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COUPLED BEST PROXIMITY POINT THEOREM IN METRIC SPACES

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Abstract. The purpose of this article is to generalized the result of W. Sintunavarat and P. Kumam [29]. We also give an example in support of our theorem for which result of W. Sintunavarat and P. Kumam [29] is not true. Moreover we establish the existence and convergence theorems of coupled best proximity points in metric spaces, we apply this results in a uniformly convex Banach space.

Contents

This article is organized in the following order.

1. Introduction and Preliminaries

Fixed point theory is one of the most useful tools in analysis. The first result of fixed point theorem is given by Banach S. [4] by the general setting of complete metric space using which is known as Banach Contraction Principle. This principle has been generalized by many researchers in many ways like by [2], [9], [10], [24], [33], [34], [40] and so on.

One of the important thing in [4] is Banach contraction principle is true for self mapping. In case of non self mapping (say T) the mapping does not has a fixed point. Then the researchers find an element x such that $d(x, Tx)$ is minimum or near to zero for a given problem which implies that x and Tx are very closed says close proximity to each other. Due to this problem the theory of fixed point is converted into the theory of best proximity point. On the other words, proximity

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theory is a generalization of fixed point theory.

Basically the proximity theory is useful tool to find proximity point when the given mapping is non self. Let A and B be two non empty subsets of X such that $T : A \to B$ then a point $x \in A$ for which $d(x, Tx) = d(A, B)$ is called a best proximity point of T. It should be noted that best proximity point reduced to fixed point when the mapping T is self mapping that is $A = B$.

In 1969, Fan [12] presented a classical result for best approximation theorem which as follows,

Theorem 1 ([12])**.** *If A is a nonempty convex subset of a Hausdorff locally convex topological vector space B and $T : A \to B$ is continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$.*

Afterword a number of authors have derived extensions of Fan's Theorem and the best approximation theorem in many directions such as Prolla [26], Sehgal and Singh [27, 28], Wlodarczyk and Plebaniak [43, 44, 45, 46], Vetrivel et al. [42], Eldred and Veramani [11], Mongkolkeha and Kumam [25] and Basha and Veeramani [5, 6, 7, 8] (see also [3, 15, 16, 17, 18, 19, 20, 21] and reference therein.)

Beside this, Bhaskar and Lakshmikantham [13] introduced the notion of mixed monotone mapping and proved some coupled fixed point theorems for mapping satisfying mixed monotone property. After the result of [13] there are lots of work presented by many authors such as [1], [14], [30], [31] (see also reference therein.)

The concept of coupled best proximity point theorem is introduced by W. Sintunavarat and P. Kumam [29] and proved coupled best proximity theorem for cyclic contraction. Our purpose of this article is to generalized the result of [29] also we give an example in support of our main theorem.

First we recall some basic definitions and examples that are related to the main results of this article. Throughout this article we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers. For nonempty subsets A and B of a metric space (X, d), we let

$$d(A, B) = \inf\{d(x, y) : x \in A \ \ \textit{and} \ \ y \in B\} \tag{1.1}$$

stands for the distance between A and B.

A Banach spaces X is said to be

(1) *strictly convex* if the following implication holds for all $x, y \in X$:

$$\|x\| = \|y\| = 1 \text{ and } x \neq y \Longrightarrow \|\tfrac{x+y}{2}\| < 1.$$

(2) *uniformly convex* if for each ϵ with $0\epsilon \leq 2$, there exists $\delta > 0$ such that thee following implication holds for all $x, y \in X$:

$$\|x\| \leq 1, \|y\| \leq 1 \text{ and } \|x - y\| \geq \epsilon \Longrightarrow \|\tfrac{x+y}{2}\| < 1 - \delta.$$

It is easily to see that a uniformly convex Banach space X is strictly but the converges is not true.

Definition 2 ([41])**.** *Let A and B be nonempty subsets of a metric space (X, d). The ordered pair (A, B) satisfies the property UC if the following holds:*

If $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $d(x_n, y_n) \to d(A, B)$ and $d(z_n, y_n) \to d(A, B)$, then $d(x_n, z_n) \to 0$.

Example 3. *Let A and B be nonempty subsets of a metric space (X, d). The following are examples of a pair of nonempty subsets (A, B) satisfying the property UC.*

(1) *Every pair of nonempty subsets A, B of a metric space (X, d) such that $d(A, B) = 0$.*
(2) *Every pair of nonempty subsets A, B of a uniformly convex Banach space X such that A is convex.*
(3) *Every pair of nonempty subsets A, B of a strictly convex Banach space which A is convex and relatively compact and the closure of B is weakly compact.*

Definition 4 ([39])**.** *Let A and B be nonempty subsets of a metric space (X, d). The ordered pair (A, B) satisfies the property UC* if (A, B) has property UC and the following condition holds:*

If $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B satisfying:

(1) $d(z_n, y_n) \to d(A, B)$
(2) *For every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$d(x_m, y_n) \leq d(A, B) + \epsilon$$

for all $m > n \geq N$. Then for every $\epsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that

$$d(x_m, z_n) \leq d(A, B) + \epsilon$$

for all $m > n \geq N_1$.

Example 5 ([39])**.** *Let A and B be nonempty subsets of a metric space (X, d). The following are examples of a pair of nonempty subsets (A, B) satisfying the property UC*.*

(1) *Every pair of nonempty subsets A, B of a metric space (X, d) such that $d(A, B) = 0$.*
(2) *Every pair of nonempty closed subsets A, B of a uniformly convex Banach space X such that A is convex.*

Definition 6. *Let A and B be nonempty subsets of a metric space (X, d) and $T : A \to B$ be a mapping. A point $x \in A$ is said to be a best proximity point of T if it satisfies the condition that*

$$d(x, Tx) = d(A, B).$$

It can be observed that a best proximity point reduces to a fixed point if the underlying mapping is a self mapping.

Definition 7 ([13])**.** *Let A be a nonempty subset of a metric space X and $F : A \times A \to A$. A point $(x, y) \in A \times A$ is called a coupled fixed point of F if*

$$x = F(x, y), \quad y = F(y, x).$$

2. **Coupled best proximity point theorems**

In this section we study the existence and convergence of coupled best proximity points for cyclic contraction pair.

Definition 8. *Let A and B be nonempty subsets of a metric space X and $F : A \times A \to B$. An ordered coupled $(x, y) \in A \times A$ is called a coupled best proximity point of F if,*

$$d(x, F(x, y)) = d(y, F(y, x)) = d(A, B).$$

It is easy to see that if $A = B$ in Definition 8, then a coupled best proximity point reduces to a coupled fixed point.

Next,W. Sintunavarat and P. Kumam [29] introduce the notion of a cyclic contraction for two mappings which as follows.

Definition 9. *Let A and B be nonempty subsets of a metric space X, $F : A\times A \to B$ and $G : B \times B \to A$. The ordered pair (F, G) is said to be a cyclic contraction if there exists a non-negative number $\alpha < 1$ such that*

$$d(F(x, y), G(u, v)) \leq \frac{\alpha}{2}[d(x, u) + d(y, v)] + (1 - \alpha)d(A, B)$$

for all $(x, y) \in A \times A$ and $(u, v) \in B \times B$.

Now we introduced the following notion of cyclic contraction for two mappings which is generalization of [29] as follows.

Definition 10. *Let A and B be nonempty subsets of a metric space X, $F : A\times A \to B$ and $G : B \times B \to A$. The ordered pair (F, G) is said to be a cyclic contraction if there exists a non-negative number $p + q < 1$ such that*

$$d(F(x, y), G(u, v)) \leq [pd(x, u) + qd(y, v)] + (1 - (p + q))d(A, B)$$

for all $(x, y) \in A \times A$ and $(u, v) \in B \times B$.

Note that if (F, G) is a cyclic contraction, then (G, F) is also a cyclic contraction. Also if we take $p = q = \frac{\alpha}{2}$ in Definition 10 then we get Definition 9.

Following example show that Definition 10 is generalization of Definition 9.

Example 11. *Let $X = \mathbb{R}$ with the usual metric $d(x, y) =\mid x - y \mid$ also $A = [6, 12]$ and $B = [-12, -6]$. It easy to see that $d(A, B) = 12$. Define $F : A \times A \to B$ and $G : B \times B \to A$ by*

$$F(x, y) = \frac{-3x - 2y - 6}{6}$$

and

$$G(x, y) = \frac{-3x - 2y + 6}{6}.$$

For arbitrary $(x, y) \in A \times A, (u, v) \in B \times B$ and fixed $k = \frac{1}{2}$, $l = \frac{1}{3}$, we get

$$\begin{aligned} d(F(x, y), G(u, v)) &= \left| \frac{-3x - 2y - 6}{6} - \frac{-3u - 2v + 6}{6} \right| \\ &\leq \frac{3 \mid x - u \mid +2 \mid y - v \mid}{6} + 2 \\ &= kd(x, u) + ld(y, v) + (1 - (k + l))d(A, B). \end{aligned}$$

This implies that (F, G) is a cyclic contraction with $p = \frac{1}{2}$ and $q = \frac{1}{3}$.

The following lemma plays an important role in our main results.

Lemma 12. *Let A and B be nonempty subsets of a metric space X, $F : A\times A \to B$ and $G : B \times B \to A$ and (F, G) be a cyclic contraction. If $(x_0, y_0) \in A \times A$ and we define*

$$\begin{aligned} x_{n+1} = F(x_n, y_n), \quad x_{n+2} = G(x_{n+1}, y_{n+1}) \\ y_{n+1} = F(y_n, x_n), \quad y_{n+2} = G(y_{n+1}, x_{n+1}) \end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$, then $d(x_n, x_{n+1}) \to d(A, B)$, $d(x_{n+1}, x_{n+2}) \to d(A, B)$, $d(y_n, y_{n+1}) \to d(A, B)$ and $d(y_{n+1}, y_{n+2}) \to d(A, B)$.

Proof. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} & d(x_n, x_{n+1}) = d(F(x_n, y_n), G(x_{n-1}, y_{n-1})) \\ \leq \quad & pd(x_n, x_{n-1}) + qd(y_n, y_{n-1}) + (1 - (p + q))d(A, B) \end{aligned}$$

Similarly we have

$$\begin{aligned} & d(y_n, y_{n+1}) = d(F(y_n, x_n), G(y_{n-1}, x_{n-1})) \\ \leq \quad & pd(y_n, y_{n-1}) + qd(x_n, x_{n-1}) + (1 - (p + q))d(A, B) \end{aligned}$$

Therefore, by letting

$$d_n = d(x_n, x_{n+1}) + d(y_n, y_{n+1})$$

by adding above inequality we have

$$d_n \leq (p + q)d_{n-1} + 2(1 - (p + q))d(A, B)$$

Similarly we can show that

$$d_{n-1} \leq (p + q)d_{n-2} + 2(1 - (p + q))d(A, B)$$

Consequently we have

$$d_1 \leq (p + q)d_0 + 2(1 - (p + q))d(A, B)$$

If $d_0 = 0$ then (x_0, y_0) is a coupled best proximity point of F and G. Now let $d_0 > 0$ for each $n \geq m$ we have

$$\begin{aligned} d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \dots\dots\dots + d(x_{m+1}, x_m) \\ d(y_n, y_m) \leq d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2}) + \dots\dots\dots + d(y_{m+1}, y_m) \end{aligned}$$

which gives

$$\begin{aligned} d(x_n, x_m) + d(y_n, y_m) \leq d_{n-1} + d_{n-2} + d_{n-3} \dots\dots + d_m \\ d_n \leq (p + q)^n d_0 + 2^n(1 - (p + q)^n)d(A, B) \end{aligned}$$

Taking $n \to \infty$ we have

$$d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \to d(A, B)$$

implies that

$$d(x_n, x_{n+1}) \to d(A, B)$$
$$d(y_n, y_{n+1}) \to d(A, B)$$

for all $n \in \mathbb{N}$.

By similar argument, we also have

$$d(x_{n+1}, x_{n+2}) \to d(A, B),$$
$$d(y_{n+1}, y_{n+2}) \to d(A, B).$$

□

Lemma 13. *Let A and B be nonempty subsets of a metric space X such that (A, B) and (B, A) have a property UC, $F : A \times A \to B$ and $G : B \times B \to A$ and let the ordered pair (F, G) is a cyclic contraction. If $(x_0, y_0) \in A \times A$ and define*

$$x_{n+1} = F(x_n, y_n), \quad x_{n+2} = G(x_{n+1}, y_{n+1})$$
$$y_{n+1} = F(y_n, x_n), \quad y_{n+2} = G(y_{n+1}, x_{n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$, then for $\epsilon > 0$, there exists a positive integer N_0 such that for all $m > n \geq N_0$

$$pd(x_m, x_{n+1}) + qd(y_m, y_{n+1}) < d(A, B) + \epsilon. \tag{2.1}$$

Proof. By Lemma 12, we have

$$d(x_n, x_{n+1}) \to d(A, B), \quad d(x_{n+1}, x_{n+2}) \to d(A, B),$$
$$d(y_n, y_{n+1}) \to d(A, B), \quad d(y_{n+1}, y_{n+2}) \to d(A, B).$$

Since (A, B) has a property UC, we get

$$d(x_n, x_{n+2}) \to 0.$$

A similar argument shows that

$$d(y_n, y_{n+2}) \to 0.$$

As (B, A) has a property UC, we also have

$$d(x_{n+1}, x_{n+3}) \to 0,$$
$$d(y_{n+1}, y_{n+3}) \to 0.$$

Suppose that (2.1) does not hold. Then there exists $\epsilon' > 0$ such that for all $k \in \mathbb{N}$, there is $m_k > n_k \geq k$ satisfying

$$pd(x_{m_k}, x_{n_k+1}) + qd(y_{m_k}, y_{n_k+1}) \geq d(A, B) + \epsilon'.$$

Further, corresponding to n_k, we can choose m_k in such a way that it is the smallest integer with $m_k > n_k$ and satisfying above relation. Then

$$pd(x_{m_k-2}, x_{n_k+1}) + qd(y_{m_k-2}, y_{n_k+1}) < d(A, B) + \epsilon'.$$

Therefore, we get

$$\begin{aligned} d(A, B) + \epsilon' &\leq pd(x_{m_k}, x_{n_k+1}) + qd(y_{m_k}, y_{n_k+1}) \\ &\leq p[d(x_{m_k}, x_{m_k-2}) + d(x_{m_k-2}, x_{n_k+1})] \\ &\quad +q[d(y_{m_k}, y_{m_k-2}) + d(y_{m_k-2}, y_{n_k+1})] \\ &< pd(x_{m_k}, x_{m_k-2}) + qd(y_{m_k}, y_{m_k-2})] + d(A, B) + \epsilon'. \end{aligned}$$

Letting $k \to \infty$, we obtain to see that

$$pd(x_{m_k}, x_{n_k+1}) + qd(y_{m_k}, y_{n_k+1}) \to d(A, B) + \epsilon'.$$

By using the triangle inequality, we get

$$
\begin{aligned}
& pd(x_{m_k}, x_{n_k+1}) + qd(y_{m_k}, y_{n_k+1}) \\
\leq\ & p[d(x_{m_k}, x_{m_k+2}) + d(x_{m_k+2}, x_{n_k+3}) + d(x_{n_k+3}, x_{n_k+1}) \\
& +q[d(y_{m_k}, y_{m_k+2}) + d(y_{m_k+2}, y_{n_k+3}) + d(y_{n_k+3}, y_{n_k+1})] \\
=\ & p[d(x_{m_k}, x_{m_k+2}) + d(G(x_{m_k+1}, y_{m_k+1}), F(x_{n_k+2}, y_{n_k+2})) + d(x_{n_k+3}, x_{n_k+1})] \\
& +q[d(y_{m_k}, y_{m_k+2}) + d(G(y_{m_k+1}, x_{m_k+1}), F(y_{n_k+2}, x_{n_k+2})) + d(y_{n_k+3}, y_{n_k+1})] \\
\leq\ & p[d(x_{m_k}, x_{m_k+2}) + pd(x_{m_k+1}, x_{n_k+2}) + qd(y_{m_k+1}, y_{n_k+2}) \\
& +(1-(p+q))d(A,B) + d(x_{n_k+3}, x_{n_k+1})] \\
& +q[d(y_{m_k}, y_{m_k+2}) + pd(y_{m_k+1}, y_{n_k+2}) + qd(x_{m_k+1}, x_{n_k+2}) + (1-(p+q))d(A,B) + d(y_{n_k+3}, y_{n_k+1}) \\
\leq\ & (p+q)[d(x_{m_k}, x_{m_k+2}) + d(x_{n_k+3}, x_{n_k+1}) + d(y_{m_k}, y_{m_k+2}) + d(y_{n_k+3}, y_{n_k+1})] \\
& +(p+q)^2[d(x_{m_k+1}, x_{n_k+2}) + d(y_{m_k+1}, y_{n_k+2})] + (1-(p+q)^2)d(A,B).
\end{aligned}
$$

Taking $k \to \infty$, we get

$$d(A,B) + \epsilon' \leq (p+q)^2[d(A,B) + \epsilon'] + (1-(p+q)^2)d(A,B) = d(A,B) + (p+q)^2\epsilon'$$

which contradicts. Therefore, we can conclude that (2.1) holds. □

Lemma 14. *Let A and B be nonempty subsets of a metric space X, (A,B) and (B,A) satisfy the property UC*. Let $F : A \times A \to B$, $G : B \times B \to A$ and (F,G) be a cyclic contraction. If $(x_0, y_0) \in A \times A$ and define*

$$
\begin{aligned}
x_{n+1} &= F(x_n, y_n) \\
y_{n+1} &= F(y_n, x_n)
\end{aligned}
$$

and

$$
\begin{aligned}
x_{n+2} &= G(x_{n+1}, y_{n+1}) \\
y_{n+2} &= G(y_{n+1}, x_{n+1})
\end{aligned}
$$

for all $n \in \mathbb{N} \cup \{0\}$, then $\{x_n\}$, $\{y_n\}$, $\{x_{n+1}\}$ and $\{y_{n+1}\}$ are Cauchy sequences.

Proof. By Lemma 12, we have $d(x_n, x_{n+1}) \to d(A,B)$ and $d(x_{n+1}, x_{n+2}) \to d(A,B)$. Since (A,B) satisfies property UC, we get $d(x_n, x_{n+2}) \to 0$. Similarly, we also have $d(x_{n+1}, x_{n+3}) \to 0$ because (B,A) satisfies property UC.

We now show that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(x_m, x_{n+1}) \leq d(A,B) + \epsilon \tag{2.2}$$

for all $m > n \geq N$

Suppose (2.2) not hold, then there exists $\epsilon > 0$ such that for all $k \in \mathbb{N}$ there exists $m_k > n_k \geq k$ such that

$$d(x_{m_k}, x_{n_k+1}) > d(A,B) + \epsilon. \tag{2.3}$$

Further, corresponding to n_k, we can choose m_k in such a way that it is the smallest integer with $m_k > n_k$ and satisfying above relation. Now we have

$$
\begin{aligned}
d(A,B) + \epsilon &< d(x_{m_k}, x_{n_k+1}) \\
&\leq d(x_{m_k}, x_{m_k-2}) + d(x_{m_k-2}, x_{n_k+1}) \\
&\leq d(x_{2m_k}, x_{2m_k-2}) + d(A,B) + \epsilon.
\end{aligned}
$$

Taking $k \to \infty$, we have $d(x_{2m_k}, x_{2n_k+1}) \to d(A,B) + \epsilon$. By Lemma 13, there exists $N \in \mathbb{N}$ such that

$$pd(x_{m_k}, x_{n_k+1}) + qd(y_{m_k}, y_{n_k+1}) < d(A,B) + \epsilon \tag{2.4}$$

for all $m > n \geq \mathbb{N}$. By using the triangle inequality, we get

$$\begin{aligned}
& d(A,B) + \epsilon \\
< \; & d(x_{m_k}, x_{n_k+1}) \\
\leq \; & d(x_{m_k}, x_{m_k+2}) + d(x_{m_k+2}, x_{n_k+3}) + d(x_{n_k+3}, x_{n_k+1}) \\
= \; & d(x_{m_k}, x_{m_k+2}) + d(G(x_{m_k+1}, y_{m_k+1}), F(x_{n_k+2}, y_{n_k+2})) \\
& + d(x_{n_k+3}, x_{n_k+1}) \\
\leq \; & d(x_{m_k}, x_{m_k+2}) + [pd(x_{m_k+1}, x_{n_k+2}) + qd(y_{m_k+1}, y_{n_k+2})] \\
& + (1-(p+q))d(A,B) + d(x_{n_k+3}, x_{n_k+1}) \\
= \; & p[d(F(x_{m_k}, y_{m_k}), G(x_{n_k+1}, y_{n_k+1}))] + q[d(F(y_{m_k}, x_{m_k}), G(y_{n_k+1}, x_{n_k+1}))] \\
& + (1-(p+q))d(A,B) + d(x_{m_k}, x_{m_k+2}) + d(x_{n_k+3}, x_{n_k+1}) \\
\leq \; & p\Big[p[d(x_{m_k}, x_{n_k+1}) + qd(y_{m_k}, y_{n_k+1}) + (1-(p+q))d(A,B)]\Big] \\
& + q\Big[[pd(y_{m_k}, y_{n_k+1}) + qd(x_{m_k}, x_{n_k+1}) + (1-(p+q))d(A,B)]\Big] \\
& + (1-(p+q))d(A,B) + d(x_{m_k}, x_{m_k+2}) + d(x_{n_k+3}, x_{n_k+1}) \\
= \; & (p+q)^2[d(x_{m_k}, x_{n_k+1}) + d(y_{m_k}, y_{n_k+1})] \\
& + (1-(p+q)^2)d(A,B) + d(x_{m_k}, x_{m_k+2}) + d(x_{n_k+3}, x_{n_k+1}) \\
< \; & (p+q)^2(d(A,B)+\epsilon) + (1-(p+q)^2)d(A,B) + d(x_{m_k}, x_{m_k+2}) + d(x_{n_k+3}, x_{n_k+1}) \\
= \; & (p+q)^2\epsilon + d(A,B) + d(x_{m_k}, x_{m_k+2}) + d(x_{n_k+3}, x_{n_k+1}).
\end{aligned}$$

Taking $k \to \infty$, we get

$$d(A,B) + \epsilon \leq d(A,B) + (p+q)^2\epsilon$$

which contradicts. Therefore, condition (2.2) holds. Since (2.2) holds and $d(x_n, x_{n+1}) \to d(A,B)$, by using property UC* of (A,B), we have $\{x_n\}$ is a Cauchy sequence. In similar way, we can prove that $\{y_n\}$,$\{x_{n+1}\}$ and $\{y_{n+1}\}$ are Cauchy sequences. □

Here we state the main results of this article in the existence and convergence of coupled best proximity points for cyclic contraction pairs on nonempty subsets of metric spaces satisfying the property UC*.

Theorem 15. *Let A and B be nonempty closed subsets of a metric space X such that (A,B) and (B,A) have a property UC*, $F : A \times A \to B$ and $G : B \times B \to A$ and let the ordered pair (F,G) is a cyclic contraction. If $(x_0, y_0) \in A \times A$ and define*

$$\begin{aligned} x_{n+1} &= F(x_n, y_n) \\ y_{n+1} &= F(y_n, x_n) \end{aligned}$$

and

$$\begin{aligned} x_{n+2} &= G(x_{n+1}, y_{n+1}) \\ y_{n+2} &= G(y_{n+1}, x_{n+1}) \end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. *Then* F *has a coupled best proximity point* $(r, s) \in A^3$ *and* G *has a coupled best proximity point* $(p', q', r') \in B^3$. *Moreover, we have* $x_n \to r, \ \ y_n \to s, \ \ x_{n+1} \to r', \ \ y_{n+1} \to s'$.

Furthermore, if $r = s$ *and* $r' = s'$, *then*

$$d(r, r') + d(s, s') = 2d(A, B).$$

Proof. By Lemma 12, we get $d(x_n, x_{n+1}) \to d(A, B)$. Using Lemma 14, we have $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Thus, there exists $r, s \in A$ such that $x_n \to r, \ \ y_n \to s$.

We obtain that

$$d(A, B) \leq d(r, x_{n-1}) \leq d(r, x_n) + d(x_n, x_{n-1}). \tag{2.5}$$

Letting $n \to \infty$ in (2.5), we have $d(r, x_{n-1}) \to d(A, B)$. By a similar argument we also have $d(s, y_{n-1}) \to d(A, B)$. It follows that

$$\begin{aligned} d(x_n, F(r, s)) &= d(G(x_{n-1}, y_{n-1}), F(r, s)) \\ &\leq pd(x_{n-1}, p) + qd(y_{n-1}, q) + (1 - (p + q))d(A, B). \end{aligned}$$

Taking $n \to \infty$, we get $d(p, F(p, q, r)) = d(A, B)$. Similarly, we can prove that

$$d(s, F(s, r)) = d(A, B)$$

Therefore, we have (r, s) is a coupled best proximity point of F.

In similar way, we can prove that there exists $r', s' \in B$ such that $x_{n+1} \to r'$ and $y_{n+1} \to s'$. Moreover, we have

$$d(r', G(r', s')) = d(A, B),$$

and

$$d(s', F(s', r')) = d(A, B)$$

and so (r', s') is a coupled best proximity point of G.

Finally, we assume that $r = s$ and $r' = s'$ and then we show that

$$d(r, r') + d(s, s') = 2d(A, B).$$

For all $n \in \mathbb{N}$, we obtain that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(G(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\leq pd(x_{n-1}, x_n) + qd(y_{n-1}, y_n) + (1 - (p + q))d(A, B). \end{aligned}$$

Letting $n \to \infty$, we have

$$d(r, r') \leq pd(r, r') + d(s, s') + (1 - (p + q))d(A, B). \tag{2.6}$$

For all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(y_n, y_{n+1}) &= d(G(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ &\leq pd(y_{n-1}, y_n) + qd(x_{n-1}, x_n) + (1 - (p + q))d(A, B). \end{aligned}$$

Letting $n \to \infty$, we have

$$d(s, s') \ \leq \ pd(s, s') + d(r, r') + (1 - (p + q))d(A, B).$$

Similarly we can write,

It follows from (2.6)and (2.7) that

$$d(r, r') + d(s, s') \leq pd(r, r') + qd(s, s') + 2(1 - (p + q))d(A, B)$$

which implies that

$$d(r,r') + d(s,s') \leq 2d(A,B). \tag{2.7}$$

Since $d(A,B) \leq d(r,r')$ and $d(A,B) \leq d(s,s')$, we have

$$d(r,r') + d(s,s') \geq 2d(A,B). \tag{2.8}$$

From (2.7) and (2.8), we get

$$d(r,r') + d(s,s') = 2d(A,B). \tag{2.9}$$

This complete the proof. □

Note that every pair of nonempty closed subsets A, B of a uniformly convex Banach space X such that A is convex satisfies the property UC. Therefore, we obtain the following corollary.

Corollary 16. *Let A and B be nonempty closed convex subsets of a uniformly convex Banach space X, $F : A \times A \to B$ and $G : B \times B \to A$ and let the ordered pair (F, G) be a cyclic contraction. Let $(x_0, y_0) \in A \times A$ and define*

$$\begin{array}{ll} x_{n+1} = F(x_n, y_n), & x_{n+2} = G(x_{n+1}, y_{n+1}) \\ y_{n+1} = F(y_n, x_n), & y_{n+2} = G(y_{n+1}, x_{n+1}) \end{array}$$

for all $n \in \mathbb{N} \cup \{0\}$. Then F has a coupled best proximity point $(r, s) \in A \times A$ and G has a coupled best proximity point $(r', s') \in B \times B$. Moreover, we have $x_n \to r,\ \ y_n \to s,\ \ x_{n+1} \to r',\ \ y_{n+1} \to s'$.

Furthermore, if $r = s$ and $r' = s'$, then

$$d(r,r') + d(s,s') = 2d(A,B).$$

Next, we give some illustrative example of Corollary 16.

Example 17. *Consider uniformly convex Banach space $X = \mathbb{R}$ with the usual norm. Let $A = [1, 2]$ and $B = [-1, -2]$.Thus $d(A, B) = 2$. Define $F : A \times A \to B$ and $G : B \times B \to A$ by*

$$F(x,y) = \frac{-2x - 3y - 1}{6}$$

and

$$G(x,y) = \frac{-2x - 3y + 1}{6}.$$

For arbitrary $(x, y) \in A \times A$ and $(u, v) \in B \times B$ and fixed $p = \frac{1}{3}$ and $q = \frac{1}{2}$ we get

$$\begin{aligned} d(F(x,y), G(u,v)) &= \left| \frac{-x-y-1}{6} - \frac{-u-v+1}{6} \right| \\ &\leq \frac{2|x-u| + 3|y-v|}{6} + \frac{1}{3} \\ &= \frac{1}{3}d(x,u) + \frac{1}{2}d(y,v) + (1-(p+q))d(A,B) \end{aligned}$$

This implies that (F, G) is a cyclic contraction with $\alpha = \frac{1}{2}$. Since A and B are closed convex, we have (A, B) and (B, A) satisfy the property UC. Therefore, all hypothesis of Corollary 16 hold. So F has a coupled best proximity point and G has a coupled best proximity point. We note that a point $(1, 1) \in A \times A$ is a unique* *coupled best proximity point of F and a point $(-1,-1,) \in B\times B$ is a unique coupled best proximity point of G. Furthermore, we get*

$$d(1,-1)+d(1,-1)=4=2d(A,B).$$

Next, we give the coupled best proximity point result in compact subsets of metric spaces.

Theorem 18. *Let A and B be nonempty compact subsets of a metric space X, $F : A\times A \to B$ and $G : B\times B \to A$ and let the ordered pair (F,G) be a cyclic contraction. Let $(x_0,y_0)\in A\times A$ and define*

$$\begin{array}{ll} x_{n+1}=F(x_n,y_n), & x_{n+2}=G(x_{n+1},y_{n+1}) \\ y_{n+1}=F(y_n,x_n), & y_{n+2}=G(y_{n+1},x_{n+1}) \end{array}$$

for all $n\in\mathbb{N}\cup\{0\}$. Then F has a coupled best proximity point $(r,s)\in A\times A$ and G has a coupled best proximity point $(r',s')\in B\times B$. Moreover, we have $x_n\to r$, $y_n\to s$, $x_{n+1}\to r'$, $y_{n+1}\to s'$.

Furthermore, if $r=s$ and $r'=s'$, then

$$d(p,p')+d(q,q')+d(r,r')=2d(A,B).$$

Proof. Since $x_0,y_0\in A$ and

$$\begin{array}{ll} x_{n+1}=F(x_n,y_n), & x_{n+2}=G(x_{n+1},y_{n+1}) \\ y_{n+1}=F(y_n,x_n), & y_{n+2}=G(y_{n+1},x_{n+1}) \end{array}$$

for all $n\in\mathbb{N}\cup\{0\}$, we have $x_n,y_n\in A$ and $x_{n+1},y_{n+1}\in A$ for all $n\in\mathbb{N}\cup\{0\}$. As A is compact, the sequences $\{x_n\}$ and $\{y_n\}$ have convergent subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ respectively, such that

$$x_{n_k}\to r\in A,\quad y_{n_k}\to s\in A.$$

Now, we have

$$d(A,B)\le d(r,x_{n_k-1})\le d(r,x_{n_k})+d(x_{n_k},x_{n_k-1}) \tag{2.10}$$

By Lemma 12, we have $d(x_{n_k},x_{n_k-1})\to d(A,B)$. Taking $k\to\infty$ in (2.10), we get

$$d(r,x_{n_k-1})\to d(A,B).$$

By a similar argument we observe that

$$d(s,x_{n_k-1})\to d(A,B).$$

Note that

$$\begin{aligned} d(A,B) &\le d(x_{n_k},F(r,s))=d(G(x_{n_k-1},y_{n_k-1}),F(r,s)) \\ &\le pd(x_{n_k-1},r)+qd(y_{n_k-1},s)+(1-(p+q))d(A,B). \end{aligned}$$

Taking $k\to\infty$, we get $d(r,F(r,s))=d(A,B)$. Similarly, we can prove that

$$d(s,F(s,r))=d(A,B).$$

Thus F has a coupled best proximity $(r,s)\in A\times A$. In similar way, since B is compact, we can also prove that G has a coupled best proximity point $(r',s')\in B\times B$. For

$$d(r,r')+d(s,s')=2d(A,B)$$

similar to the final step of the proof of Theorem 15. This complete the proof. □

3. **Coupled Fixed Point Theorems**

In this section, we give the new coupled fixed point theorem for a cyclic contraction pair.

Theorem 19. *Let A and B be nonempty closed subsets of a metric space X, $F : A \times A \to B$ and $G : B \times B \to A$ and let the ordered pair (F, G) be a cyclic contraction. Let $(x_0, y_0) \in A \times A$ and define*

$$\begin{aligned} x_{n+1} &= F(x_n, y_n), \quad x_{n+2} = G(x_{n+1}, y_{n+1}) \\ y_{n+1} &= F(y_n, x_n), \quad y_{n+2} = G(y_{n+1}, x_{n+1}) \end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. If $d(A, B) = 0$, then F has a coupled fixed point $(r, s) \in A \times A$ and G has a coupled fixed point $(r', s') \in B \times B$. Moreover, we have $x_n \to r$, $y_n \to s$, $x_{n+1} \to r'$, $y_{n+1} \to s'$.

Furthermore, if $r = r'$ and $s = s'$, then F and G have a common coupled fixed point in $(A \cap B)^2$.

Proof. Since $d(A, B) = 0$, we get (A, B) and (B, A) satisfy the property UC. Therefore, by Theorem 15, claim that F has a coupled best proximity point $(r, s) \in A \times A$ that is

$$d(r, F(r, s)) = d(s, F(s, r)) = d(A, B) \tag{3.1}$$

and G has a coupled best proximity point $(r', s') \in B \times B$ that is

$$d(r', G(r', s')) = d(s', G(s', r')) = d(A, B). \tag{3.2}$$

From (3.1) and $d(A, B) = 0$, we conclude that

$$r = F(r, s), \quad s = F(s, r).$$

that is (r, s) is a coupled fixed point of F. It follows from (3.2) and $d(A, B) = 0$, we get

$$r' = G(r', s'), \quad \textit{and} \quad s' = G(s', r')$$

that is (r', s') is a coupled fixed point of G.

Next, we assume that $r = r'$ and $s = s'$ and then we show that F and G have a unique common coupled fixed point in $(A \cap B)^2$. From Theorem 15, we get

$$d(r, r') + d(s, s') = 2d(A, B). \tag{3.3}$$

Since $d(A, B) = 0$, we get

$$d(r, r') + d(s, s') = 0$$

which implies that $r = r'$ and $s = s'$. Therefore, we conclude that $(r, s) \in (A \cap B)^2$ is common coupled fixed point of F and G.

□

Example 20. *Consider $X = \mathbb{R}$ with the usual metric, $A = [-2, 0]$ and $B = [0, 2]$. Define $F : A \times A \to B$ and $G : B \times B \to A$ by*

$$F(x, y) = -\frac{2x + 3y}{6}$$

and

$$G(u, v) = -\frac{2u + 3v}{6}.$$

Then $d(A,B)=0$ *and* (F,G) *is a cyclic contraction with* $p=\frac{1}{3}$ *and* $q=\frac{1}{2}$. *Indeed, for arbitrary* $(x,y)\in A\times A$ *and* $(u,v)\in B\times B$, *we have*

$$\begin{aligned} d(F(x,y),G(u,v)) &= \left|-\frac{2x+3y}{6}+\frac{2u+3v}{6}\right| \\ &\leq \frac{1}{6}(2\mid x-u\mid+3\mid y-v\mid) \\ &\leq pd(x,u)+qd(y,v)+(1-(p+q))d(A,B). \end{aligned}$$

Therefore, all hypothesis of Theorem 19 hold. So F *and* G *have a common coupled fixed point and this point is* $(0,0)\in(A\cap B)^2$.

If we take $A=B$ in Theorem 19, then we get the following results.

Corollary 21. *Let* A *be a nonempty closed subset of a complete metric space* X, $F:A\times A\to A$ *and* $G:A\times A\to A$ *and let the ordered pair* (F,G) *be a cyclic contraction. Let* $(x_0,y_0)\in A\times A$ *and define*

$$x_{n+1}=F(x_n,y_n)$$
$$y_{n+1}=F(y_n,x_n)$$

and

$$x_{n+2}=G(x_{n+1},y_{n+1})$$
$$y_{n+2}=G(y_{n+1},x_{n+1})$$

for all $n\in\mathbb{N}\cup\{0\}$. *Then* F *has a coupled fixed point* $(r,s)\in A\times A$ *and* G *has a coupled fixed point* $(r',s')\in B\times B$. *Moreover, we have* $x_n\to r,\;\; y_n\to s,\;\; x_{n+1}\to r',\;\; y_{n+1}\to s'$.

Furthermore, if $r=r'$ *and* $s=s'$, *then* F *and* G *have a common coupled fixed point in* $A\times A$.

We take $F=G$ in Corollary 21, then we get the following results

Corollary 22. *Let* A *be nonempty closed subsets of a complete metric space* X, $F:A\times A\to A$ *and*

$$d(F(x,y),F(u,v))\leq pd(x,u)+qd(y,v)$$

for all $(x,y),(u,v)\in A\times A$. *Then* F *has a coupled fixed point* $(r,s)\in A\times A$.

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