International Journal of Analysis and Applications
Volume 18, Number 1 (2020), 85-98
URL: https://doi.org/10.28924/2291-8639
DOI: 10.28924/2291-8639-18-2020-85

THE OPTIMAL HOMOTOPY ASYMPTOTIC METHOD WITH APPLICATION TO SECOND KIND OF NONLINEAR VOLTERRA INTEGRAL EQUATIONS

H. ULLAH[1,*], S. MUKHTAR[2], M. NAWAZ[1], M. ADNAN[3]

[1] *Department of Mathematics, Abdul Wali Khan University Mardan Kpk, Pakistan*

[2] *Department of Basic Sciences, Deanship of Preparatory Year, King Faisal University, Hofuf 31982, Al Ahsa, Saudi Arabia*

[3] *Department of Mathematics, Islamia College (Chartered University) Peshawar Kpk, Pakistan*

[*] *Corresponding author:* hakeemullah1@gmail.com

Abstract. In this paper, we solved some problems of nonlinear second kind of Volterra integral equations by Optimal Homotopy Asymptotic Method(OHAM). We compared the results obtained by OHAM with the exact solutions of the problems. We find that the results obtained by OHAM are effective, simple and explicit from others analytical methods. We also showed the fast convergence of OHAM and list some examples to show the effectiveness of this method. In graphical analysis, we can see the exactness, accuracy and convergence of the method.The OHAM has mechanized steps that can be easily achieved with the help of Mathematica. All computational work and graphs are obtained by Mathematica 9.

1. Introduction

Most of the problems are nonlinear in nature, especially in engineering and applied sciences. There are many applications of Volterra integral equations (VIE's) in applied field including bio-mechanics, fluid mechanics, demography and the study of viscoelastic materials. An Italian mathematician and physicist Vito Volterra invented these equations in his mathematical physics research in 1908 [1]. There are several analytical and numerical methods, such as finite difference method, finite element method, perturbation

Received 2019-06-17; accepted 2019-07-22; published 2020-01-02.

2010 *Mathematics Subject Classification.* 45D05.

Key words and phrases. nonlinear; Volterra integral equation; OHAM; explicit; convergence; mathematica.

method, etc which can be used to obtain an approximate solutions of the nonlinear problems. However, there are several complications such as in grid modification, selection of stability conditions and selection of small and large parameters etc. In order to avoid these complications, decomposition method [2] was introduced, which is an exceptionally effective and powerful method for solving linear and nonlinear problems in various fields. The researchers introduced some others methods to deals such type of problems with easy way and less efforts. There are some analytical methods for solving such type of problems as we have; Homotopy Perturbation Method (HPM) [3], Group Analysis Method (GAM) [4], Differential Transform Method (DTM) [5], Variational Iterative Method (VIM) [6] and Adomian Decomposition Method (ADM) [7]. Here we discussed some nonlinear Volterra Integral Equations of the second kind.
The general form of the nonlinear Volterra integral equation is;

$$\psi(x) = f(x) + \lambda \int_0^x K(x,t)G(\psi(t))dt \tag{1.1}$$

The function $G(\psi(x))$ is nonlinear in $\psi(x)$ such as $\psi^2(x)$, $\psi^3(x)$, $e^{\psi(x)}$, $sin\psi(x)$ and many others. In eq. (1.1), λ is a parameter and $K(x,t)$ is the kernel of integral equation [8].The integration limit for volterra integral equations are function of 'x' and not a constant value like in Fredholm integral equations.The kernel $K(x,t)$ in eq.(1.1) will be assuming a separable kernel.

2. Optimal Homotopy Asymptotic Method

Recently, engineers and scientists known the applications of OHAM in linear and nonlinear problems [9] and [10], because this method continuously deforms complex problems into simple problems which can be solved very easily. This method gives a quick way to the convergence of approximate series and keep more proficiency and high potentiality in science and engineering for solving nonlinear problems. Several researchers have broadly studied different mathematical methods for integral equations such as [12] and [13]. Here, we discuss OHAM which is proposed by Marinca and Herianu [11].
Consider a general nonlinear problem [14].

$$\tau\{\alpha(x)\} + f(x) + \aleph\{a(x)\} = 0 \tag{2.1}$$

where τ is known as function which is called linear operator, $f(x)$ is a given function, $\aleph$ is a nonlinear operator and $\alpha(x)$ is unknown function. According to OHAM [12], we construct a Homotopy: $\Omega \times [0,1] \longrightarrow \Re$ for (2.1) which satisfy

$$(1-\rho)[\tau\{\alpha(x,\rho)\} + f(x)] = H(\rho)[\tau\{\alpha(x,\rho)\} + f(x) + \aleph\{\alpha(x,\rho)\}] \tag{2.2}$$

where $H(\rho)$ represents a nonzero auxiliary function for $\rho \neq 0$ and $H(0) = 0$. Obviously,
when, $\rho = 0$ then it holds that

$$\alpha(x,0) = \alpha_0(x) \tag{2.3}$$

and when, $\rho = 1$ then it holds that

$$\alpha(x, 1) = \alpha_1(x) \tag{2.4}$$

Suppose that the auxiliary function $H(\rho)$ can be expressed as;

$$H(\rho) = \Sigma_{j=1}^{m} c_j \rho^j \tag{2.5}$$

where c_j, $j = 1, 2, 3, ...$ are constant. Putting $\rho = 0$ in eq.(2.2), it holds that

$$\tau\{\alpha_0(x)\} + f(x) = 0 \tag{2.6}$$

By Taylor‘s series, the OHAM solution can be calculated as;

$$\alpha(x, \rho, c_j) = \alpha_0(x) + \Sigma_{k=1}^{m} \alpha_k(x, c_j)\rho^m \tag{2.7}$$

where $j = 1, 2, 3, ...$

When $\rho = 1$, then eq. (2.7) becomes

$$\alpha(x, \rho, c_j) = \alpha_0(x) + \Sigma_{k=1}^{m} \alpha_k(x, c_j) \tag{2.8}$$

Substituting eq. (2.8) into eq. (2.2) and equating the coefficient of the same power of ρ, we get;

$$\tau\{\alpha_1(x)\} = c_1 \aleph\{\alpha_0(x)\} \tag{2.9}$$

$$\tau\{\alpha_m(x) - \alpha_{m-1}(x)\} = c_m \aleph\{\alpha_0(x)\} + \Sigma_{j=1}^{m-1} c_j[\tau\{\alpha_{m-j}(x)\} + \aleph_{m-j}\{\alpha_0(x) + \alpha_1(x) + ... + \alpha_{m-1}(x)\}] \tag{2.10}$$

where $m = 2, 3, ...$ and

$$\aleph_m\{\alpha_0(x) + \alpha_1(x) + ... + \alpha_{m-1}(x)\}$$

are the coefficient of ρ^m in the expansion of

$$N\{a(x, \rho)\}$$

about ρ.

$$\aleph\{\alpha(x, \rho, c_j)\} = \aleph_0\{\alpha_0(x)\} + \Sigma_{m=1}^{\infty} \aleph_m\{\alpha_0(x), \alpha_1(x), ... + \alpha_m(x)\}\rho^m \tag{2.11}$$

The result of m^{th} order approximation are follow;

$$\alpha^m(x, c_{i,j}) = \alpha_0(x) + \Sigma_{k=1}^{m} \alpha_k(x, c_j), j = 1, 2, ..., m \tag{2.12}$$

Substituting eq. (2.12) into (2.1), we get residual equation.

$$\Re(x, c_j) = \tau\{\alpha^m(x, c_j)\} + f(x) + \aleph\{\alpha^m(x, c_j)\} \tag{2.13}$$

If $\Re(x, c_j) = 0$ then $\alpha^m(x, c_j)$ will be the exact solution. For finding the constants c_j $, j = 1, 2, 3, ...$ Using Least Square Method, at first consider.

$$\Im(c_j) = \int_a^b \Re^2(X, c_j)dx \tag{2.14}$$

then the constants c_j $, j = 1, 2, 3, ...$ can be identified as follow.

$$\frac{\partial \Im}{\partial c_1} = \frac{\partial \Im}{\partial c_2} = \frac{\partial \Im}{\partial c_3} \tag{2.15}$$

Replacing the values of $c_j, j = 1, 2, 3, ...$ in eq. (2.13), we get the approximate solution.

3. Some numerical examples of nonlinear Volterra integral equations.

In this section we used OHAM to solve some nonlinear Volterra integral equations while the exact solution is also given.

Example 1. Consider a nonlinear second kind of Volterra integral equation with the exact solution $\psi(x) = x^2$ [15]

$$\psi(x) = x^2 + \frac{x^5}{10} - \frac{1}{2}\int_0^x \psi^2(t)dt. \tag{3.1}$$

we start from zero order solution and proceed similarly step by step.

$$\psi_0(x) = x^2 + \frac{x^5}{10} \tag{3.2}$$

which is the solution.

$$\psi_0(x) = \frac{1}{10}(10x^2 + x^5) \tag{3.3}$$

$$\psi_1(x) = -x^2 - \frac{x^5}{10} - x^2 c_1 - \frac{x^5 c_1}{10} + \psi_0 + c_1\psi_0 + \frac{1}{2}xc_1\psi_0^2 \tag{3.4}$$

$$\psi_1(x) = \frac{1}{200}x^5(10 + x^3)^2 c_1 \tag{3.5}$$

$$\psi_2(x) = -x^2 c_2 - \frac{x^5 c_2}{10} + c_2\psi_0 + \frac{1}{2}xc_2\psi_0^2 + \psi_1 + c_1\psi_1 + xc_1\psi_0\psi_1 \tag{3.6}$$

$$\psi_2(x) = \frac{x^5(10 + x^3)^2(10c_1 + (10 + 10x^3 + x^6)c_1^2 + 10c_2)}{2000} \tag{3.7}$$

$$\psi_3(x) = -x^2 c_3 - \frac{x^5 c_3}{10} + c_3\psi_0 + \frac{1}{2}xc_3\psi_0^2 + c_2\psi_1 + xc_2\psi_0\psi_1 + \frac{1}{2}xc_1\psi_1^2 + \psi_2 + c_1\psi_2 + xc_1\psi_0\psi_1 \tag{3.8}$$

$$\psi_3(x) = \frac{1}{16000} x^5 (10 + x^3)^2 (16(10 + 10x^3 + x^6)c_1^2$$
$$+(80 + 160x^3 + 116x^6 + 20x^9 + x^12)c_1^3 + \tag{3.9}$$
$$16c_1(5 + (10 + 10x^3 + x^6)c_2) + 80(c_2 + c_3)$$

The series solution is given as;

$$\psi(x) = \psi_0(x) + \psi_1(x) + \psi_2(x) + \psi_3(x) \tag{3.10}$$

That is,

$$\psi(x) = \frac{1}{16000} x^2 (10 + x^3)(24x^3(100 + 110x^3 + 20x^6 + x^9)c_1^2 + x^3$$
$$(800 + 1680x^3 + 1320x^6 + 316x^9 + 30x^{12} + x^{15})c_1^3 + 16x^3(10 + x^3)c_1(15 + (10 + 10x^3 + x^6)c_2) + \tag{3.11}$$
$$80(20 + 2x^3(10 + x^3)c_2 + x^3(10 + x^3)c_3)$$

For finding the values of c_i, we use the Least Square Method.

$c_1 = -0.1677940548, c_2 = 0.1114129522, c_3 = 0.0386473756.$

By putting the constant values of c_i in eq.(3.11), we get.

$$\psi(x) = -2.95263 \times 10^{-7} x^2 (10 + x^3)(-338681 + 33793x^3 - 2992.5x^6 - 274.362x^9 + 236.282x^{12} + 30x^{15} + x^{18}) \tag{3.12}$$

TABLE 1. In this table, we compared OHAM solution and exact solution of eq. (3.1), where λ represents the absolute error of OHAM.

x	OHAM solution	Exact solution	λ
0.0	0.0	0.0	0.0
0.1	0.01	0.01	2.20146×10^{-9}
0.2	0.0400001	0.04	6.79223×10^{-8}
0.3	0.0900005	0.09	4.65761×10^{-7}
0.4	0.160002	0.16	1.58682×10^{-6}
0.5	0.250003	0.25	3.24191×10^{-6}
0.6	0.360004	0.36	3.65805×10^{-6}
0.7	0.49	0.49	2.79695×10^{-7}
0.8	0.639996	0.64	4.44442×10^{-6}
0.9	0.810002	0.81	1.57289×10^{-6}
1.0	0.999986	1.0	0.0000142671

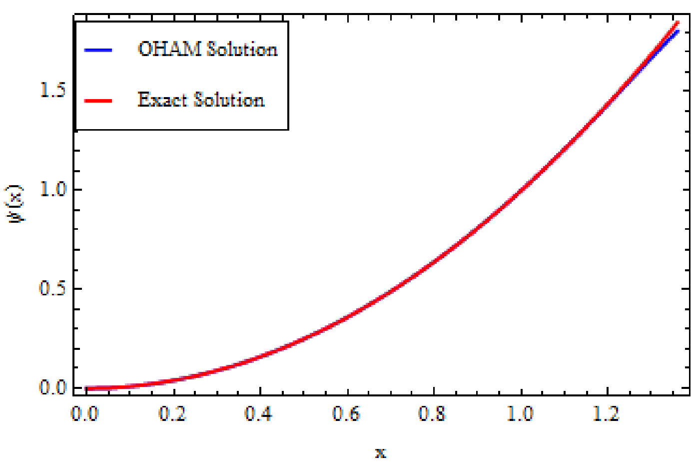

FIGURE 1. Shows the comparison of OHAM and Exact solution of the eq.(3.1)

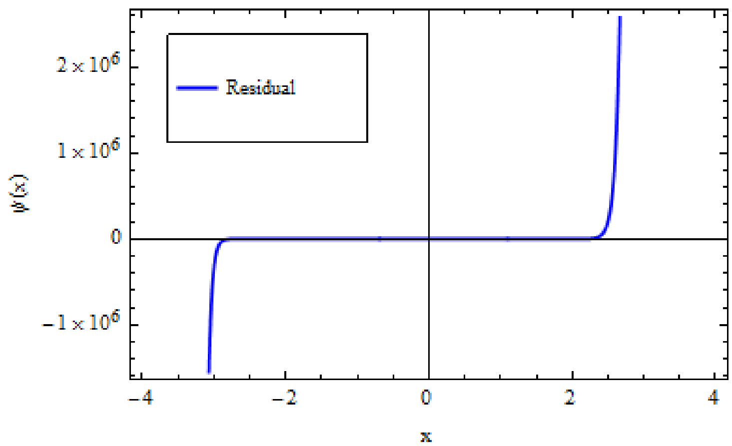

FIGURE 2. Shows the Residual solution of the problem.

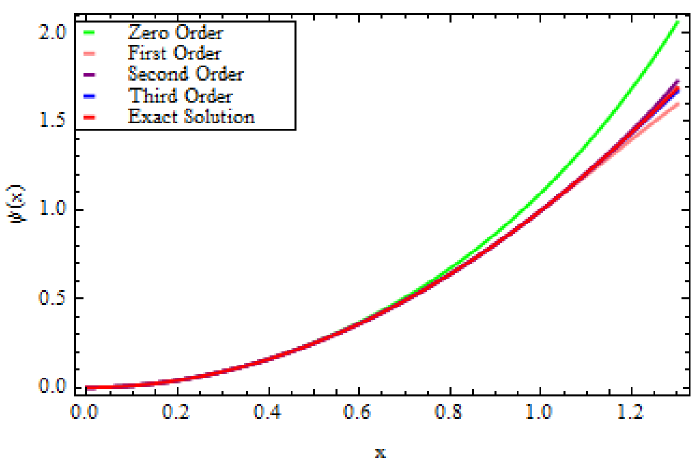

FIGURE 3. Shows the comparison of zero order, first order, second order and third order of OHAM solution and exact solution of eq.(3.1)

Example 2. Consider a nonlinear second kind of VIE with exact solution $\psi(x) = x$ [15]

$$\psi(x) = x - \frac{x^4}{4} + \int_0^x t\psi^2(t)dt \tag{3.13}$$

We used OHAM to find analytical solution.

$$\psi_0(x) = x - \frac{x^4}{4} \tag{3.14}$$

$$\psi_0(x) = \frac{1}{4}(4x - x^4) \tag{3.15}$$

$$\psi_1(x) = -x + \frac{x^4}{4} - xc_1 + \frac{x^4 c_1}{4} + \psi_0 + c_1\psi_0 - \frac{1}{2}x^2 c_1 \psi_0^2 \tag{3.16}$$

$$\psi_1(x) = -\frac{1}{32}x^4(-4+x^3)^2c_1 \tag{3.17}$$

$$\psi_2(x) = -xc_2 + \frac{x^4c_2}{4} + c_2\psi_0 - \frac{1}{2}x^2c_2\psi_0^2 + \psi_1 + c_1\psi_1 - x^2c_1\psi_0\psi_1 \tag{3.18}$$

$$\psi_2(x) = -\frac{1}{128}x^4(-4+x^3)^2(4c_1 + (-2+x^3)^2c_1^2 + 4c_2) \tag{3.19}$$

$$\begin{aligned} \psi_3(x) = -xc_3 + \frac{x^4c_3}{4} + c_3\psi_0 - \frac{1}{2}x^2c_3\psi_0^2 + c_2\psi_1 - \\ x^2c_2\psi_0\psi_1 - \frac{1}{2}x^2c_1\psi_1^2 + \psi_2 + c_1\psi_2 - x^2c_1\psi_0\psi_2 \end{aligned} \tag{3.20}$$

$$\begin{aligned} \psi_3(x) = -\frac{1}{2048}x^4(-4+x^3)^2(32(-2+x^3)^2c_1^2 + (64 - 128x^3 + 112x^6 - 40x^9 + 5x^12)c_1^3 \\ +32c_1(2 + (-2+x^3)^2c_2) + 64(c_2+c_3) \end{aligned} \tag{3.21}$$

The series solution is;

$$\psi(x) = \psi_0(x) + \psi_1(x) + \psi_2(x) + \psi_3(x) \tag{3.22}$$

That is,

$$\begin{aligned} \psi(x) = x - \frac{x^4}{4} - \frac{1}{32}x^4(-4+x^3)^2c_1 - \frac{1}{128}x^4(-4+x^3)^2(4c_1 + (-2+x^3)^2c_1^2 + 4c_2) - \\ \frac{1}{2048}x^4(-4+x^3)^2 32(-2+x^3)^2c_1^2 + (64 - 128x^3 + 112x^6 - 40x^9 + 5x^12)c_1^3 + \\ 32c_1(2 + (-2+x^3)^2c_2) + 64(c_2+c_3) \end{aligned} \tag{3.23}$$

For finding values of c_i , using Least Square Method.

$c_1 = -0.8102578861, c_2 = 0.5900091712, c_3 = 0.2244268082.$

By putting these values in eq.(3.23), we get

$$\psi(x) = x(1 + 0.0224215x^3 - 0.161442x^6 + 0.368418x^9 - 0.337199x^{12} + 0.12507x^{15} - 0.0207792x^{18} + 0.0012987x^{21}) \tag{3.24}$$

TABLE 2. In this table, we compared OHAM solution and exact solution of eq. (3.13), where λ represents the absolute error of OHAM.

x	OHAM solution	Exact solution	λ
0.0	0.0	0.0	0.0
0.1	0.100002	0.1	2.22604×10^{-6}
0.2	0.200034	0.2	0.0000338453
0.3	0.300148	0.3	0.000148429
0.4	0.400346	0.4	0.000345904
0.5	0.500461	0.5	0.000460563
0.6	0.600208	0.6	0.000207781
0.7	0.69962	0.7	0.000379802
0.8	0.799578	0.8	0.00042168
0.9	0.900738	0.9	0.000737938
1.0	0.997787	1.0	0.00221256

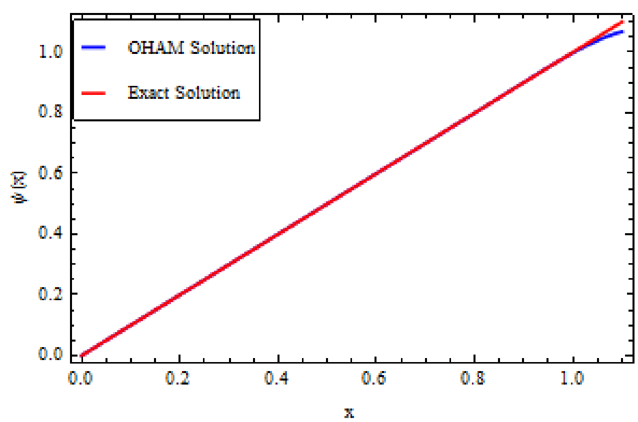

FIGURE 4. Shows the comparison of OHAM and Exact solution of the eq.(3.13)

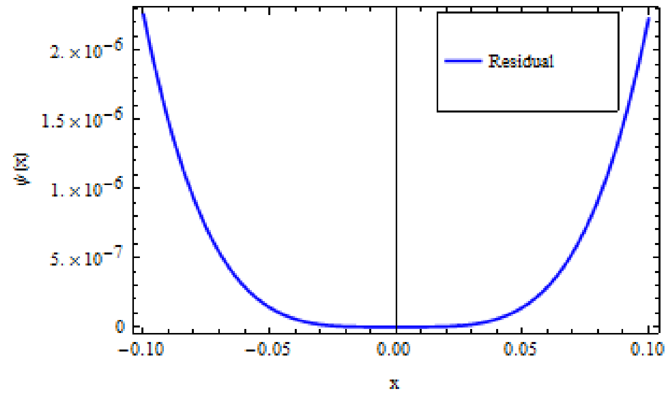

FIGURE 5. Shows the Residual solution of the problem.

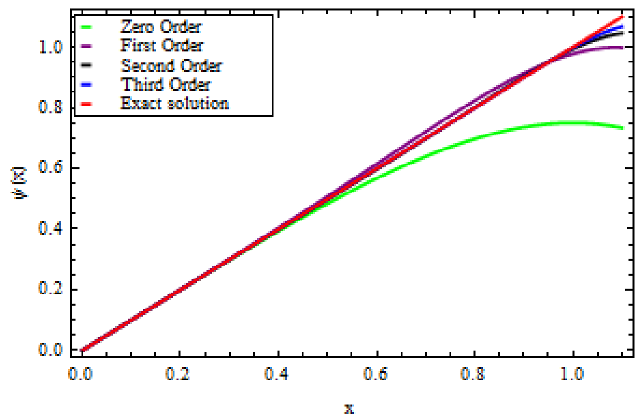

FIGURE 6. Shows the comparison of zero order, first order, second order and third order of OHAM solution and exact solution of eq.(3.13)

Example 3. Consider a nonlinear VIE with exact solution $\psi(x) = x^2$. [15]

$$\psi(x) = x^2 + \frac{x^6}{12} - \frac{1}{2}\int_0^x t\psi^2(t)dt \tag{3.25}$$

OHAM Solution:

$$\psi_0(x) = x^2 + \frac{x^6}{12} \tag{3.26}$$

$$\psi_0(x) = \frac{1}{12}(12x^2 + x^6) \tag{3.27}$$

$$\psi_1(x) = -x^2 - \frac{x^6}{12} - x^2c_1 - \frac{x^6c_1}{12} + \psi_0 + c_1\psi_0 + \frac{1}{4}x^2c_1\psi_0^2 \tag{3.28}$$

$$\psi_1(x) = \frac{1}{576}x^6(12 + x^4)^2c_1 \tag{3.29}$$

$$\psi_2(x) = -x^2c_2 - \frac{x^6c_2}{12} + c_2\psi_0 + \frac{1}{4}x^2c_2\psi_0^2 + \psi_1 + c_1\psi_1 + \frac{1}{2}x^2c_1\psi_0\psi_1 \tag{3.30}$$

$$\psi_2(x) = \frac{x^6(12+x^4)^2(24c_1 + (24+12x^4+x^8)c_1^2 + 24c_2)}{13824} \tag{3.31}$$

$$\psi_3(x) = -x^2c_3 - \frac{x^6c_3}{12} + c_3\psi_0 + \frac{1}{4}x^2c_2\psi_0^2 + \psi_1 + c_1\psi_1 + \frac{1}{2}x^2c_1\psi_0\psi_1 \tag{3.32}$$

$$\begin{aligned}\psi_3(x) = \frac{1}{1327104}x^6(12+x^4)(192(24+12x^4+x^8)c_1^2 + (2304+2304x^4+912x^8+120x^{12}+5x^{16})c_1^3 \\ +192c_1(12+(24+12x^4+x^8)c_2) + 2304(c_2+c_3)\end{aligned} \tag{3.33}$$

The series solution is given below;

$$\psi_0(x) = \psi_0(x) + \psi_1(x) + \psi_2(x) + \psi_3(x) \tag{3.34}$$

That is,

$$\begin{aligned}\psi(x) = \frac{1}{1327104}x^2(12+x^4)(288x^4(12+x^4)(24+12x^4+x^8)c_1^2 + x^4(12+x^4)(2304+2304x^4+ \\ 912x^8+120x^{12}+5x^{16})c_1^3 + 192x^4(12+x^4)c_1(36+(24-12x^4+x^8)c_2)+ \\ 2304(48+2x^4(12+x^4)c_2 + x^4(12+x^4)c_3))\end{aligned} \tag{3.35}$$

To find the values of c_i, where $c_i = 1, 2, 3, ...$, we use Least Square Method.

$c_1 = -0.2887286851, c_2 = 0.1652477727, c_3 = 0.0721620291.$

Put the values of c_i in eq.(3.35), we get.

$$\begin{aligned}\psi(x) = -9.06849\times 10^{-8}x^2(12+x^4)\times \\ (-918933 + 76484.9x^4 - 5863x^4 - 5863x^8 - 311.456x^{12} + 347.023x^{16} + 36x^{20} + x^{24})\end{aligned} \tag{3.36}$$

TABLE 3. In this table, we compared OHAM solution and exact solution of eq. (3.25), where λ represents the absolute error of OHAM.

x	OHAM solution	Exact solution	λ
0.0	0.0	0.0	0.0
0.1	0.01	0.1	1.01049×10^{-10}
0.2	0.04	0.2	6.41391×10^{-9}
0.3	0.0900001	0.3	7.04647×10^{-8}
0.4	0.16	0.4	3.58157×10^{-7}
0.5	0.250001	0.5	1.08877×10^{-6}
0.6	0.360002	0.6	2.00225×10^{-6}
0.7	0.490002	0.7	1.50262×10^{-6}
0.8	0.639998	0.8	1.71305×10^{-6}
0.9	0.809999	0.9	5.7885×10^{-7}
1.0	0.999991	1.0	8.55913×10^{-6}

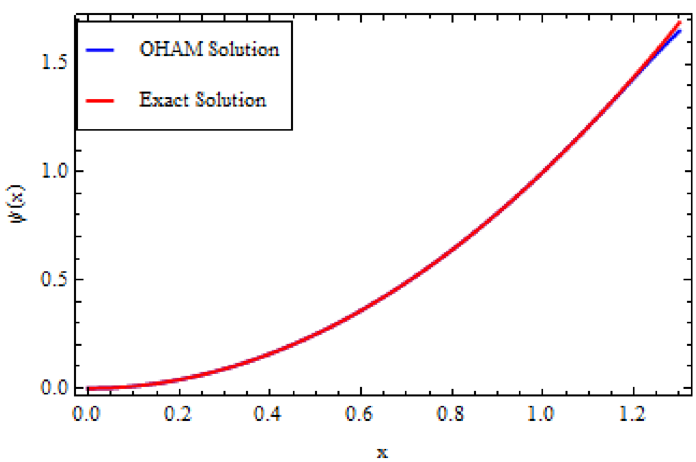

FIGURE 7. Shows the comparison of OHAM and Exact solution of the eq.(3.25)

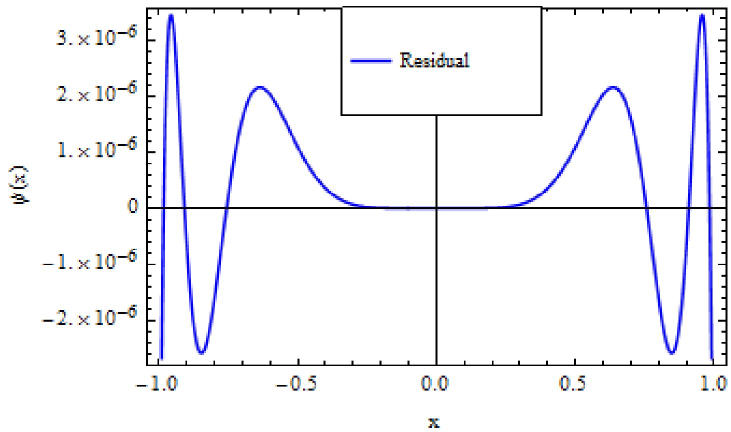

FIGURE 8. Shows the Residual solution of the problem.

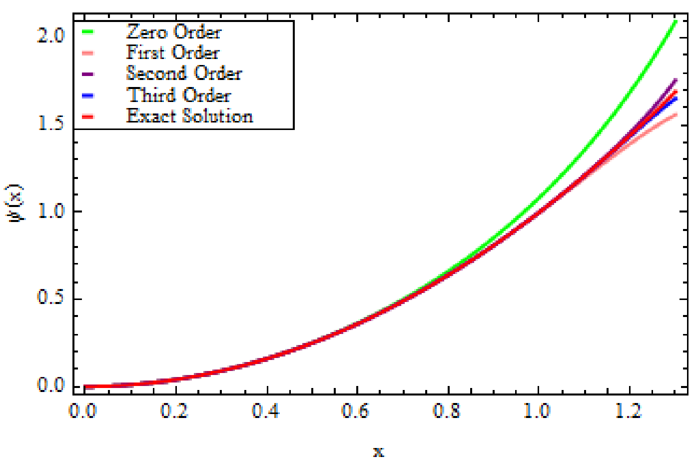

FIGURE 9. Shows the comparison of zero order, first order, second order and third order of OHAM solution and exact solution of eq.(3.25)

4. CONCLUSION

In this research article, we presented the application of (OHAM) by solving some examples of nonlinear Volterra integral equations of the second kind. This technique is verified on three different problems.The technique showed to be an accurate and well-organized method for finding approximate solutions for the nonlinear Volterra integral equations of the second kind. The (OHAM) is relatively simple to apply. It is shown that, with few terms, the method is capable of giving sufficient accuracy.This method can be a promising tool for solving strongly nonlinear problems. The convergence of (OHAM) to exact solution is very excellent and quick.

5. ACKNOWLEDGMENT

The authors would like to thanks the reviewers for the valuable comments and suggestion which help in the improvement of this paper.

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] Kumar, S., Singh, OmP. Dixit, S., Generalized Abel inversion using Homotopy perturbation method, Appl. Math. 2 (2011), 254-257.

[2] Jafari, H., Gejji, V.D., Revised Adomian decomposition method for solving a system of nonlinear equations, Appl. Math. Comput. 175 (2006), 1-7.

[3] Ghoreishi, M., Ismail, A.M. and Alomari, A.K., Comparison between Homotopy analysis method and optimal Homotopy asymptotic method for nthorder Integrodifferential equation. Math. Methods Appl. Sci. 34 (15) (2011), 1833-1842.

[4] Ayub M, Haq S, Siddiqui A.M, Hayat T., Group analysis of two dimensional flow of a non-Newtonian fluid, Islamabad J. Sci. 13 (1) (2003), 47-59.

[5] Ganji D.D, Afrouzi R. A., The application of differential transformation method to nonlinear equation arising in heat transfer, Int. Commun. Heat Mass Transfer, 38 (6), 815-820 (2011).

[6] Avaji, M., Hafshejani, J.S., Dehcheshmeh, S.S. and Ghahfarokhi, D.F., Solution of delay Volterra integral equations using the Variational iteration method. J. Appl. Sci. 12 (2) (2012), 196-200.

[7] Feng, J.Q. and Sun, S., Numerical Solution of Volterra Integral Equation by Adomian Decomposition Method. Asian Res. J. Math. 4 (1) (2017), Article no. ARJOM.33105.

[8] Wazwaz, A. M. Linear and nonlinear integral equations (Vol. 639). Heidelberg: Springer. (2011).

[9] Yang, L. H., Li, H. Y., and Wang, J. R. Solving a system of linear Volterra integral equations using the modified reproducing kernel method. Abstr. Appl. Anal. 2013 (2013), Art. ID 196308.

[10] Wazwaz, A. M. Linear and nonlinear integral equations (Vol. 639). Heidelberg: Springer. (2011).

[11] Marinca, V. and Herianu, N., Application of optimal Homotopy asymptotic method for solving nonlinear equations arising in heat transfer. International Commun. Heat Mass Transfer, 35 (6) (2008), 710-715.

[12] Yang, L. H., Li, H. Y., and Wang, J. R., Solving a system of linear Volterra integral equations using the modified reproducing kernel method. Abstr. Appl. Anal. 2013 (2013), Art. ID 196308.

[13] Maleknejad, K., Mahmoudi, Y., Taylor polynomial Solution of high-Order nonlinear Volterra Fredholm Integro-diffrential equations, Appl. Math. Comput. 145 (2-3) (2003), 641-653.

[14] Almousa, M.S.T., Approximate Analytical Methods for Solving Fredholm Integral Equations, Doctoral dissertation, University Sains Malaysia, 2015.

[15] Wazwaz, A.M., A first course in integral equations. World Scientific, Singapore, 2015.