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Dual Notions of Prime Modules

I. M. Ali, R. I. Khalaf
Department of Mathematics, College of Education Ibn-Al-Haitham, University of Baghdad

Abstract

Let R be a commutative ring with unity .M an R-Module. M is called coprime module (dual notion of prime module) if ann M =**ann** M/N for every proper submodule N of M In this paper we study coprime modules we give many basic properties of this concept. Also we give many characterization of it under certain of module.

Introduction

Let R be a commutative ring with unity. M an R-Module. N is called prime submodule of an R-modules M if N ≠ M and whenever rx ∈ N such that r ∈ R, x ∈ M, then either x ∈N or r ∈ [N: M], see [1], [2] .M is called a prime module if $\mathbf{ann}_R M = \mathbf{ann}_R N$ for every non-zero submodule N of M, see [3],[4] .It is clear that M is prime iff <0> is Prime submodule of M.

Yassemi.s. in [5], introduced the notion of second submodule (as dual notion of prime submodule) as follows: N is second submodule of an R-module M if for every r ∈ R, the homothety r* on N is either zero or surjective, where if M is an R-module and r ∈R, then an R-endomorphism r* is called homothety if r*(x) = rx for all x ∈ M. M is second module if it is second submodule of M.

Annin. S. in [6] introduced the notion of coprime modules (as dual notion prime modules) as follows: An R-modules M is called coprime if $\mathbf{ann}_R M = \mathbf{ann}_R M/N$ for every proper submodule N of M .Not that $[N: M] = \mathbf{ann}_R M/N$ .Specially a ring R is coprime iff R is coprime R-module.

Abuhilail J.in [7] in introduced a notion coprime submodules (as dual notion prime submodules) as follows: N is called coprime submodule of an R-module M if $\mathbf{ann}_R M = \mathbf{W}(M/N)$ = {a ∈ R; the homothety r* on M/N is not surjective}, see [5].We notice that M is coprime module iff M is second module iff (0) is coprime submodule.

The main purpose of this paper is to give basic properties of coprime (second) modules and study the relationships between coprime modules and other modules
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We show that a submodule of coprime module need not be coprime module and we give certain conditions to make a submodule of coprime module is coprime. Moreover we obtained the relationships between coprime modules and divisible modules, principally injective and injective modules. Finally we investigate the behavior of coprime modules under localization.
Definition 1:[6] An R-module M is called ***coprime*** if for every proper submodule N of M, $\operatorname{ann}_R N = \operatorname{ann}_R (M / N)$.

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Specially, a ring R is coprime iff R is a coprime R-module.

Recall that a proper submodule N of an R-module M is called ***invariant*** if for each $f \in \underset{R}{\mathrm{End}}(M)$, $f(N) \subseteq N$. M is called ***fully invariant*** if every submodule of M is invariant, see [8]. .Wijayanti I.E in [9], gave the following characterization for coprime modules.

Theorem 2: [9]

Let M be an R-module, then the following statements are equivalent:

1. M is a coprime R-module.
2. $\underset{R}{\mathrm{ann}}\, M = \underset{R}{\mathrm{ann}}\,(M / N)$ for every proper invariant submodule N of M.

Proof:

$(1) \longrightarrow (2)$ is obvious.

$(2) \longrightarrow (1)$. To prove M is a coprime R-module. Assume that $\underset{R}{\mathrm{ann}}\,(M / N) \not\subseteq \underset{R}{\mathrm{ann}}\, M$ for some proper submodule N of M. Let $I = \underset{R}{\mathrm{ann}}\,(M / N)$. Then $I \not\subseteq \underset{R}{\mathrm{ann}}\, M$ and $IM \subseteq N$. But IM is invariant, since for each $f \in \underset{R}{\mathrm{End}}(M)$, $f(IM) = If(M) \subseteq IM$. Hence $I \subseteq \underset{R}{\mathrm{ann}}\,(M / IM) = \underset{R}{\mathrm{ann}}\, M$, which is a contradiction. ■

Note that, statement (2) in theorem (2) is used to define coprime modules in [27, Definition 1.3.1].

First, we give some remarks and examples of coprime modules.

Remarks and Examples 3*:*

1. Z as Z-module is not coprime.
2. Q as Z-module is coprime module.
3. Z_{p^∞} as a Z-module is a coprime module [9].

4. Every simple R-module is a coprime module, however the converse is not true for example: Q as Z-module is coprime and not simple.

Recall that an R-module M is called multiplication if for any submodule N of M, there exist an ideal I of R such that IM=N, equivalently for every submodule N of M, **N=** **[**N: M**]**M, see [10].

5. If M is a multiplication coprime module, then M is simple, and hence M is prime.

Proof: Since M is a coprime R-module, then $\underset{R}{\mathrm{ann}}\, M = [N \underset{R}{:} M]$ for all proper submodule N of M. Hence $(\underset{R}{\mathrm{ann}}\, M)\cdot M = [N \underset{R}{:} M]\cdot M$. Thus $(0) = [N \underset{R}{:} M]\cdot M$. But M is a multiplication R-module, so $[N \underset{R}{:} M]\cdot M = N$, and hence $N = (0)$. Thus M is a simple R-module, and M is a prime R-module. ■

The condition "M is multiplication" can not be dropped from previous remark for example: Z_{p^∞} is a coprime Z-module, and by [2, Remark 1.1.3(11)] Z_{p^∞} is not prime and not multiplication.

6. If M is a cyclic coprime R-module, then M is simple and so prime.

7. R is a coprime ring if and only if R is a field.

Proof: The proof follows by (Rem. and Ex. 2. (5), (4)). ■

8. Z_n is a coprime Z-module iff n is a prime number.

Proof: The proof follows directly by (Rem. and Ex. 3 (4), (5)). ■

9. For any $n, m \in Z$; $n \neq m$, the Z-module $M = Z_n \oplus Z_m$ is not coprime.

10. Every vector space M over a field R is a coprime R-module.

We have the following proposition:

Proposition (4):

Let M be an R-module. Consider the following statements:

1. M is a coprime R-module.
2. $\underset{R}{\text{ann}}\, M = [x \underset{R}{:} M]$ for every $x \in M$ such that (x) is a proper submodule of M.
3. For every ideal I of R and for every $x \in M$, (x) is a proper submodule of M such that $I \cdot \frac{M}{(x)} = (x)$, implies I = 0 or I M = 0.

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3), and (2) $\Rightarrow$ (1) if $\bigcap_{i\in\wedge}[x_i \underset{R}{:} M] = [\sum_{i\in\wedge} Rx_i \underset{R}{:} M]$, where $x_i \in M$ and $\wedge$ is any index set.

Proof: (1) $\Rightarrow$ (2). It is clear.

(2) $\Rightarrow$ (3). Let $I \cdot \frac{M}{(x)} = (x)$ and assume $I \neq 0$, then $I M \subseteq (x)$, that is $I \subseteq [x \underset{R}{:} M]$ and by (2) $I \subseteq \underset{R}{\text{ann}}\, M$. Thus I M = (0).

(3) $\Rightarrow$ (2). Let $r \in [x \underset{R}{:} M]$; $x \in M$ and (x) proper submodule of M. Then $r \in \underset{R}{\text{ann}} \frac{M}{(x)}$, that is $(r) \frac{M}{(x)} = 0$, hence by (3), either $r = 0$ or (r) M = 0. Thus $r \in \underset{R}{\text{ann}}\, M$.

(2) $\Rightarrow$ (1). Let N be a proper submodule of M. Then $N = \sum R x_i$, $x_i \in N$. So that $[N \underset{R}{:} M] = [\sum R x_i \underset{R}{:} M] = \bigcap_{x_i \in N} [x_i \underset{R}{:} M] = \underset{R}{\text{ann}}\, M$. ■

The following proposition is a characterization of coprime module under the class of finitely generated (multiplication) modules.

Proposition (5):

Let M be a finitely generated (or multiplication) R-module, then M is a coprime R-module if and only if $\underset{R}{\text{ann}}\, M = [N \underset{R}{:} M]$, for every prime submodule N of M.

Proof: ($\Rightarrow$) It is clear.

To prove the converse, let W be a proper submodule of M. Since M is finitely generated (or multiplication) R-module, then by [15], [1] there exists a maximal submodule N of M(which is prime by [30, Corollary 2.5, ch.1]) such that $W \subseteq N \neq M$. Hence $[W \underset{R}{:} M] \subseteq [N \underset{R}{:} M]$. But $[N \underset{R}{:} M] = \underset{R}{\text{ann}}\, M$ by assumption. Thus $[W \underset{R}{:} M] \subseteq \underset{R}{\text{ann}}\, M$, and so $\underset{R}{\text{ann}}\, M = [W \underset{R}{:} M]$.

Recall that a submodule N of an R-module M is called second submodule if for every $r \in R$, the homothety r* on N is either zero or surjective, where if M is an R-module and $r \in R$, then an R-endomorphism r* is called homothety if r*(x) = rx for all $x \in M$; see [29, Definition 2.1. (b)]. M is second module if it is second submodule of M

Remark (6): It is clear that N is a second submodule of an R-module iff for every $r \in R$, $r \neq 0$, either rN=N or rN=0.

The following result is an interesting characterization of coprime modules.

Theorem (7):

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Let M be an R-module, then M is coprime R-module iff M is a second R-module.

Proof: (⇒) Let M be a coprime R-module. Let $r \in$ R such that $r \neq 0$. Suppose the homothety r* on M is not surjective, so r M ≠ M. Let r M = N, then it is clear that $r \in [N \underset{R}{:} M]$. Since M is a coprime, then $[N \underset{R}{:} M] = \underset{R}{\text{ann}} M$. Hence $r \in \underset{R}{\text{ann}} M$, that is r M = 0. Thus r * = 0.

(⇐) To prove $\underset{R}{\text{ann}} (M / N) = \underset{R}{\text{ann}} M$ for every proper submodule N of M. Let $r \in [N \underset{R}{:} M]$, then r M ⊆ N ⊆ M. Since M is a second submodule, then the homothety r * on M is either zero or surjective. If r * is surjective, then r *(M) = r M = M, implies M ⊆ N which is a contradiction. Thus r M = 0 and so r ∈ $\underset{R}{\text{ann}}$ M. Therefore $\underset{R}{\text{ann}} \frac{M}{N} = \underset{R}{\text{ann}} M$. ■

The following result is an immediate consequence of Theorem (7) and [16, Rem. and Ex. 1.1.4 (3)).

Note (8):

If M is a coprime R-module, then $\underset{R}{\text{ann}} M$ is a prime ideal, and R / $\underset{R}{\text{ann}} M$ is an integral domain.

So that we shall say that M is ***P-coprime*** if M is coprime with $\underset{R}{\text{ann}} M$ = P.

A series of results follows by using theorem (7).

Corollary (9): [28]

Let M be an R-module, then M is coprime iff for every $r \in$ R, $r \neq 0$, either r M = 0 or r M = M (i.e. M is a second module).

Proof: It follows directly by Theorem (7) and Remark (6). ■

Corollary (10):

Let M be an R-module, let I be an ideal of R such that I ⊆ $\underset{R}{\text{ann}} M$. Thus M is a coprime R-module if and only if M is a coprime R / I-module.

Proof: It follows by Theorem (7) and [16, Rem. and Ex. (1.1.4(8)). ■

Corollary (11):

Let M be an R-module. Then M is a coprime R-module iff M is a coprime $\overline{R}$ = R / $\underset{R}{\text{ann}} M$-module.

Proof: It follows directly by previous corollary. ■

Corollary (12):

If R is an integral domain, then Q(R) [the total quotient field of R] is a coprime R-module.

Proof: Is obvious.

Corollary (13):

The homomorphic image of coprime R-module is coprime.

Proof: It follows by Theorem (7) and [16.Rem. and Ex. (1.1.4(5)]. ■

Note that,Wijayanti I.E. proved that if M is a coprime R-module and N is an invariant submodule of M, then M / N is a coprime R-module, see [9, Prop. 1.3.8]. Hence the following corollary is a stronger result.

Corollary (14):

If M is a coprime R-module, then M / W is a coprime R-module.

Proof: Let π: M ⟶ M / W be the natural projection. Hence the result follows by previous corollary. ■

Corollary (15):

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Let M, W be two R-modules such that M $\cong$ W, then M is coprime iff W is coprime.

Proof: It is immediate by corollary (14). ■

By considering (Rem. and Ex. 3 (1), (2)) a submodule of coprime R-module (second module) need not be coprime R-module.

However, in the following proposition, this is true under certain condition.

Proposition (16):

Let N be a non-zero proper submodule of an R-module M such that $r\,M \cap N = r\,N$, for every $r \in R$, then M is P-coprime iff N and M / N are P-coprime R-modules.

Proof: If M is a P-coprime R-module, then M is coprime with $\underset{R}{\text{ann}}\,M = P$. Thus by corollary (14) M / N is coprime. Since $P = \underset{R}{\text{ann}}\,M = \underset{R}{\text{ann}}\,(M / N)$, then M /N is P-coprime.

Now, to prove N is a P-coprime module. Since M is coprime, then for any $r \in R$, $r \neq 0$, either $r\,M = 0$ or $r\,M = M$. If $r\,M = 0$, then $r\,M \cap N = 0$. But $r\,M \cap N = r\,N$, then $r\,N = 0$. If $r\,M = M$, so $r\,M \cap N = N$. But $r\,M \cap N = r\,N$, hence $r\,N = N$. Thus N is second.

To prove $N = P = \underset{R}{\text{ann}}\,M$. It is clear that $\underset{R}{\text{ann}}\,M \subseteq \underset{R}{\text{ann}}\,N$. Let $r \in \underset{R}{\text{ann}}\,N$, thus $r\,N = 0$. If $r\,M = 0$, there is nothing to prove. If $r\,M = M$, then $r\,M \cap N = N$. But $r\,M \cap N = r\,N$. Thus $r\,N = N$ and so N = (0) which is a contradiction.

Conversely, if N and M / N are P-coprime. Then $P = \underset{R}{\text{ann}}\,N = \underset{R}{\text{ann}}\,\frac{M}{N}$ and $r\,N = N$, $r\,\frac{M}{N} = \frac{M}{N}$ for every $r \notin P$. To prove M is P-coprime. It is clear $\underset{R}{\text{ann}}\,M \subseteq \underset{R}{\text{ann}}\,N = \underset{R}{\text{ann}}\,\frac{M}{N} = P$. Let $r \in P$, so $r\,N = 0$ and $r\,M \subseteq N$. Hence $r\,M \cap N = r\,M$, but $r\,M \cap N = r\,N$, so $r\,M = r\,N = 0$. Thus $r \in \underset{R}{\text{ann}}\,M$, hence $P = \underset{R}{\text{ann}}\,M$. Let $r \notin \underset{R}{\text{ann}}\,M = P$ and let $m \in M$, then $m + N \in \frac{M}{N} = r\,\frac{M}{N}$, hence $m + N = r\,(m' + N)$ for some $m' \in M$. Thus $m - r\,m' \in N = r\,N$, so $m - r\,m' = r\,n$ for some $n \in N$ and hence $m = r\,(m' + n) \in r\,M$. Thus $M = r\,M$ for every $r \notin P$, and so M is P-coprime. ■

Recall that a ring R is said to be ***regular*** (in sense of Von Neumann) if for each $x \in R$, there exists $a \in R$ such that $x = x^2 \cdot a$, see [6].

Corollary (17): Let M be a module over a regular ring (in sense of Von Neumann) and let N be a submodule of M. Then M is a P-coprime module iff N and M /N are P-coprime R-modules.

Proof: Since R is a regular ring, then for every $r \in R$, $r\,M \cap N = r\,N$. Hence the result follows by proposition (16). ■

Now, we have the following result.

Proposition (18): Let N be a finitely generated submodule of an R-module M such that $\underset{R}{\text{ann}}\,N = \underset{R}{\text{ann}}\,M$. If N is a coprime R-module, then M is a coprime R-module.

Proof: Let $r \in R$ and $r \notin \underset{R}{\text{ann}}\,M$. Since N is a coprime module and $\underset{R}{\text{ann}}\,N = \underset{R}{\text{ann}}\,M$, $r\,N = N$. But N is a finitely generated submodule, so by [25, p.50], there exists $r' \in R$ such that $(1 - r\,r')\,N = 0$. Hence $(1 - r\,r')\,M = 0$, and so $M = r\,M$. Therefore M is coprime. ■

Note that, the condition $\underset{R}{\text{ann}}\,N = \underset{R}{\text{ann}}\,M$ can not dropped from the previous proposition as the following example shows:

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$(\bar{2})$ in Z_6 as Z-module is coprime module, and $\underset{Z}{\text{ann}}(\bar{2}) = 3Z \neq \underset{Z}{\text{ann}}(Z_6) = 6Z$. However Z_6 is not coprime Z-module, see (Rem. and Ex. 3 (8)).

Next, we can give the following proposition.

Proposition (19): Let N be a finitely generated submodule of M and $\underset{R}{\text{ann}} M$ is a prime ideal. If N is a coprime module, then $\underset{R}{\text{ann}} M = [N \underset{R}{:} M]$.

Proof: Let $r \in [N \underset{R}{:} M]$, then $r M \subseteq N$. It is clear that $r \in \underset{R}{\text{ann}} N$ or $r \notin \underset{R}{\text{ann}} N$. If $r \in \underset{R}{\text{ann}} N$, then $r N = 0$ and hence $r^2 M \subseteq r N = 0$; that is $r^2 \in \underset{R}{\text{ann}} M$. Since $\underset{R}{\text{ann}} M$ is prime, then $r \in \underset{R}{\text{ann}} M$. If $r \notin \underset{R}{\text{ann}} N$, then $r N = N$ because N is a coprime R-module.

On the other hand, N is finitely generated, there exists $r' \in R$ such that $(1 - r r') N = 0$, see [25, p.50]. It follows that $r(1 - r r') M \subseteq (1 - r r') N = 0$. Thus $r(1 - r r') M = 0$ and so $r(1 - r r') \in \underset{R}{\text{ann}} M$, which implies either $r \in \underset{R}{\text{ann}} M$ or $(1 - r r') \in \underset{R}{\text{ann}} M$. If $(1 - r r') \in \underset{R}{\text{ann}} M$, then $M = r M \subseteq N$ which is a contradiction. Thus $r \in \underset{R}{\text{ann}} M$ and so $[N \underset{R}{:} M] = \underset{R}{\text{ann}} M$.

Recall that an R-module M is said to be ***Noetherian*** if every submodule of M is finitely generated, see [19, Prop. 6.2, p.75].

The following result is an immediate consequence of proposition (19).

Corollary (20): Let M be a Noetherian R-module such that $\underset{R}{\text{ann}} M$ is a prime ideal of R, if every submodule of M is a coprime R-module, then M is coprime.

Recall that a submodule N of an R- module M is called ***primary*** if for every $r \in R$, $x \in M$ such that $rx \in N$, then either $x \in N$ or $r \in \sqrt{[N \underset{R}{:} M]} = \{s \in R;\ s^n \in [N \underset{R}{:} M]$ for some $n \in Z_+\}$, see [20].

We know that every prime submodule is primary; however the converse is not true in general. The following result shows that the two concepts are equivalent in the class of coprime modules.

Proposition (21): Let M be a coprime R-module, let N be a proper submodule of M, then N is primary iff N is prime.

Proof: Let N be a primary submodule of M, since M is coprime, then $\underset{R}{\text{ann}} M = [N \underset{R}{:} M]$, so $[N \underset{R}{:} M]$ is a prime ideal, hence by [14, Prop. 2.10, ch.1], we have N is prime. ■

S.Yassem in [5] gave the following result. We give its proof for the sake of completeness.

Recall that a submodule N of an R-module M is called **secondary submodule** if for each $r \in R$,the homothety r^* on N is either surjective or nilpotent, where r^* is nilpotent if there exist $k \in Z+$ such that $(r^*) = 0$,see[21].

Proposition (22): Let N be a submodule of an R-module M such that M is P-second (P-coprime), then N is P-secondary iff N is P-second (P-coprime).

Proof: $(\Rightarrow)$ Since M is P-coprime, $\underset{R}{\text{ann}} M = P$. Suppose that N is P-secondary, then $P = \sqrt{\underset{R}{\text{ann}} N}$. Thus $P = \sqrt{\underset{R}{\text{ann}} N} = \underset{R}{\text{ann}} M$. To prove N is P-second. Since N is a submodule of

M, then P = $\underset{R}{\text{ann}} \subseteq \underset{R}{\text{ann}}$N, but $\underset{R}{\text{ann}}N \subseteq \sqrt{\underset{R}{\text{ann}}N}$ = P which is a prime ideal. Thus by proposition (7) N is a second submodule with P = $\underset{R}{\text{ann}}$N; that is N is P-second submodule.

The converse is obvious by [15, Rem. and Ex. 1.2.2 (1)]. ■

Recall that if M is an R-module, then ***J(M)*** is the intersection of all maximal submodules of M (if exist) or J(M) = M if M has no maximal submodule, see [12, Definition 9.1.2,p.214].

The following result appeared in [27]. However we give another proof.

Proposition (23): If M is a coprime R-module and J(M) ≠ M, then $\overline{R}$ is a coprime ring, where $\overline{R} = R / \underset{R}{\text{ann}}M$.

Proof: Since J(M) ≠ M, then there exists a maximal submodule N of M. By [15, Propo. 2.2, ch. 1] the ideal $[N \underset{R}{:} M]$ is a maximal ideal of R. Since M is coprime, then $[N \underset{R}{:} M] = \underset{R}{\text{ann}}M$ and so $\underset{R}{\text{ann}}M$ is a maximal ideal of R, so $\overline{R} = R / \underset{R}{\text{ann}}M$ is a field. Thus $\overline{R}$ is a coprime ring. ■

We note that the condition J(M) ≠ M in proposition (23) is necessary for example:

Q as Z-module is coprime module, which has no maximal submodule; that is J(Q) = Q and Z is not a field.

Recall that an R-module M is called **divisible** if for each non zero divisor r of R, rM=M ,see[22].

The following proposition is an immediate result by theorem (7) and [16, prop 1.1.7].

Proposition (24): Let M be an R-module, then the following statements are equivalent:

1. M is a coprime R-module.
2. M is a divisible $R / \underset{R}{\text{ann}}M$-module.
3. $rM = M$ for every $r \in R \setminus \underset{R}{\text{ann}}M$.
4. $IM = M$ for every ideal $I \not\subseteq \underset{R}{\text{ann}}M$.
5. $W(M) = \underset{R}{\text{ann}}M$.

Note that statement (5) in proposition (24) was given as definition of coprime module by Abuihlial J. in [7].He defined that an R-submodule N of M is a coprime submodule if for each homothety r^* on M / N is either zero or surjective. Then it is clear that M is a coprime R-module if and only if (0) is a coprime submodule. By combining this result with theorem (7) we have the following corollary.

Corollary (25): Let M be an R-module, then the following statements are equivalent:

1. M is a coprime R-module.
2. M is a second submodule of itself.
3. (0) is a coprime submodule of M.

The following remark is clear.

Remark (26): Let N be a proper submodule of an R-module M. Then N is a coprime submodule of M iff M / N is a coprime R-module.

The following result follows by Remark (26) and Note (8).

Corollary (27): Let N be a proper submodule of an R-module M, if N is a coprime submodule then $\underset{R}{\text{ann}}(M / N)$ is a prime ideal.

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Recall that an R-module M is called ***divisible*** if for each non zero divisor r of R, rM=M, see [22].

Now, we list some consequences of proposition (24).

Corollary (28): Every faithful coprime R-module is divisible.

Proof: Is clear. ■

Remark (29): Every divisible module over an integral domain is faithful coprime.

By combining corollary (28) and remark (29) we get:

Corollary (30): Let M be a module over an integral domain R. Then M is a faithful coprime R-module iff M is a divisible R-module.

Recall that an R-module M is called ***principally injective*** if for every principal ideal I of R and every monomorphism f: I ⟶ M, there is a homomorphism g: R ⟶ M such that $g/_I = f$, see [23].

Proposition (31): Let M be a module over an integral domain R. Then the following statements are equivalent:

1. M is a faithful coprime R-module.
2. M is a divisible R-module.
3. M is a principally injective R-module.

Proof: (1) ⟶ (2) follows by corollary (30).

(2) ↔ (3) follows by [12, Exc. 9(a), P.104].

Recall that an R-module M is said to be ***injective*** if and only if for any monomorphism f: A ⟶ B where A and B are any two R-modules and for any homomorphism g: A ⟶ M there exists a homomorphism h: B ⟶ M such that $h \circ f = g$, see [24, p.28].

Corollary (32): If R is PID and M is an R-module, then the following statements are equivalent:

1. M is a faithful coprime R-module.
2. M is a divisible R-module.
3. M is an injective R-module.

Proof:

(1) ⇔ (2) follows by corollary (.30).

(2) ⇔ (3). By [26, Th 2.8, P.35]. ■

Corollary (33):

Let M be an injective module over an integral domain, then M is coprime.

Proof: Since every injective R-module is divisible by [26, Theorem 2.6, p. 33]. Hence the result obtained by Remark (29). ■

Recall that a module M over an integral domain is called **torsion free** if τ(M) = 0, where τ(M) = {$m \in$ M; $\exists\, r \in$ R, $r \neq 0$; $r\,m = 0$}, see [19, p.45].

Under the class of torsion free modules over an integral domain, we have the following result.

Proposition (34): Let M be torsion free over an integral domain R. Then the following statements are equivalent:

1. M is a coprime R-module.
2. M is a divisible R-module.
3. M is an injective R-module.

Proof:

(1) ⇔ (2). Since M is torsion free, then M is faithful. Thus the result follows by corollary (30).

(2) ⇔ (3) follows by [26, Th 2.7, p.34]. ■

Recall that an integral domain R is called a ***Dedekind domain*** if every non-zero ideal of R is invertible and an ideal I is called ***invertible***, when $I^{-1} = \{x \in R_S : x\,I \subseteq R\}$ and S is the set of non-zero divisors of R, then $I^{-1} \cdot I = R$, see [25].

Next, we prove the following.

Proposition (35): If R is a Dedekind domain and M is an R-module. Then the following statements are equivalent:

1. M is a faithful coprime R-module.
2. M is a divisible R-module.
3. M is an injective R-module.

Proof:

(1) $\longrightarrow$ (2) follows by corollary (30).

(2) $\Leftrightarrow$ (3) follows by [24, Prop. 2.10, p.36]. ■

Yassemi S. in [5] introduced the following result without proof. We give its proof for sake of completeness.

Theorem (36): Let M be a prime R-module. The following statements are equivalent:

1. M is a coprime R-module.
2. M is an injective $R/\operatorname{ann}_R M$-module.

Proof: Since M is a prime R-module, then by [2, Rem. and Ex. (1.1.3 (3)], $\operatorname{ann}_R M$ is a prime ideal, so $\overline{R}$ is an integral domain and M is a torsion free $\overline{R} = R / \operatorname{ann}_R M$-module, see [20], [11].

(1) $\longrightarrow$ (2). Since M is a coprime R-module, then by corollary (10) M is a coprime $\overline{R}$-module. Hence by proposition (34) M is an injective $\overline{R}$-module.

(2) $\longrightarrow$ (1). If M is an injective $\overline{R}$-module, then by corollary (33) M is a coprime $\overline{R}$-module. Hence by corollary (10) M is a coprime R-module. ■

Recall that an R-module M is called ***flat*** if $\sum_{k=1}^{n} \lambda_k a_k = 0$, where $\lambda_k \in R$, $a_k \in M$, then there exist $b_1, \ldots, b_n \in M$ and $\{u_{ik}\} \subseteq R$, $i = 1, \ldots, n$, $k = 1,2,\ldots,r$ such that $\sum_{i=1}^{n} u_{ik} b_i = a_k$ and $\sum_{k=1}^{r} u_{ik} \lambda_k = 0$, see [2].

The following theorem appeared in [18], we give the details of the proof for completeness.

Recall that a subset A of an R-module M is called a ***basis*** of M, if A generates M and A is R-linearly independent. M is said to be a ***free*** R-module, if M has a basis [25, p.190].

Theorem (37): Let M be a coprime R-module. Then the following statements are equivalent:

1. M is a prime R-module.
2. -M is a flat $\overline{R}$-module, where $\overline{R} = R / \operatorname{ann}_R M$.

Proof: Since M is a coprime R-module, then $\overline{R} = R/\operatorname{ann}_R M$ is an integral domain. Also by proposition (24), M is a divisible $\overline{R}$-module.

(1) ⟶ (2). Since M is a prime R-module, then M is a torsion free $\overline{R}$-module, where $\overline{R} = R / \underset{R}{\text{ann}} M$.

Now, we can show that M is a vector space over K = the total quotient field of $\overline{R}$ as follows: Let $\frac{r + \text{ann}\, M}{s + \text{ann}\, M} \in$ K, where $r, s \in$ R, $s \notin \underset{R}{\text{ann}} M$. Let $m \in M$, $m = (s + \underset{R}{\text{ann}} M)\, m'$, for some $m' \in M$ since M is a divisible $\overline{R}$-module. It follows that $\frac{r + \text{ann}\, M}{s + \text{ann}\, M} \cdot m = (r + \underset{R}{\text{ann}} M)\, m' = r\, m' \in M$. Thus M is a vector space over K, so it has a basis. It follows that M is a free K-module, hence M is a flat K-module [25, Prop 1.26, p.22]. Then K is a flat $\overline{R}$-module by [27, Ex. 20, p.319]. Thus by [19, Exc. 9(a), p.32] M is flat $\overline{R}$-module.

(2) ⟶ (1). [The proof is different from the proof given in [29].

Since $\overline{R}$ is an integral domain, $(\overline{0})$ is a prime ideal of $\overline{R}$. Hence by [30, Corollary 4.9, ch.1] $(\overline{0})$ M = (0) is a prime $\overline{R}$-submodule of M. Thus (0) is a prime $\overline{R}$-submodule. Then it is easy to check that (0) is a prime R-submodule of M and hence a prime R-module.

Finally, we turn our attention to the localization of coprime modules. First we have.

Proposition (38): Let M be a coprime R-module, then M_S is a coprime R_S-module, where S is a multiplicatively closed subset of R.

Proof:

It follows by theorem () and [16, prop 1.1.20].

We notice that the converse of proposition (38) is not true as the following example shows:

Let M be Z-module Z and S = Z – {0}, it is clear that S is a multiplicatively closed subset of Z. Z_S = Q and Q as Q-module is coprime, but Z is not coprime Z-module.

The following corollary follows immediately from proposition (38).

Corollary (39):

Let M be a coprime R-module,then M_p is a coprime R_p-module for any prime ideal P of R.

Next, we have the following result.

Corollary (40): Let M be a finitely generated R-module. Let S be a multiplicatively closed subset of R. If M is P-coprime R-module, then M_S is P_S coprime R_S –module.

Proof: Since M is P-coprime, then M is coprime with $\underset{R}{\text{ann}} M$ = P.Therefore by proposition (2.1.38), M_S is coprime R_S –module.

Now, $\underset{R}{\text{ann}} M = [0 \underset{R}{:} M] = P$, that is $[0 \underset{R}{:} M]_S = P_S$ and by [28, p.152], $[0_S : M_S] = P_S$. Thus $\underset{R_S}{\text{ann}} M_S = P_S$. Therefore M_S is P_S-coprime R_S-module. ■

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المفهوم الرديف للمقاسات الاولية

انعام محمد علي ، رشا ابراهيم خلف
قسم الرياضيات ، كلية التربية – ابن الهيثم ، جامعة بغداد

الخلاصة

لتكن R حلقة إبدالية ذات محايد ؛M مقاسا على R.يسمى M مقاسا أوليا مضادا (كمفهوم مضاد للمقاس الأولي) إذا كان تلّف M_R= تلّف $_R\frac{M}{N}$ لكل مقاس جزئي فعلي N في M .

في هذا البحث درسنا المقاسات الأولية المضادة ةأعطينا عددا من الخواص الأساسية لهذا المفهوم وكذلك أعطينا عددا من المميزات له تحت أصناف معينة من المقاسات