International Journal of Analysis and Applications
ISSN 2291-8639
Volume 15, Number 1 (2017), 23-30
http://www.etamaths.com

STEFFENSEN'S INTEGRAL INEQUALITY FOR CONFORMABLE FRACTIONAL INTEGRALS

MEHMET ZEKI SARIKAYA, HATICE YALDIZ* AND HÜSEYIN BUDAK

Abstract. The aim of this paper is to establish some Steffensen's type inequalities for conformable fractional integral. The results presented here would provide generalizations of those given in earlier works.

1. Introduction

The most basic inequality which deals with the comparison between integrals over a whole interval $[a, b]$ and integrals over a subset of $[a, b]$ is the following inequality, which was estab-lished by J.F. Steffensen in 1919,(see [10]).

Theorem 1.1 (Steffensen's inequality)**.** *Let a and b be real numbers such that $a < b$, f and g be integrable functions from $[a, b]$ into $\mathbb{R}$ such that f is nonincreasing and for every $x \in [a, b]$, $0 \le g(x) \le 1$. Then*

$$\int_{b-\lambda}^{b} f(x)\,dx \le \int_{a}^{b} f(x)\,g(x)\,dx \le \int_{a}^{a+\lambda} f(x)\,dx, \tag{1.1}$$

where $\lambda = \int_a^b g(x)\,dx$.

A comprehensive survey on this inequality can be found in [9]. Steffensen's inequality plays an important role in the study of integral inequalities. For more results concerning new proofs, generalizations, weaker hypothesis or different forms were emerging one after another see [6]– [11], and the references therein.

2. Definitions and properties of conformable fractional derivative and integral

The following definitions and theorems with respect to conformable fractional derivative and integral were referred in (see, [1]- [5]).

Definition 2.1 (Conformable fractional derivative)**.** *Given a function $f : [0, \infty) \to \mathbb{R}$. Then the "conformable fractional derivative" of f of order α is defined by*

$$D_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f\left(t + \epsilon t^{1-\alpha}\right) - f(t)}{\epsilon} \tag{2.1}$$

for all $t > 0$, $\alpha \in (0, 1)$. If f is $\alpha-$differentiable in some $(0, a)$, $\alpha > 0$, $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exist, then define

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t). \tag{2.2}$$

We can write $f^{(\alpha)}(t)$ for $D_\alpha(f)(t)$ to denote the conformable fractional derivatives of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is $\alpha-$differentiable.

Theorem 2.1. *Let $\alpha \in (0, 1]$ and f, g be $\alpha-$differentiable at a point $t > 0$. Then*

Received 1st May, 2017; accepted 9th July, 2017; published 1st September, 2017.
2010 *Mathematics Subject Classification.* 26D15.
Key words and phrases. Steffensen inequality; conformable fractional integral.

i. $D_\alpha (af + bg) = aD_\alpha (f) + bD_\alpha (g)$, for all $a, b \in \mathbb{R}$,

ii. $D_\alpha (\lambda) = 0$, for all constant functions $f(t) = \lambda$,

iii. $D_\alpha (fg) = fD_\alpha (g) + gD_\alpha (f)$,

iv. $D_\alpha \left(\frac{f}{g}\right) = \frac{fD_\alpha (g) - gD_\alpha (f)}{g^2}$.

If f is differentiable, then

$$D_\alpha (f)(t) = t^{1-\alpha} \frac{df}{dt}(t). \tag{2.3}$$

Definition 2.2 (Conformable fractional integral). *Let $\alpha \in (0,1]$ and $0 \le a < b$. A function $f : [a,b] \to \mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral*

$$\int_a^b f(x)\, d_\alpha x := \int_a^b f(x)\, x^{\alpha-1} dx \tag{2.4}$$

exists and is finite. All α-fractional integrable on $[a,b]$ is indicated by $L_\alpha^1([a,b])$.

Remark 2.1.

$$I_\alpha^a (f)(t) = I_1^a \left(t^{\alpha-1} f\right) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

Theorem 2.2. *Let $f : (a,b) \to \mathbb{R}$ be differentiable and $0 < \alpha \le 1$. Then, for all $t > a$ we have*

$$I_\alpha^a D_\alpha^a f(t) = f(t) - f(a). \tag{2.5}$$

Theorem 2.3 (Integration by parts). *Let $f, g : [a,b] \to \mathbb{R}$ be two functions such that fg is differentiable. Then*

$$\int_a^b f(x)\, D_\alpha^a (g)(x)\, d_\alpha x = fg|_a^b - \int_a^b g(x)\, D_\alpha^a (f)(x)\, d_\alpha x. \tag{2.6}$$

Theorem 2.4. *Assume that $f : [a,\infty) \to \mathbb{R}$ such that $f^{(n)}(t)$ is continuous and $\alpha \in (n, n+1]$. Then, for all $t > a$ we have*

$$D_\alpha^a f(t)\, I_\alpha^a = f(t).$$

Theorem 2.5 (Fractional Steffensen's inequality). *([4]) Let $\alpha \in (0,1]$ and a and b be real numbers such that $0 \le a < b$. Let $f : [a,b] \to [0,\infty)$ and $g : [a,b] \to [0,1]$ be α-fractional integrable functions on $[a,b]$ with f is decreasing. Then*

$$\int_{b-\ell}^{b} f(x)\, d_\alpha x \le \int_a^b f(x)\, g(x)\, d_\alpha x \le \int_a^{a+\ell} f(x)\, d_\alpha x, \tag{2.7}$$

where $\ell := \frac{\alpha(b-a)}{b^\alpha - a^\alpha} \int_a^b g(x)\, d_\alpha x$.

The aim of this paper is to establish some Steffensen's type inequalities for conformable fractional integral. The results presented here would provide generalizations of those given in earlier works.

3. Steffensen's type inequalities for conformable fractional integrals

Lemma 3.1. *Let $\alpha \in (0,1]$ and $a,b \in \mathbb{R}$ with $0 \leq a < b$, g and h be $\alpha-$fractional integrable function on $[a,b]$, $0 \leq g(t) \leq h(t)$ all $t \in [a,b]$, and define*

$$l := \frac{(b-a)}{\int_a^b h(t)\, d_\alpha(t)} \int_a^b g(t)\, d_\alpha(t) \in [0, b-a]. \tag{3.1}$$

Then, we have

$$\int_{b-l}^{b} h(t)\, d_\alpha(t) \leq \int_a^b g(t)\, d_\alpha(t) \leq \int_a^{a+l} h(t)\, d_\alpha(t). \tag{3.2}$$

Proof. Since $0 \leq g(t) \leq h(t)$ for all $t \in [a,b]$, l given in (3.1) satisfies,

$$0 \leq l = \frac{(b-a)}{\int_a^b h(t)\, d_\alpha(t)} \int_a^b g(t)\, d_\alpha(t) \leq \frac{(b-a)}{\int_a^b h(t)\, d_\alpha(t)} \int_a^b h(t)\, d_\alpha(t) = b-a,$$

and by average values, we get the following inequalities

$$\frac{1}{l} \int_{b-l}^{b} h(t)\, d_\alpha(t) \leq \frac{1}{b-a} \int_a^b h(t)\, d_\alpha(t) \leq \frac{1}{l} \int_a^{a+l} h(t)\, d_\alpha(t)$$

and then

$$\int_{b-l}^{b} h(t)\, d_\alpha(t) \leq \frac{l}{b-a} \int_a^b h(t)\, d_\alpha(t) \leq \int_a^{a+l} h(t)\, d_\alpha(t).$$

By (3.1), we obtain the following inequalities

$$\int_{b-l}^{b} h(t)\, d_\alpha(t) \leq \int_a^b g(t)\, d_\alpha(t) \leq \int_a^{a+l} h(t)\, d_\alpha(t).$$

This completes the proof. □

Remark 3.1. *If we take $h(t) = 1$ in Lemma 3.1, then Lemma 3.1 reduces to the Lemma 2.1 in [4].*

Theorem 3.1. *Let $\alpha \in (0,1]$ and $a,b \in \mathbb{R}$ with $0 \leq a < b$, $f,g,h : [a,b] \to [0,\infty)$ be $\alpha-$fractional integrable function on $[a,b]$, $0 \leq g(t) \leq h(t)$ all $t \in [a,b]$, with f decreasing function. Then*

$$\int_{b-l}^{b} h(t) f(t)\, d_\alpha(t) \leq \int_a^b f(t) g(t)\, d_\alpha(t) \leq \int_a^{a+l} h(t) f(t)\, d_\alpha(t) \tag{3.3}$$

where l is given by (3.1).

Proof. We will prove only the case in (3.3) for right inequality; the proof for the left inequality is similar, and relies on (3.2). By definition of l and the conditions on g,h the inequality (3.2) holds.

Since f is decreasing function, we obtain that

$$\begin{aligned}
& \int_a^{a+l} h(t) f(t) d_\alpha(t) - \int_a^b f(t) g(t) d_\alpha(t) \\
= & \int_a^{a+l} f(t) [h(t) - g(t)] d_\alpha(t) - \int_{a+l}^b f(t) g(t) d_\alpha(t) \\
\geq & f(a+l) \int_a^{a+l} [h(t) - g(t)] d_\alpha(t) - \int_{a+l}^b f(t) g(t) d_\alpha(t) \\
= & f(a+l) \left[\int_a^{a+l} h(t) d_\alpha(t) - \int_a^{a+l} g(t) d_\alpha(t) \right] - \int_{a+l}^b f(t) g(t) d_\alpha(t) \\
\geq & f(a+l) \left[\int_a^b g(t) d_\alpha(t) - \int_a^{a+l} g(t) d_\alpha(t) \right] - \int_{a+l}^b f(t) g(t) d_\alpha(t) \\
= & f(a+l) \int_{a+l}^b g(t) d_\alpha(t) - \int_{a+l}^b f(t) g(t) d_\alpha(t) \\
= & \int_{a+l}^b [f(a+l) - f(t)] g(t) d_\alpha(t) \\
\geq & 0.
\end{aligned}$$

This completes the proof. □

Remark 3.2. *If we take $h(t) = 1$ in Theorem 3.1, then the inequality (3.3) reduces to the inequality (2.7).*

Remark 3.3. *If we take $h(t) = 1$ and $\alpha = 1$ in Theorem 3.1, then the inequality (3.3) reduces to the inequality (1.1).*

In order to obtain our other results, we need the following lemma.

Lemma 3.2. *Under the assumptions of Lemma 3.1* and l *is defined by*

$$\int_a^{a+l} h(t) d_\alpha(t) = \int_a^b g(t) d_\alpha(t) = \int_{b-l}^b h(t) d_\alpha(t). \tag{3.4}$$

Then, we have

$$\begin{aligned}
\int_a^b f(t) g(t) d_\alpha(t) = & \int_a^{a+l} (f(t) h(t) - [f(t) - f(a+l)] [h(t) - g(t)]) d_\alpha(t) \\
& + \int_{a+l}^b [f(t) - f(a+l)] g(t) d_\alpha(t),
\end{aligned} \tag{3.5}$$

and

$$\int_a^b f(t)\,g(t)\,d_\alpha(t) = \int_{b-l}^b \left(f(t)\,h(t) - [f(t) - f(b-l)]\,[h(t) - g(t)]\right) d_\alpha(t) + \int_a^{b-l} [f(t) - f(b-l)]\,g(t)\,d_\alpha(t)\,. \tag{3.6}$$

Proof. We know that $a \le a+l \le b$, $a \le b-l \le b$. Firstly, we calculate identity (3.5). By direct computation, we have

$$\begin{aligned}
&\int_a^{a+l} \left(f(t)\,h(t) - [f(t) - f(a+l)]\,[h(t) - g(t)]\right) d_\alpha(t) - \int_a^b f(t)\,g(t)\,d_\alpha(t) \\
= &\int_a^{a+l} \left(f(t)\,h(t) - f(t)g(t) - [f(t) - f(a+l)]\,[h(t) - g(t)]\right) d_\alpha(t) \\
&+ \int_a^{a+l} f(t)\,g(t)\,d_\alpha(t) - \int_a^b f(t)\,g(t)\,d_\alpha(t) \\
= &\int_a^{a+l} f(a+l)\,[h(t) - g(t)]\,d_\alpha(t) - \int_{a+l}^b f(t)\,g(t)\,d_\alpha(t) \\
= &\ f(a+l)\left(\int_a^{a+l} h(t)\,d_\alpha(t) - \int_a^{a+l} g(t)\,d_\alpha(t)\right) - \int_{a+l}^b f(t)\,g(t)\,d_\alpha(t) \\
= &\ f(a+l)\left(\int_a^b g(t)\,d_\alpha(t) - \int_a^{a+l} g(t)\,d_\alpha(t)\right) - \int_{a+l}^b f(t)\,g(t)\,d_\alpha(t) \\
= &\ f(a+l)\int_{a+l}^b g(t)\,d_\alpha(t) - \int_{a+l}^b f(t)\,g(t)\,d_\alpha(t)\,.
\end{aligned}$$

which completes the proof. Similarly, the second part is obtained. The proof of the Lemma is completed. □

Theorem 3.2. *Under the assumptions of Theorem 3.1. Then*

$$\begin{aligned}\int_{b-l}^{b} f(t)\,h(t)\,d_{\alpha}(t) &\leq \int_{b-l}^{b} \left(f(t)\,h(t)-\left[f(t)-f(b-l)\right]\left[h(t)-g(t)\right]\right) d_{\alpha}(t) \\ &\leq \int_{a}^{b} f(t)\,g(t)\,d_{\alpha}(t) \\ &\leq \int_{a}^{a+l} \left(f(t)\,h(t)-\left[f(t)-f(a+l)\right]\left[h(t)-g(t)\right]\right) d_{\alpha}(t) \\ &\leq \int_{a}^{a+l} f(t)\,h(t)\,d_{\alpha}(t)\end{aligned}$$

where l is given by (3.4).

Proof. From $0 \leq g(t) \leq h(t)$ and f is decreasing function on $[a,b]$, then we have

$$\int_{a}^{b-l} \left[f(t)-f(b-l)\right] g(t)\,d_{\alpha}(t) \geq 0 \tag{3.7}$$

and

$$\int_{b-l}^{b} \left[f(b-l)-f(t)\right]\left[h(t)-g(t)\right] d_{\alpha}(t) \geq 0. \tag{3.8}$$

Using the identity (3.6) together with the inequalities (3.7) and (3.8), we obtain

$$\begin{aligned}&\int_{b-l}^{b} f(t)\,h(t)\,d_{\alpha}(t) \\ \leq &\int_{b-l}^{b} \left(f(t)\,h(t)-\left[f(t)-f(b-l)\right]\left[h(t)-g(t)\right]\right) d_{\alpha}(t) \\ \leq &\int_{a}^{b} f(t)\,g(t)\,d_{\alpha}(t).\end{aligned}$$

In the same way as above, we can prove that

$$\begin{aligned}&\int_{a}^{b} f(t)\,g(t)\,d_{\alpha}(t) \\ \leq &\int_{a}^{a+l} \left(f(t)\,h(t)-\left[f(t)-f(a+l)\right]\left[h(t)-g(t)\right]\right) d_{\alpha}(t) \\ \leq &\int_{a}^{a+l} f(t)\,h(t)\,d_{\alpha}(t).\end{aligned}$$

This completes the proof. □

Theorem 3.3. *Let $\alpha \in (0,1]$ and $g \in L^1([0,1])$ such that $0 \leq g(x) \leq 1$ for all $x \in [0,1]$. If $\varphi : [0,1] \to [0,\infty)$ is a convex, α-fractional differentiable function with $\varphi(0) = 0$, then*

$$\varphi\left(\alpha \int_0^1 g(x)\, d_\alpha x\right) \leq \int_0^1 g(x)\, D_\alpha \varphi(x)\, d_\alpha x. \tag{3.9}$$

Proof. The function φ is convex and α-fractional differentiable on $[0,1]$ and $D_\alpha \varphi$ is nondecreasing for all $x \in [0,1]$. Then $-D_\alpha \varphi$ is decreasing and we take $f(x) = -D_\alpha \varphi$, $a = 0$ and $b = 1$ in the Fractional Steffensen's inequality (2.7) it follows that

$$\int_0^\ell D_\alpha \varphi(x) d_\alpha x \leq \int_0^1 g(x)\, D_\alpha \varphi(x) d_\alpha x \leq \int_{1-\ell}^1 D_\alpha \varphi(x) d_\alpha x.$$

By simple computation, we have

$$\varphi(\ell) - \varphi(0) \leq \int_0^1 g(x)\, D_\alpha \varphi(x) d_\alpha x \leq \varphi(1) - \varphi(1-\ell).$$

Since $\ell := \alpha \int_a^b g(x)\, d_\alpha x$ and $\varphi(0) = 0$, we obtain the desired result (3.9). □

Now, we give the new inequality for functions $g \in L_\alpha^1([0,1])$ as follows:

Theorem 3.4. *Let $\alpha \in (0,1]$ and $g \in L_\alpha^1([0,1])$ such that $0 \leq g(x) \leq 1$ for all $x \in [0,1]$. If $\varphi : [0,1] \to [0,\infty)$ is a convex, α-fractional differentiable function with $\varphi(0) = 0$, then*

$$\varphi\left(\alpha \int_0^1 g(x)\, d_\alpha x\right) \leq \int_0^1 g(x)\, D_\alpha \varphi(x)\, d_\alpha x$$

for all $x \in [0,1]$.

Proof. Let $g \in L_\alpha^1([0,1])$ and $\varepsilon = \frac{1}{n} > 0$, there exists a sequence $(g_n)_{n \in N}$ of functions which are continuous on $[0,1]$ such that $\|g_n - g\|_{\alpha,1} < \frac{1}{n}$. Since g_n is continuous, then by Theorem 3.3, we obtain that

$$\begin{aligned} \varphi\left(\alpha \int_0^1 g_n(x)\, d_\alpha x\right) &\leq \int_0^1 g_n(x)\, D_\alpha \varphi(x)\, d_\alpha x \\ &= \int_0^1 g(x)\, D_\alpha \varphi(x)\, d_\alpha x + \int_0^1 [g_n(x) - g(x)]\, D_\alpha \varphi(x)\, d_\alpha x. \end{aligned}$$

Since

$$\left| \int_0^1 g_n(x)\, d_\alpha x - \int_0^1 g(x)\, d_\alpha x \right| \leq \int_0^1 |g_n(x) - g(x)|\, d_\alpha x < \frac{1}{\alpha n} \to 0 \text{ as } n \to \infty,$$

it follows that

$$\varphi\left(\alpha \int_0^1 g(x)\, d_\alpha x\right) \leq \int_0^1 g(x)\, D_\alpha \varphi(x)\, d_\alpha x$$

which is completed the proof. □

References

[1] T. Abdeljawad, On conformable fractional calculus, J. Comput. Appl. Math. 279 (2015) 57–66.
[2] M. Abu Hammad, R. Khalil, conformable fractional heat differential equations, Int. J. Pure Appl. Math. 94(2) (2014), 215-221.
[3] M. Abu Hammad, R. Khalil, Abel's formula and wronskian for conformable fractional differential equations, Int. J. Differ. Equ. Appl. 13(3) 2014, 177-183.
[4] D. R. Anderson, Taylor's formula and integral inequalities for conformable fractional derivatives, Contrib. Math. Eng. Springer, (2016).
[5] R. Khalil, M. Al horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, J. Comput. Appl. Math. 264 (2014), 65-70.
[6] P. Cerone, On some generalizations of Steffensen's inequality and related results, J. Ineq. Pure Appl. Math. 3 (2) (2001), Art. ID 28.
[7] Z. Liu, More on Steffensen type inequalities, Soochow J. Math., 31 (3) (2005), 429–439.
[8] Z. Liu, On Steffensen type inequalities, J. Nanjing Univ. Math. Biquart. 19 (2) (2002), 25-30.
[9] D.S. Mitrinovic, J.E. Pecaric and A.M. Fink, Classical and new inequalities in analysis, Kluwer, Dordrecht (1993).
[10] J. F. Steffensen, On certain inequalities and methods of approximation, J. Inst. Actuaries 51(1919), 274–297.
[11] S.-H. Wu and H. M. Srivastava, Some improvements and generalizations of Steffensen's integral inequality, Appl. Math. Comput. 192 (2007), 422-428.

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey

*Corresponding author: yaldizhatice@gmail.com