

QUASI-SLOWLY OSCILLATING SEQUENCES IN LOCALLY NORMAL RIESZ SPACES

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ABSTRACT. In this paper, we study quasi-slowly oscillating sequences, on quasi-slowly oscillating compactness and quasi-slowly oscillating continuous functions in locally normal Riesz space.

1. INTRODUCTION

The concept of continuity and any concept involving continuity play a very important role not only in pure mathematics but also in other branches of sciences involving mathematics especially in computer science, information theory, biological science, speech analysis, bioinformatics.

A real valued function is continuous on the set of real numbers if and only if it preserves Cauchy sequences. Using the idea of continuity of a real function and the idea of compactness in terms of sequences, many kinds of continuities were introduced and investigated, not all but some of them we recall in the following: forward continuity [6], slowly oscillating continuity [9, 12, 14, 15, 35], statistical ward continuity [7], δ -ward continuity [11], ideal ward continuity [5, 22]. The concept of a Cauchy sequence involves far more than that the distance between successive terms is tending to zero. Nevertheless, sequences which satisfy this weaker property are interesting in their own right. A sequence (x_n) of points in \mathbb{R} is called quasi-Cauchy if (Δx_n) is a null sequence where $\Delta x_n = x_{n+1} - x_n$. In [4] Burton and Coleman named these sequences as "quasi-Cauchy" and in [8] Çakallı used the term "ward convergent to 0" sequences. In terms of quasi-Cauchy we restate the definitions of ward compactness and ward continuity as follows: a function f is ward continuous if it preserves quasi-Cauchy sequences, i.e. $(f(x_n))$ is quasi-Cauchy whenever (x_n) is, and a subset E of \mathbb{R} is ward compact if any sequence $\mathbf{x} = (x_n)$ of points in E has a quasi-Cauchy subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of the sequence \mathbf{x} .

A Riesz space is an ordered vector space which is a lattice at the same time. It was first introduced by F. Riesz [32] in 1928. Riesz spaces have many applications in measure theory, operator theory and optimization. They have also some applications in economics (see [2]), and we refer to [1, 3, 21, 23, 25, 27, 28, 30, 31, 33, 36] for more details.

2. PRELIMINARIES AND NOTATIONS

It is known that a sequence $\mathbf{x} = (x_n)$ of points in \mathbb{R} , the set of real numbers, is slowly oscillating, denoted by $\mathbf{x} \in \mathbf{so}$, if

$$\lim_{\lambda \rightarrow 1^+} \overline{\lim}_n \max_{n+1 \leq k \leq [\lambda n]} |x_k - x_n| = 0$$

where $[\lambda n]$ denotes the integer part of λn . This is equivalent to the following if $(x_m - x_n) \rightarrow 0$ whenever $1 \leq \frac{m}{n} \rightarrow 1$ as $m, n \rightarrow \infty$. Using $\varepsilon > 0$ and δ this is also equivalent to the case when for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and $N = N(\varepsilon)$ such that $|x_m - x_n| < \varepsilon$ if $n \geq N(\varepsilon)$ and $n \leq m \leq (1 + \delta)n$ (see [9]).

A function defined on a subset E of \mathbb{R} is called slowly oscillating continuous if it preserves slowly oscillating sequences, i.e. $(f(x_n))$ is slowly oscillating whenever (x_n) is.

Connor and Grosse-Erdman [13] gave sequential definitions of continuity for real functions calling G -continuity instead of A -continuity and their results covers the earlier works related to A -continuity

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where a method of sequential convergence, or briefly a method, is a linear function G defined on a linear subspace s , space of all sequences, denoted by c_G , into \mathbb{R} . A sequence $\mathbf{x} = (x_n)$ is said to be G -convergent to ℓ if $\mathbf{x} \in c_G$ and $G(\mathbf{x}) = \ell$. In particular, \lim denotes the limit function $\lim \mathbf{x} = \lim_n x_n$ on the linear space \mathbf{c} .

A method G is called regular if every convergent sequence $\mathbf{x} = (x_n)$ is G -convergent with $G(\mathbf{x}) = \lim \mathbf{x}$. A method is called subsequential if whenever \mathbf{x} is G -convergent with $G(\mathbf{x}) = \ell$, then there is a subsequence (x_{n_k}) of \mathbf{x} with $\lim_k x_{n_k} = \ell$ (for details see [10]).

Let X be a real vector space and \leq be a partial order on this space. Then X is said to be an *ordered vector space* if it satisfies the following properties:

- (i) if $x, y \in X$ and $y \leq x$, then $y + z \leq x + z$ for each $z \in X$.
- (ii) if $x, y \in X$ and $y \leq x$, then $ay \leq ax$ for each $a \geq 0$.

If, in addition, X is a lattice with respect to the partial ordered, then X is said to be a *Riesz space* (or a *vector lattice*) (see [36]).

For an element x of a Riesz space X , the *positive part* of x is defined by $x^+ = x \vee \bar{0} = \sup\{x, \bar{0}\}$, the *negative part* of x by $x^- = -x \vee \bar{0}$ and the *absolute value* of x by $|x| = x \vee (-x)$, where $\bar{0}$ is the zero element of X .

A subset S of a Riesz space X is said to be *normal* if $y \in S$ and $|x| \leq |y|$ implies $x \in S$.

A *topological vector space* (X, τ) is a vector space X which has a topology (linear) τ , such that the algebraic operations of addition and scalar multiplication in X are continuous. Continuity of addition means that the function $f : X \times X \rightarrow X$ defined by $f(x, y) = x + y$ is continuous on $X \times X$, and continuity of scalar multiplication means that the function $f : \mathbb{R} \times X \rightarrow X$ defined by $f(a, x) = ax$ is continuous on $\mathbb{R} \times X$.

Every linear topology τ on a vector space X has a base N for the neighborhoods of $\bar{\theta}$ satisfying the following properties:

- (1) Each $Y \in N$ is a *balanced set*, that is, $ax \in Y$ holds for all $x \in Y$ and for every $a \in \mathbb{R}$ with $|a| \leq 1$.
- (2) Each $Y \in N$ is an *absorbing set*, that is, for every $x \in X$, there exists $a > 0$ such that $ax \in Y$.
- (3) For each $Y \in N$ there exists some $E \in N$ with $E + E \subseteq Y$.

A linear topology τ on a Riesz space X is said to be *locally normal or solid* if τ has a base at zero consisting of normal sets. A *locally normal (solid) Riesz space* (X, τ) is a Riesz space equipped with a locally normal (solid) topology τ .

Recall that a first countable space is a topological space satisfying the "first axiom of countability". Specifically, a space X is said to be first countable if each point has a countable neighborhood basis (local base). That is, for each point x in X there exists a sequence V_1, V_2, \dots of open neighborhoods of x such that for any open neighborhood V of x there exists an integer j with V_j contained in V .

The idea of statistical convergence first appeared, under the name of almost convergence, in the first edition Zygmund [37] of celebrated monograph [38] of Zygmund. Later, this idea was introduced by Fast [16] and Steinhaus [34] and many authors. Actually, this concept is based on the natural density of subsets of \mathbb{N} of positive integers. A subset E of \mathbb{N} is said to have natural or asymptotic density $\delta(E)$, if

$$\delta(E) = \lim_{n \rightarrow \infty} \frac{|E(n)|}{n} \text{ exists,}$$

where $E(n) = \{k \leq n : k \in E\}$ and $|E|$ denotes the cardinality of the set E .

A sequence $\mathbf{x} = (x_n)$ of points in X is said to be *statistically convergent* (see [1]) to an element L in X if for each τ -neighborhood V of zero, $\delta(\{n \in \mathbb{N} : x_n - L \notin V\}) = 0$, i.e.

$$\lim_{k \rightarrow \infty} \frac{1}{k} |\{n \leq k : x_n - L \notin V\}| = 0.$$

Kostyrko et al. [26] introduced the notion of ideal convergence which is a generalization of statistical convergence (see [16, 18]) based on the structure of the admissible ideal I of subsets of natural numbers \mathbb{N} .

A family of sets $I \subset P(\mathbb{N})$ (the power sets of \mathbb{N}) is said to be an *ideal* on \mathbb{N} if and only if $\phi \in I$ for each $A, B \in I$, we have $A \cup B \in I$ for each $A \in I$ and each $B \subset A$, we have $B \in I$. A non-empty family of sets $F \subset P(\mathbb{N})$ is said to be a *filter* on \mathbb{N} if and only if $\phi \notin F$ for each $A, B \in F$, we have $A \cap B \in F$ each $A \in F$ and each $B \supset A$, we have $B \in F$. An ideal I is called *non-trivial* ideal if $I \neq \phi$

and $\mathbb{N} \notin I$. Clearly $I \subset P(\mathbb{N})$ is a non-trivial ideal if and only if $F = F(I) = \{\mathbb{N} - A : A \in I\}$ is a filter on \mathbb{N} . A non-trivial ideal $I \subset P(\mathbb{N})$ is called *admissible* if and only if $\{\{n\} : n \in \mathbb{N}\} \subset I$. Throughout we assume I is a non-trivial admissible ideal in \mathbb{N} .

A sequence $\mathbf{x} = (x_n)$ of points in a locally normal Riesz space X is said to be ideally convergent to $x_0 \in X$ if for every τ -neighborhood V of zero, the set $\{n \in \mathbb{N} : x_n - x_0 \notin V\} \in I$. In this case we write $x_n \xrightarrow{I_\tau} \ell$ i.e. $I_\tau\text{-lim } x_n = \ell$ (for details see [21]).

The notion of lacunary statistical convergence was introduced by Fridy and Orhan [19] and has been investigated for the real case in [20]. A sequence $\mathbf{x} = (x_n)$ in X is called lacunary statistically convergent to an element L in X (see [29]) if for every τ -neighborhood V of zero,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{n \in J_r : x_n - L \notin V\}| = 0$$

where $J = (k_{r-1}, k_r]$ and $k_0 = 0, h_r := k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$ and $(\theta) = (k_r)$ is an increasing sequence of positive integers.

Throughout the article, the symbol N_{nor} we will denote any base at zero consisting of normal sets and satisfying the conditions (1), (2) and (3) in a locally normal topology. Also (X, τ) a locally normal Riesz space (in short LNRS) and \mathbb{N} and \mathbb{R} will denote the set of all positive integers, and the set of all real numbers, respectively. We will use boldface letters $\mathbf{x}, \mathbf{y}, \mathbf{z}, \dots$ for sequences $\mathbf{x} = (x_n), \mathbf{y} = (y_n), \mathbf{z} = (z_n), \dots$ of points in X .

3. QUASI-SLOWLY OSCILLATING SEQUENCES IN LNRS

In this section we introduce the concepts of quasi-slowly oscillating continuity and quasi-slowly oscillating compactness in LNRS and establish some interesting results related to these notions.

A sequence $\mathbf{x} = (x_n)$ of points in X is called quasi-Cauchy if for each τ -neighborhood V of zero, there exists an $m_0 \in \mathbb{N}$ such that $x_{n+1} - x_n \in V$ for $n \geq m_0$. It is clear that Cauchy sequences are slowly oscillating not only the real case but also in the LNRS setting. It is easy to see that any slowly oscillating sequence of points in X is quasi-Cauchy and therefore Cauchy sequence is quasi-Cauchy. The converses are not always true. There are quasi-Cauchy sequences which are not Cauchy. There are quasi-Cauchy sequences which are not slowly oscillating. Any subsequence of Cauchy sequence is Cauchy. The analogous property fails for quasi-Cauchy sequences and slowly oscillating sequences as well. A sequence $\mathbf{x} = (x_n)$ of points in X is said to be slowly oscillating, denoted by $\mathbf{x} \in \mathbf{so}(X)$ if (x_n) is a slowly oscillating sequences, i.e. for each τ -neighborhood V of zero, there exist $\delta = \delta(V) > 0$ and $m = m(V)$ such that

$$x_k - x_n \in V \text{ for } n \geq m(V) \text{ and } n \leq k \leq (1 + \delta)n.$$

A sequence $\mathbf{x} = (x_n)$ of points in X is called ideally quasi-Cauchy if for each τ -neighborhood V of zero, the set

$$\{n \in \mathbb{N} : x_{n+1} - x_n \notin V\} \in I.$$

It is clear that slowly oscillating sequence of points in X is ideally quasi-Cauchy.

Now we introduce the notion of quasi-slowly oscillating sequences and quasi-slowly oscillating continuity in LNRS.

Definition 3.1. A sequence $\mathbf{x} = (x_n)$ of points in X is said to be quasi-slowly oscillating, denoted by $\mathbf{x} \in \mathbf{qso}(X)$ if (Δx_n) is a slowly oscillating sequences, i.e. for each τ -neighborhood V of zero, there exist $\delta = \delta(V) > 0$ and $m = m(V)$ such that

$$\Delta x_k - \Delta x_n \in V \text{ for } n \geq m(V) \text{ and } n \leq k \leq (1 + \delta)n.$$

It is clear that a convergent sequence is slowly oscillating, since every convergent sequence is a Cauchy sequence, and any slowly oscillating sequence is quasi-Cauchy, but the converse need not to be true in general. For examples, if $X = \mathbb{R}$, then $(\sum_{n=1}^{\infty} \frac{1}{n}), (\ln n), (\ln \ln n)$ are slowly oscillating, but not Cauchy. The sequence $(\sum_{n=1}^k \frac{1}{n})$ is quasi-Cauchy, but not slowly oscillating.

Theorem 3.1. If a sequence is slowly oscillating then it is a quasi-slowly oscillating.

Proof. Let (x_n) be a slowly oscillating sequence. For each τ -neighborhood V of zero, there exists a $Y \in N_{nor}$ such that $Y \subseteq V$. Choose $W \in N_{nor}$ such that $W - W \subseteq Y$. Since (x_n) is a slowly oscillating sequence, there exist $\delta = \delta(W) > 0$ and a positive integer $n_1 = n_1(W)$ such that

$$x_k - x_n \in W \text{ for all } n \geq n_1 \text{ and } n \leq k \leq (1 + \delta)n.$$

Hence for all $n \geq n_1(W)$ and $n \leq k \leq (1 + \delta)n$ we have

$$\Delta x_k - \Delta x_n = (x_k - x_{k+1}) - (x_n - x_{n+1}) = (x_k - x_n) - (x_{k+1} - x_{n+1}) \in W - W \subseteq Y \subseteq V.$$

It implies that (x_n) is a quasi-slowly oscillating sequence. □

Definition 3.2. A function f defined on a subset E of X is called quasi-slowly oscillating continuous if it transforms quasi-slowly oscillating sequences to quasi-slowly oscillating sequences of points in E , that is, $(f(x_n))$ is quasi-slowly oscillating whenever (x_n) is quasi-slowly oscillating sequences of points in E .

We note that sum of two quasi-slowly oscillating continuous functions is quasi-slowly oscillating continuous and the composite of two quasi-slowly oscillating continuous functions is quasi-slowly oscillating continuous in LNRS.

In connection with slowly oscillating sequences, quasi-slowly oscillating sequences and convergent sequences the problem arises to investigate the following types of continuity of functions on X .

- (qso-qso): : $(x_n) \in \mathbf{qso}(X) \Rightarrow (f(x_n)) \in \mathbf{qso}(X)$
- (qso-c): : $(x_n) \in \mathbf{qso}(X) \Rightarrow (f(x_n)) \in \mathbf{c}(X)$
- (c-c): : $(x_n) \in \mathbf{c}(X) \Rightarrow (f(x_n)) \in \mathbf{c}(X)$
- (c-qso): : $(x_n) \in \mathbf{c}(X) \Rightarrow (f(x_n)) \in \mathbf{qso}(X)$
- (qso-so): : $(x_n) \in \mathbf{qso}(X) \Rightarrow (f(x_n)) \in \mathbf{so}(X)$
- (so-qso): : $(x_n) \in \mathbf{so}(X) \Rightarrow (f(x_n)) \in \mathbf{qso}(X)$
- (u): : uniform continuity of f .

It is clear that (qso-qso) implies (so-qso), but (so-qso) need not imply (qso-qso). Also (qso-c) implies (c-qso) and (qso-c) implies (c-c) and we see that (c-c) need not imply (qso-c), because identity function is an example for it. We also see that (u) implies (so-qso).

Theorem 3.2. If f is quasi-slowly oscillating continuous on a subset E of X then it is continuous on E in the ordinary sense.

Proof. Suppose that f is quasi-slowly oscillating continuous on E and let (x_n) be any convergent sequence of points in E with $\lim_{n \rightarrow \infty} x_n = x_0$. Then the sequence

$$(x_1, x_1, x_0, x_0, x_2, x_2, x_0, x_0, \dots, x_{n-1}, x_{n-1}, x_0, x_0, x_n, x_n, x_0, x_0, \dots)$$

is also convergent to x_0 and hence (y_n) is quasi-slowly oscillating. Since f is quasi-slowly oscillating continuous, the sequence

$$(f(x_1), f(x_1), f(x_0), f(x_0), f(x_2), f(x_2), f(x_0), f(x_0), \dots, \\ f(x_{n-1}), f(x_{n-1}), f(x_0), f(x_0), f(x_n), f(x_n), f(x_0), f(x_0), \dots)$$

is also quasi-slowly oscillating. From the definition of quasi-slowly oscillation, we have that the sequence

$$(0, f(x_1) - f(x_0), 0, f(x_0) - f(x_2), 0, f(x_2) - f(x_0), 0, f(x_0) - f(x_3), 0, f(x_3) - f(x_0), 0, \dots, \\ 0, f(x_0) - f(x_{n-1}), 0, f(x_{n-1}) - f(x_0), f(x_0) - f(x_n), 0, f(x_n) - f(x_0), \dots)$$

is a slowly oscillating. Since any slowly oscillating sequence is quasi-Cauchy, then the sequence

$$(f(x_0) - f(x_1), f(x_1) - f(x_0), f(x_0) - f(x_2), f(x_2) - f(x_0), f(x_0) - f(x_3), f(x_3) - f(x_0), \dots, \\ f(x_0) - f(x_{n-1}), f(x_{n-1}) - f(x_0), f(x_0) - f(x_n), f(x_n) - f(x_0), \dots)$$

is a null sequence. Now it follows that if for each τ -neighborhood V of zero, there exists $m = m(V)$ such that

$$f(x_n) - f(x_0) \in V \text{ for } n \geq m.$$

This completes the proof of theorem. □

In general the converse is not true. If $X = \mathbb{R}$. Then, it follows from the function $f(x) = x^2 + 1$ and the sequence $(x_n) = (\sqrt{n})$.

Corollary 3.1. *Any quasi-slowly oscillating continuous function is G -continuous for any regular sub-sequential method G .*

Corollary 3.2. *If f is quasi-slowly oscillating continuous on a subset E of X , then it is ideally continuous on E .*

Corollary 3.3. *If f is quasi-slowly oscillating continuous on a subset E of X , then it is statistically continuous on E .*

Corollary 3.4. *If f is quasi-slowly oscillating continuous on a subset E of X , then it is lacunary statistically continuous on E .*

Theorem 3.3. *If f is a uniformly continuous function defined on a subset E of X , then it is quasi-slowly oscillating continuous on E .*

Proof. Let f be uniformly continuous function and $\mathbf{x} = (x_n)$ be any quasi-slowly oscillating sequence in E . Let W be a τ -neighborhood of zero. Since f is uniformly continuous on E , then there exists a τ -neighborhood V of zero such that $f(x) - f(y) \in W$ whenever $x - y \in V$. Since (x_n) is slowly oscillating, for the same τ -neighborhood W of zero, there exist $m = m(V)$ and $\delta = \delta(V) > 0$ such that $\Delta x_k - \Delta x_n \in V$ for $n \geq m(V)$ and $n \leq k \leq (1 + \delta)n$. Hence we have $\Delta f(x_k) - \Delta f(x_n) \in W$ whenever $n \geq m(V)$ and $n \leq k \leq (1 + \delta)n$. It follows that $(f(x_n))$ is quasi-slowly oscillating. This completes the proof of theorem. \square

Definition 3.3. *A sequence (x_n) of points in X is called Cesáro quasi-slowly oscillating if (t_n) is -quasi-slowly oscillating, where $t_n = \frac{1}{n} \sum_{k=1}^n x_k$, is the Cesáro means (see [17]) of the sequence (x_n) . Also a function f defined on a subset E of X is called Cesáro quasi-slowly oscillating continuous if it preserves Cesáro quasi-slowly oscillating sequences of points in E .*

By using the similar argument used in proof of Theorem 3.3, we immediately have the following result.

Theorem 3.4. *If f is a uniformly continuous on a subset E of X and (x_n) is a quasi-slowly oscillating sequence in E , then $(f(x_n))$ is Cesáro quasi-slowly oscillating.*

Definition 3.4. *A sequence of functions (f_n) defined on a subset E of X is said to be uniformly convergent to a function f if for each τ -neighborhood V of zero, there exists an integer $n_0 = n_0(V)$ such that $f_n(x) - f(x) \in V$ for all $n \geq n_0$ and $x \in E$.*

Theorem 3.5. *If (f_n) is a sequence of quasi-slowly oscillating continuous functions defined on a subset E of X and (f_n) is uniformly convergent to a function f on E , then f is quasi-slowly oscillating continuous on E .*

Proof. Let (x_n) be any quasi-slowly oscillating sequence of points in E . By uniform convergence of (f_n) , if for each τ -neighborhood V of zero, there exists a $Y \in N_{nor}$ such that $Y \subseteq V$. Choose $W \in N_{nor}$ such that $-W + W + W - W + W \subseteq Y$. Then there exists $n_1 = n_1(W)$ such that

$$f_n(x) - f(x) \in W$$

for each $x \in E$ and for all $n \geq n_1(W)$. Also since f_{n_1} is quasi-slowly oscillating continuous, there exist $n_2 = n_2 > n_1$ and $\delta = \delta(W) > 0$ such that

$$\Delta f_{n_1}(x_k) - \Delta f_{n_1}(x_n) \in W$$

whenever $n \geq n_2(W)$ and $n \leq k \leq (1 + \delta)n$. Therefore if $n \geq n_1(W)$ and $n \leq k \leq (1 + \delta)n$ we have

$$\begin{aligned} \Delta f(x_k) - \Delta f(x_n) &= [f(x_k) - f(x_{k+1})] - [f(x_n) - f(x_{n+1})] \\ &= [f(x_k) - f_{n_1}(x_k)] + [f_{n_1}(x_{k+1}) - f(x_{k+1})] + [f_{n_1}(x_n) - f(x_n)] + [f(x_{n+1}) - f_{n_1}(x_{n+1})] \\ &\quad + [f_{n_1}(x_k) - f_{n_1}(x_{k+1}) - f_{n_1}(x_n) + f_{n_1}(x_{n+1})] \in -W + W + W - W + W \subseteq Y \subseteq V. \end{aligned}$$

Thus it implies that $\Delta f(x_k) - \Delta f(x_n) \in V$ if $n \geq n_1$ and $n \leq k \leq (1 + \delta)n$. It follows that $(f(x_n))$ is a quasi-slowly oscillating sequences of points in E which completes the proof of theorem. \square

Using the same techniques as in the Theorem 3.5, the following result can be obtained easily.

Theorem 3.6. *If (f_n) is a sequence of Cesáro quasi-slowly oscillating continuous functions defined on a subset E of X and (f_n) is uniformly convergent to a function f on E , then f is Cesáro quasi-slowly oscillating continuous on E .*

Theorem 3.7. *The set of all quasi-slowly oscillating continuous functions defined on a subset E of X is a closed subset of all continuous functions on E , that is $\overline{\mathbf{qso}(E)} = \mathbf{qso}(E)$, where $\mathbf{qso}(E)$ is the set of all quasi-slowly oscillating continuous functions defined on E and $\overline{\mathbf{qso}(E)}$ denotes the set of all cluster points of $\mathbf{qso}(E)$.*

Proof. Let f be any element of $\overline{\mathbf{qso}(E)}$. Then there exists a sequence of points (f_n) in $\mathbf{qso}(E)$ such that $\lim_{n \rightarrow \infty} f_n = f$. To show that f is quasi-slowly oscillating sequence on E . Now let (x_n) be any quasi-slowly oscillating sequence in E . Let V be an arbitrary τ -neighborhood of zero. There exists a $Y \in N_{nor}$ such that $Y \subseteq V$. Choose $W \in N_{nor}$ such that $W+W+W+W+W \subseteq Y$. Since (f_n) converges to f , there exists a positive integer n_1 such that for all $x \in E$ and for all $n \geq n_1$, $f_n(x) - f(x) \in W$. Also since f_{n_1} is quasi-slowly oscillating continuous, there exist an integer $n_2 = n_2 > n_1$ and $\delta > 0$ such that

$$\Delta f_{n_1}(x_k) - \Delta f_{n_1}(x_n) \in W$$

whenever $n \geq n_2$ and $n \leq k \leq (1 + \delta)n$. Hence, for all $n \geq n_1$ and $n \leq k \leq (1 + \delta)n$ we have

$$\begin{aligned} \Delta f(x_k) - \Delta f(x_n) &= [\Delta f(x_k) - \Delta f_{n_1}(x_n)] + [\Delta f_{n_1}(x_n) - \Delta f_{n_1}(x_k)] + [\Delta f_{n_1}(x_k) - \Delta f(x_n)] \\ &= [f(x_k) - f(x_{k+1})] - [f_{n_1}(x_n) - f_{n_1}(x_{n+1})] + [f_{n_1}(x_n) - f_{n_1}(x_{n+1})] - [f_{n_1}(x_k) - f_{n_1}(x_{k+1})] \\ &\quad + [f_{n_1}(x_k) - f_{n_1}(x_{k+1})] - [f(x_n) - f(x_{n+1})] \\ &\in W + W + W + W + W \subseteq Y \subseteq V. \end{aligned}$$

Thus it implies that $f(x_k) - f(x_n) \in V$ for all $n \geq n_1$ and $n \leq k \leq (1 + \delta)n$. Thus f is slowly oscillating continuous function on E and this completes the proof of theorem. \square

Corollary 3.5. *The set of all quasi-slowly oscillating continuous functions defined on a subset E of X is a complete subspace of the space of all continuous functions on E .*

An element x_0 in X is called an ideal limit point of a subset E of X if there is an E -valued sequence of points with ideal limit x_0 . It follows that the set of all ideal limit points of E is equal to the set of all limit points of E in the ordinary sense. An element x_0 in X is called an ideal accumulation point of a subset E if it is an ideal limit point of the set $E - \{x_0\}$. The set of all ideal accumulation points of E is equal to the set of all accumulation points of E in the ordinary sense.

A function f on X is said to have an ideally sequential limit at a point x_0 of X if the image sequence $(f(x_n))$ is ideally convergent to x_0 for any ideally convergent sequence $\mathbf{x} = (x_n)$ with ideal limit x_0 and a function f is to be ideally sequentially continuous at a point x_0 of X if the sequence $(f(x_n))$ is ideally convergent to $f(x_0)$ for any ideally convergent sequence $\mathbf{x} = (x_n)$ with ideal limit x_0 (for details see [5]).

Lemma 3.1. *A function f on X has an ideally sequential limit at a point x_0 of X if and only if it has an ideal limit at a point x_0 of X in ordinary sense.*

Proof. The proof follows from the fact that any ideally convergent sequence has a convergent subsequence (also see [5]). \square

Next we define the concept of quasi-slowly oscillating compactness in LNRS.

Definition 3.5. *A subset E of X is called quasi-slowly oscillating compact if any sequence of points in E has a quasi-slowly oscillating subsequence.*

We see that any compact subset of X is quasi-slowly oscillating compact, union of two quasi-slowly oscillating compact subsets of X is quasi-slowly oscillating compact. Any subset of quasi-slowly oscillating compact set is also quasi-slowly oscillating compact and so intersection of any quasi-slowly oscillating compact subsets of X is quasi-slowly oscillating compact.

Theorem 3.8. *A quasi-slowly oscillating continuous image of a quasi-slowly oscillating compact subset of X is quasi-slowly oscillating compact.*

Proof. Let f be a quasi-slowly oscillating continuous function on X and E be a quasi-slowly oscillating compact subset of X . Let $\mathbf{y} = (y_n)$ be a sequence of points in $f(E)$. Then we can write $y_n = f(x_n)$ where (x_n) is sequence of points in E for each $n \in \mathbb{N}$. Since E is quasi-slowly oscillating compact, there is a quasi-slowly oscillating subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of (x_n) . Then, quasi-slowly oscillating continuity of f implies that $f(z_k)$ is a quasi-slowly oscillating subsequence of $f(x_n)$. Hence $f(E)$ is quasi-slowly oscillating compact. \square

Corollary 3.6. *For any regular subsequential method G , if E is G -sequentially compact subset of X , then it is quasi-slowly oscillating compact.*

Proof. The proof of the result follows from the regularity and subsequence property of G . \square

Corollary 3.7. *A real valued function defined on a bounded subset of \mathbb{R} is uniformly continuous if and only if it is slowly oscillating continuous.*

Proof. The proof of the result follows from the fact that totally boundedness coincides with slowly oscillating compactness and boundedness coincides with totally boundedness in \mathbb{R} . \square

Corollary 3.8. *Any totally bounded subset of X is quasi-slowly oscillating compact.*

Proof. The proof of the result follows from the fact that any sequence of points in a totally bounded subset of X has a Cauchy subsequence, which is quasi-slowly oscillating. \square

Now we give the definition on ideal continuous function in LNRS.

Definition 3.6. *Let (X, τ_1) and (Y, τ_2) be LNR spaces and $E \subset Y$. A function $f : E \rightarrow Y$ is called ideally continuous at a point $x_0 \in E$ if $x_n \xrightarrow{I\tau_1} x_0$ in E implies $f(x_n) \xrightarrow{I\tau_2} f(x_0)$ in Y .*

Theorem 3.9. *Let (X, τ_1) and (Y, τ_2) be LNR spaces. If a function $f : X \rightarrow Y$ is uniformly continuous, then f is ideally continuous.*

Proof. Let $f : X \rightarrow Y$ be uniformly continuous and $x_n \xrightarrow{I\tau_1} x_0$ in X . Let θ_1 and θ_2 be denote the zeros in X and Y , respectively. Let W be an arbitrary τ_2 -neighborhood of θ_2 . Since f is uniformly continuous, there exists some τ_1 -neighborhood V of θ_1 such that

$$x - y \in V \Rightarrow f(x) - f(y) \in W. \tag{3.1}$$

Since $x_n \xrightarrow{I\tau_1} x_0$, we put $K = \{n \in \mathbb{N} : x_n - x_0 \notin V\}$, so $K \in I$.

Then from (3.1) we have

$$f(x_n) - f(x_0) \notin W \text{ for all } n \in K.$$

Therefore we have

$$\{n \in \mathbb{N} : f(x_n) - f(x_0) \notin W\} \subseteq K$$

and hence

$$\{n \in \mathbb{N} : f(x_n) - f(x_0) \notin W\} \in I.$$

i.e. we have $f(x_n) \xrightarrow{I\tau_2} f(x_0)$, which shows that f is ideally continuous. \square

Theorem 3.10. *A function f on X is ideally sequentially continuous at a point x_0 of X if and only if it is continuous at a point x_0 in ordinary sense.*

Proof. The proof follows from the fact that any ideally convergent sequence has a convergent subsequence and from the above Lemma 3.1. \square

Theorem 3.11. *Let $f : X \rightarrow X$ be any function and (x_n) be a sequence of points in X such that $I_\tau - \lim_{n \rightarrow \infty} x_n = x_0$ implies $\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$, then it is a constant function.*

Proof. For the proof of the theorem follows form Theorem 3 in [12]. \square

Theorem 3.12. *If a function is quasi-slowly oscillating continuous on a subset E of X , then it is ideally sequentially continuous on E .*

Proof. Let f be any quasi-slowly oscillating continuous on E . By Theorem 3.2, we have f is continuous on E . Also from Theorem 3.10, we see that f is ideally sequentially continuous on E . This completes the proof. \square

Theorem 3.13. *If a function is δ -ward continuous on a subset E of X , then it is ideally sequentially continuous on E .*

Proof. Let f be any δ -ward continuous function on E . It follows from Corollary 2 in [11] that f is continuous. By Theorem 3.10 we obtain that f is ideally sequentially continuous on E . This completes the proof of the theorem. \square

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