha_k,\eta) \le$$
$$\Big\{ \max[S_m(\alpha_k,\alpha_k,\eta)S_m(\alpha_k,\alpha_k,\eta))S_m(\eta,\eta,\eta), S_m(\alpha_k,\alpha_k,\eta)S_m(\alpha_k,\alpha_k,\eta),$$
$$S_m(\alpha_k,\alpha_k,\eta)S_m(\eta,\eta,\eta), S_m(\alpha_k,\alpha_k,\alpha_k)S_m(\eta,\eta,\alpha_k)]\Big\}^{\lambda} S_m(G\alpha_k,G\alpha_k,\alpha_k)$$

therefore

$$S_m(\alpha_k,\alpha_k,\eta) \le \Big\{S_m(\alpha_k,\alpha_k,\eta)\Big\}^{\lambda} S_m(G\alpha_k,G\alpha_k,\alpha_k)$$

$$S_m(\alpha_k,\alpha_k,\eta)^{1-\lambda} \le S_m(G\alpha_k,G\alpha_k,\alpha_k)$$

$$S_m(\alpha_k,\alpha_k,\eta) \le S_m^{\frac{1}{1-\lambda}}(G\alpha_k,G\alpha_k,\alpha_k)$$

$$S_m(\alpha_k,\alpha_k,\eta) = 1 \text{ as } k \to \infty.$$

Thus G,H,I and J is well-posed.

4. Example

Suppose $X=[0,1]$,S_m- metric space by $S_m(\alpha,\beta,\gamma)=e^{|\alpha-\beta|+|\beta-\gamma|+|\gamma-\alpha|}$,
when $\alpha,\beta,\gamma\in X$. Define G ,I ,H J:$XxX\to X$ as follows

$$G(\alpha)=\begin{cases}\frac{1-\alpha}{2} & \text{if } 0\le\alpha\le\frac{1}{3};\\ \frac{3-2\alpha}{4} & \text{if } \frac{1}{3}<\alpha\le 1.\end{cases} \qquad J(\alpha)=\begin{cases}\frac{1+4\alpha}{7} & \text{if } 0\le\alpha\le\frac{1}{3};\\ \frac{2\alpha+1}{4} & \text{if } \frac{1}{3}<\alpha\le 1.\end{cases}$$

and

$$H(\alpha)=\begin{cases}\frac{2-\alpha}{5} & \text{if } 0\le\alpha\le\frac{1}{3};\\ \frac{3-2\alpha}{4} & \text{if } \frac{1}{3}<\alpha\le 1.\end{cases} \qquad I(\alpha)=\begin{cases}\frac{2\alpha+1}{5} & \text{if } 0\le\alpha\le\frac{1}{3};\\ \frac{\alpha+5}{11} & \text{if } \frac{1}{3}<\alpha\le 1.\end{cases}$$

Then G(X)=$(\frac{7}{12},\frac{1}{4}]$ and J(X) =$[\frac{1}{7},\frac{1}{3}]\cup(\frac{5}{12},\frac{3}{4}]$.

And also H(X)=$(\frac{7}{21},\frac{1}{4}]$ and I(X)=$[\frac{1}{5},\frac{1}{3}]\cup(\frac{16}{33},\frac{6}{11}]$

$thisimpliesimpliesG(X)\subseteq J(X)$ and $H(X)\subseteq I(X)$ hence the inequality (3.1.1) holds.

Take a sequence $\{\alpha_k\}$ as $\alpha_k=\frac{1}{2}+\frac{1}{k}$ as k ≥ 0.

Now $G(\alpha_k)=G(\frac{1}{2}+\frac{1}{k})$=$\frac{3-2(\frac{1}{2}+\frac{1}{k})}{4}$=$\frac{1}{2}$ and $J(\alpha_k)=J(\frac{1}{2}+\frac{1}{k})$=$\frac{2(\frac{1}{2}+\frac{1}{k})+1}{4}$=$\frac{1}{2}$.

And also $H(\alpha_k)=H(\frac{1}{2}+\frac{1}{k})$=$\frac{3-2(\frac{1}{2}+\frac{1}{k})}{4}$=$\frac{1}{2}$ and $I(\alpha_k)=I(\frac{1}{2}+\frac{1}{k})$=$\frac{5+(\frac{1}{2}+\frac{1}{k})}{11}$=$\frac{1}{2}$.

$\therefore G\alpha_k$=$J\alpha_k$=$\frac{1}{2}$ and $H\alpha_k$=$J\alpha_k$=$\frac{1}{2}$ as $k\to\infty$.

then $GI(\alpha_k)=GI(\frac{1}{2}+\frac{1}{k})=G(\frac{1}{2}+\frac{1}{11k})=\frac{3-2(\frac{1}{2}+\frac{1}{11k})}{4}=\frac{1}{2}$ and $IG(\alpha_k)=IG(\frac{1}{2}+\frac{1}{k})=I(\frac{1}{2}-\frac{1}{2k})=$ $\frac{5+(\frac{1}{2}-\frac{1}{2k})}{11}$=$\frac{1}{2}$ as $k\to\infty$.

$HJ(\alpha_k)=HJ(\frac{1}{2}+\frac{1}{k})=H(\frac{5+(\frac{1}{2}+\frac{1}{k})}{11})=H(\frac{1}{2}+\frac{1}{11k})=\frac{3-2(\frac{1}{2}+\frac{1}{11K})}{4}=\frac{1}{2}$

and $JH(\alpha_k) = JH(\frac{1}{2}+\frac{1}{k}) = J(\frac{3-2(\frac{1}{2}+\frac{1}{k})}{4}) = I(\frac{1}{2}-\frac{1}{2k}) = \frac{1+2(\frac{1}{2}-\frac{1}{2k})}{4}=\frac{1}{2}$ as $k \to \infty$.

$\lim_{k\to\infty} S_m(GI\alpha_k, GI\alpha_k, IG\alpha_k) = S_m(\frac{1}{2},\frac{1}{2},\frac{1}{2}) = 1$ and

$\lim_{k\to\infty} S_m(HJ\alpha_k, HJ\alpha_k, JH\alpha_k) = S_m(\frac{1}{2},\frac{1}{2},\frac{1}{2}) = 1$

Hence the pairs (G,I) and (H,J) are satisfies compatible property .

Take another sequence $\{\beta_k\}$ as $\beta_k = \frac{1}{3}-\frac{1}{k}$ as k ≥ 0.

Now $G(\beta_k) = G(\frac{1}{3}-\frac{1}{k})=\frac{1-(\frac{1}{3}-\frac{1}{k})}{2}=\frac{1}{3}=$ and $I(\beta_k) = I(\frac{1}{3}-\frac{1}{k})=\frac{2+(\frac{1}{3}-\frac{1}{k})}{5}=\frac{1}{3}$.

And also $H(\beta_k) = H(\frac{1}{3}-\frac{1}{k})=\frac{2-(\frac{1}{3}-\frac{1}{k})}{5}=\frac{1}{3}$ and $J(\beta_k) = J(\frac{1}{3}-\frac{1}{k})=\frac{4(\frac{1}{3}-\frac{1}{k})+1}{7}=\frac{1}{3}$as $k \to \infty$.

$\therefore G\beta_k=I\beta_k=\frac{1}{3}=\eta$ similarly $H\beta_k=J\beta_k=\frac{1}{3}=\eta$ as $k\to\infty$.

Further $GI(\beta_k) = GI(\frac{1}{3}-\frac{1}{k})=G(\frac{2(\frac{1}{3}-\frac{1}{k})+1}{5})=G(\frac{1}{3}-\frac{2}{5k})=\frac{1-(\frac{1}{3}-\frac{2}{5k})}{2}=\frac{1}{3}$

and $IG((\beta_k) = IG(\frac{1}{3}-\frac{1}{k})=I(\frac{1-(\frac{1}{3}-\frac{1}{k}}{2})=I(\frac{1}{3}+\frac{1}{2k})=\frac{5+(\frac{1}{3}+\frac{1}{2k})}{11}=\frac{16}{33}$ $HJ(\beta_k) = HJ(\frac{1}{3}-\frac{1}{k}) = H(\frac{4(\frac{1}{3}-\frac{1}{k})+1}{7} = H(\frac{1}{3}-\frac{4}{7k}) = \frac{2-(\frac{1}{3}+\frac{4}{7k})}{11} = \frac{1}{3}$

and $JH(\beta_k) = JH(\frac{1}{3}-\frac{1}{k}) = J(\frac{2-(\frac{1}{3}-\frac{1}{k}}{5}) = J(\frac{1}{3}+\frac{1}{5k}) = \frac{2(\frac{1}{3}+\frac{1}{5k})+1}{4} = \frac{5}{12}$ as $k\to\infty$.

this implies $\lim_{k\to\infty} S_m(GI\beta_k, GI\beta_k, IG\beta_k) = S_m(\frac{1}{3},\frac{1}{3},\frac{66}{33}) \neq 1$ which shows that the pairs (G,I) is faintly compatible mappings. Moreover $I(\frac{1}{2}) = \frac{1}{2}$ and $G(\frac{1}{2}) = \frac{1}{2}$ and also

$\lim_{k\to\infty} S_m(GI\alpha_k, GI\alpha_k, Iw) = S_m(\frac{1}{2},\frac{1}{2},\frac{1}{2}) = 1$ $\lim_{k\to\infty} S_m(IG\alpha_k, IG\alpha_k, Gw) = S_m(\frac{1}{2},\frac{1}{2},\frac{1}{2}) = 1$ this shows that the pairs (G,I) is reciprocally continuous. The inequity (3.1.4) holds.

Further $H(\frac{1}{2}) = \frac{5+\frac{1}{2}}{11} = \frac{1}{2}$ and $J(\frac{1}{2}) = \frac{2\frac{1}{2}+1}{4} = \frac{1}{2}$. $\therefore H(\frac{1}{2}) = J(\frac{1}{2}) = (\frac{1}{2})$ where $\frac{1}{2}\in X$.

And also $HJ(\frac{1}{2}) = \frac{1}{2}$ and $JH(\frac{1}{2}) = \frac{1}{2}$, $\implies$ $HJ(\frac{1}{2}) = JH(\frac{1}{2}) = \frac{1}{2}$. Moreover, $H(\frac{1}{3}) = \frac{2-\frac{1}{3}}{5} = \frac{1}{3}$ and $J(\frac{1}{3}) = \frac{4\frac{1}{3}+1}{7} = \frac{1}{3}$. $\therefore H(\frac{1}{3}) = J(\frac{1}{3}) = (\frac{1}{3})$ where $\frac{1}{3}\in X$.

And also $HJ(\frac{1}{3}) = \frac{1}{3}$ and $JH(\frac{1}{3}) = \frac{1}{3}$, $\implies HJ(\frac{1}{3}) = JH(\frac{1}{3}) = \frac{1}{3}$. Which shows that the pair(H,J) satisfies OWC.

So that the inequity (3.1.3) holds.

Further more $S_m(G\beta_k, G\beta_k, \beta_k) = S_m(\frac{1}{3},\frac{1}{3},\frac{1}{3}) = 1, S_m(I\beta_k, I\beta_k, \beta_k) = S_m(\frac{1}{3},\frac{1}{3},\frac{1}{3}) = 1$, when $k\to\infty$. Which implies $lim_{k\to\infty} S_m(\beta_k,\beta_k,\eta) = lim_{k\to\infty} S_m(\frac{1}{3},\frac{1}{3},\frac{1}{3}) = 1$,

CASE-I

Let $\alpha,\beta \in [0,\frac{1}{3}]$,while we have $S_m(\alpha,\beta,\gamma) = e^{|\alpha-\gamma|+|\beta-\gamma|}$

In the inequality**(3.1.2)** putting $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{4}$ implies

$$S_m(0.375, 0.375, 0.36) \leq$$

$$\Big\{max[S_m(0.375, 0.375, 0.28)S_m(0.36, 0.36, 0.286)S_m(0.375, 0.375, 0.286)S_m(0.36, 0.36, 0.28),$$

$$S_m(0.375, 0.375, 0.286)S_m(0.36, 0.36, 0.286), S_m(0.375, 0.375, 0.28)S_m(0.36, 0.36, 0.28)]\Big\}^{\lambda}$$

$$\implies e^{0.03} \leq \Big\{ \max[e^{0.19}e^{0.15}, e^{0.18}e^{0.16}, e^{0.18}e^{0.15}, e^{0.19}e^{0.16}]\Big\}^{\lambda}$$

$$e^{0.03} \leq \{\ \max[e^{0.34}, e^{0.34}, e^{0.33}, e^{0.35}]\}^{\lambda} \implies e^{0.03} \leq e^{0.35\lambda}$$

which gives $\lambda = 0.08$, where $\lambda \in (0, \frac{1}{2})$.

CASE-II

Let $\alpha, \beta \in [\frac{1}{2}, 1]$, then $S_m(\alpha, \beta, \gamma) = e^{|\alpha-\gamma|+|\beta-\gamma|}$

In the inequality **(3.1.2)** putting $\alpha = \frac{2}{3}$ and $\beta = \frac{3}{4}$ implies

$$S_m(0.42, 0.42, 0.375) \leq$$

$$\Big\{ max[S_m(0.42, 0.42, 0.52) S_m(0.375, 0.375, 0.47) S_m(0.42, 0.42, 0.47) S_m(0.375, 0.375, 0.52),$$

$$S_m(0.42, 0.42, 0.47) S_m(0.375, 0.375, 0.47), S_m(0.42, 0.42, 0.52) S_m(0.375, 0.375, 0.52)] \Big\}^{\lambda}$$

$$\implies e^{0.09} \leq \Big\{\ \max[e^{0.2}e^{0.19}, e^{0.1}e^{0.29}, e^{0.1}e^{0.19}, e^{0.2}e^{0.29}] \Big\}^{\lambda}$$

$$e^{0.09} \leq \{\ \max[e^{0.39}, e^{0.39}, e^{0.29}, e^{0.49}]\}^{\lambda} \implies e^{0.09} \leq e^{0.49\lambda}$$

which gives $\lambda = 0.18$, where $\lambda \in (0, \frac{1}{2})$.

Hence the inequality **(3.1.2)** holds.

The verification in the remaining intervals is also simple. It can be observed that $\frac{1}{2}$ is a unique common fixed point of G,H,I and J.

5. CONCLUSION:

This article, aimed to prove a common fixed point theorem in S_m-metric space using conditions OWC, reciprocally continuous and faintly compatible mappings. Also proved the well- posed property. Further our result is supported with a suitable example.

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