Classification and Construction of (k,3)-Arcs on Projective Plane Over Galois Field GF(9)

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Abstract

In this work, we construct and classify the projectively distinct (k,3)-arcs in PG(2,9), where $k \geq 5$, and prove that the complete (k,3)-arcs do not exist, where $5 \leq k \leq 13$. We found that the maximum complete (k,3)-arc in PG(2,q) is the (16,3)-arc and the minimum complete (k,3)-arc in PG(2,q) is the (14,3)-arc. Moreover, we found the complete (k,3)-arcs between them.

Keywords: arcs, secant, Projective plane ,Galois Field

Introduction

A (k,n)-arc K [1] in PG(2,q) is a set of k points, such that some n, but no n + 1 of them are collinear. A (k,3)-arc is a set of k points no four of them are collinear.

A (k,n)-arc K is complete if it is not contained in a (k + 1,n)-arc. A line of the plane containing exactly i points of a (k,n)-arc is called an i-secant of K. For a (k,3)-arc each line of PG(2,q) is a 3-secant, 2-secant, 1 secant, or 0-secant. A 3-secant is called a trisecant.

A point N not on a (k,n)-arc K has index i if there are exactly i trisecants of K through N. Let c_i be the set of points N of index i and let $C_i = |c_i|$ is the number of points N of index i. The (k,n)-arc K is complete if every point of PG(2,q) lies on some trisecant of K. Thus K is complete iff $C_0 = 0$ [2,3].

Theorem 1: [2]

Let r_i be the total number of the i-secants of a (k,n)-arc K in PG(2,q), then the following equations are hold:

$$\sum_{i=0}^{n} r_i = q^2 + q + 1$$

$$\sum_{i=1}^{n} i r_i = k(q+1)$$

$$\sum_{i=2}^{n} i(i-1) r_i = k(k-1)$$

Notation 2: [2]

Let r_i be the total number of i-secants of a (k,n)-arc K in PG(2,q), then the type of a (k,n)-arc K with respect to its lines is denoted by $(r_n, r_{n-1}, \ldots, r_0)$.

Definition 3: [2]

Let K_1 is of type $(r_n, r_{n-1}, \ldots, r_0)$ and K_2 is of type $(t_n, t_{n-1}, \ldots, t_0)$, then K_1 and K_2 have the same type iff $r_i = t_i$ for all i, and when K_1 and K_2 have the same type then they are projectively equivalent.

Definition 4: [2]:

Let Q_1 and Q_2 be two points in PG(2,q) which are not in arc K and let $K_1 = K \cup \{Q_1\}$, $K_2 = K \cup \{Q_2\}$, then Q_1 and Q_2 lie in the same set iff K_1 and K_2 are projectively equivalent under type of lines.

The Projective Plane PG(2,9): [2]

PG(2,9) contains 91 points, 91 lines, 10 points on every line and 10 lines through every point. Let P_i and L_i, i = 1, 2, …, 91, be the points and the lines of PG(2,9), resp.

Let i stands for the point P_i and all the points and lines of PG(2,9) are given in the table.

The Classification of (5,3)-Arcs in PG(2,9):

Let A ={1,2,11,21} be a set of the reference points in PG(2,9), no three of them are collinear. The distinct (5,3)-arcs can be constructed by adding to A each time one point from the remaining 87 points of PG(2,9). There are only two projectively distinct (5,3)-arcs which are:

$B_1 = \{1,2,11,21,3\}$ and $B_2 = \{1,2,11,21,4\}$

The group $G(B_1)$ has eight projectivities which has five elements of order two and two elements of order four, hence $G(B_1) \cong D_4$. The group $G(B_2) \cong D_6$.

The Classification of (6,3)-Arcs:

There are 72 points of index zero for B_1. Then $G(B_1)$ partitions 72 points into 13 orbits. So we have 13(6,3)-arcs to be constructed by adding one point from each of the 13 orbits to B_1.

We have 79 points of index zero for B_2. $G(B_2)$ partitions these points into 12 orbits. So we have 12 (6,3)-arcs to be constructed by adding one point from each of the 12 orbits to B_2. So there exist twenty five (6,3)-arcs to be constructed and by projectively equivalent arcs, we have four projectivity distinct (6,3)-arcs, these arcs are:

$C_1 = \{1,2,11,21,3,12\}$, $C_2 = \{1,2,11,21,3,13\}$,
$C_3 = \{1,2,11,21,3,32\}$, $C_4 = \{1,2,11,21,4,32\}$.

The group $G(C_1) \cong S_4$, the group $G(C_2) \cong S_3$, $G(C_3)$ is isomorphic to the identity group and $G(C_4) \cong Z_2$.

The Classification of (7,3)-Arcs:

There groups $G(C_i)$, $i = 1, 2, 3, 4$ partition the points of index zero for C_i into 133 orbits. So there exist 133 (7,3)-arcs to be constructed and by projectively equivalent arcs, we have only six (7,3)-arcs which are projectively distinct. These are:

$D_1 = \{1,2,11,21,3,12,20\}$, $D_2 = \{1,2,11,21,3,12,23\}$, $D_3 = \{1,2,11,21,3,12,33\}$,
$D_4 = \{1,2,11,21,3,13,32\}$, $D_5 = \{1,2,11,21,3,32,40\}$, $D_6 = \{1,2,11,21,4,32,42\}$.

Each one of them is incomplete arc.

$G(D_1) \cong S_4$, $G(D_2) \cong Z_2$, $G(D_3) \cong Z_2$, $G(D_4) \cong I$, $G(D_5) \cong Z_2$, $G(D_6) \cong I$.

The Classification of (8,3)-Arcs:

There groups $G(D_i)$, $i = 1, \ldots, 6$ partition the points of index zero for D_i into 239 orbits and so there exist 239 (8,3)-arcs to be constructed and by projectively equivalent of arcs, we have only seven projectively distinct (8,3)-arcs which are:

$E_1 = \{1,2,11,21,3,12,20,42\}$, $E_2 = \{1,2,11,21,3,12,20,32\}$, $E_3 = \{1,2,11,21,3,12,23,34\}$,
$E_4 = \{1,2,11,21,3,12,33,45\}$, $E_5 = \{1,2,11,21,3,13,32,53\}$, $E_6 = \{1,2,11,21,3,32,40,54\}$,
$E_7 = \{1,2,11,21,4,32,42,53\}$.

$G(E_1) \cong Z_2$, $G(E_2)$, $G(E_3)$, $G(E_4)$, $G(E_5)$, $G(E_6)$ and $G(E_7)$ are isomorphic to the identity group. Each one of these arcs is incomplete.

The Classification of (9,3)-Arcs:

There groups $G(E_i)$ partition the points of index zero for E_i $i = 1, \ldots, 7$ into 369 orbits. So there exist 369 (9,3)-arcs to be constructed and by projectively equivalent of arcs, we have only eight projectively distinct (9,3)-arcs which are:

$F_1 = \{1,2,11,21,3,12,20,42,43\}$, $F_2 = \{1,2,11,21,3,12,20,42,33\}$,
$F_3 = \{1,2,11,21,3,12,20,42,32\}$, $F_4 = \{1,2,11,21,3,12,20,32,40\}$,
$F_5 = \{1,2,11,21,3,12,23,34,45\}$, $F_6 = \{1,2,11,21,3,12,33,45,79\}$,
$F_7 = \{1,2,11,21,3,32,40,54,37\}$, $F_8 = \{1,2,11,21,3,32,40,54,79\}$.

Each one is an incomplete arc.

The groups $G(F_1)$, $G(F_2)$, $G(F_3)$, $G(F_5)$, $G(F_7)$ and $G(F_8)$ consist of the identity. $G(F_4) \cong Z_2$, $G(F_6) \cong Z_3$.

The Classification of (10,3)-Arcs:

There groups $G(F_i)$, i = 1, …, 8 partition the points of index zero for F_i into 307 orbits. So there exist 307 (10,3)-arcs and by projectively equivalent of arcs, we have only nine (10,3)-arcs which are projectively distinct and each one of them is an incomplete arc

H_1 = {1,2,11,21,3,12,20,42,43,69}, H_2 = {1,2,11,21,3,12,20,42,43,33},
H_3 = {1,2,11,21,3,12,20,42,43,35}, H_4 = {1,2,11,21,3,12,20,42,43,32},
H_5 = {1,2,11,21,3,12,20,42,33,86}, H_6 = {1,2,11,21,3,12,20,32,40,54},
H_7 = {1,2,11,21,3,12,23,34,45,60}, H_8 = {1,2,11,21,3,32,40,54,37,15},
H_9 = {1,2,11,21,3,32,40,54,37,79}.

$G(H_1) \cong Z_3$, $G(H_2)$, $G(H_3)$, $G(H_4)$, $G(H_5)$, $G(H_6)$, $G(H_7)$ and $G(H_8)$ are isomorphic to the identity group $G(H_9) \cong Z_2$.

The Classification of (11,3)-Arcs:

There groups $G(H_i)$, i = 1, …, 9 partition the points of index zero for H_i into 258 orbits. So we have 258 (11,3)-arcs to be constructed. By projectively equivalent of arcs, we have only nine (11,3)-arcs which are projectively distinct :

K_1 = {1,2,11,21,3,12,20,42,43,69,34}, K_2 = {1,2,11,21,3,12,20,42,43,69,32},
K_3 = {1,2,11,21,3,12,20,42,43,33,32}, K_4 = {1,2,11,21,3,12,20,42,43,35,32},
K_5 = {1,2,11,21,3,12,20,42,33,86,32}, K_6 = {1,2,11,21,3,12,20,32,40,54,37},
K_7 = {1,2,11,21,3,12,23,34,45,60,36}, K_8 = {1,2,11,21,3,32,40,54,37,15,72},
K_9 = {1,2,11,21,3,32,40,54,37,79,50}.

The groups $G(K_1)$, $G(K_2)$, $G(K_3)$, $G(k_4)$, $G(K_5)$, $G(K_6)$, $G(K_8)$ and $G(K_9)$ are isomorphic to the identity group, the group $G(K_7) \cong Z_2$.

The Classification of (12,3)-Arcs:

There groups $G(K_i)$, i = 1, …, 9 partition the points of index zero for K_i into 196 orbits. So there exist 196 (12,3)-arcs to be constructed. By projectively equivalent of arcs, we have only nine projectively distinct (12,3)-arcs which are:

L_1 = {1,2,11,21,3,12,20,42,43,69,34,35}, L_2 = {1,2,11,21,3,12,20,42,43,69,34,32},
L_3 = {1,2,11,21,3,12,20,42,43,69,32,54}, L_4 = {1,2,11,21,3,12,20,42,43,33,32,49},
L_5 = {1,2,11,21,3,12,20,42,43,35,32,63}, L_6 = {1,2,11,21,3,12,20,32,40,54,37,42},
L_7 = {1,2,11,21,3,12,23,34,45,60,36,46}, L_8 = {1,2,11,21,3,32,40,54,37,15,72,50},
L_9 = {1,2,11,21,3,32,40,54,37,15,72,79}.

Each one is an incomplete arc.

The groups $G(L_i)$, i = 1, …, 9 are isomorphic to the identity group.

The Classification of (13,3)-Arcs:

There groups $G(L_i)$ partition the points of index zero for L_i, i = 1, …, 9 into 137 orbits. So there exist 137 (13,3)-arcs to be constructed and by projectively equivalent of arcs, we have only nine projectively distinct (13,3)-arcs which are:

M_1 = {1,2,11,21,3,12,20,42,43,69,34,35,62}, M_2 = {1,2,11,21,3,12,20,42,43,69,34,35,58},
M_3 = {1,2,11,21,3,12,20,42,43,69,34,32,54}, M_4 = {1,2,11,21,3,12,20,42,43,69,32,54,85},

M_5 = {1,2,11,21,3,12,20,42,43,33,32,49,86}, M_6 = {1,2,11,21,3,12,20,42,43,35,32,63,64},
M_7 = {1,2,11,21,3,12,20,32,40,54,37,42,53}, M_8 = {1,2,11,21,3,12,23,34,45,60,36,46,49},
M_9 = {1,2,11,21,3,32,40,54,37,15,72,50,79}.
Each one is an incomplete arc.

The groups $G(M_i)$, i = 1, 2, 3, 5, …, 9 are isomorphic to the identity group $G(M_4) \cong Z_3$.

The Classification of (14,3)-Arcs:

There groups $G(M_i)$ partition the points of index zero for M_i, i = 1, …, 9 into eight projectively distinct (14,3)-arcs which are:

N_1={1,2,11,21,3,12,20,42,43,69,34,35,62,58}, N_2= {1,2,11,21,3,12,20,42,43,69,34,35,58,68},
N_3= {1,2,11,21,3,12,20,42,43,69,34,32,54,68}, N_4={1,2,11,21,3,12,20,42,43,33,32,49,86,53},
N_5={1,2,11,21,3,12,20,42,43,33,32,49,86,64}, N_6={1,2,11,21,3,12,20,42,43,35,32,63,64,86},
N_7={1,2,11,21,3,12,20,32,40,54,37,42,53,63}, N_8= {1,2,11,21,3,32,40,54,37,15,72,50,79,22}.

N_1, N_2, N_3 are complete, while N_4, …, N_8 are incomplete arcs. The groups $G(N_i)$, i = 1,2,…, 8 are isomorphic to the identity group.

The Classification of (15,3)-Arcs:

There groups $G(N_i)$, i = 4, …, 8 partition the points of index zero for N_i, into 21 orbits. So there exist 21 (15,3)-arcs to be constructed and by projectively equivalent of arcs, we have only six projectively distinct (15,3)-arcs which are:

Q_1 = {1,2,11,21,3,12,20,42,43,33,32,49,86,53,70},
Q_2 = {1,2,11,21,3,12,20,42,43,33,32,49,86,53,76},
Q_3 = {1,2,11,21,3,12,20,42,43,33,32,49,86,64,70},
Q_4 = {1,2,11,21,3,12,20,42,43,35,32,63,64,86,78},
Q_5 = {1,2,11,21,3,12,20,32,40,54,37,42,53,63,68},
Q_6 = {1,2,11,21,3,32,40,54,37,15,72,50,79,22,56}.

The groups $G(Q_i)$, i = 1, …, 6 are isomorphic to the identity group. Q_1, Q_2, Q_4 and Q_5 are complete since there are no points of index zero for them. The groups $G(Q_3)$ and $G(Q_6)$ partition the points of index zero for Q_3 and Q_6 into 5 orbits. So, there exist five (16,3)-arcs and by projectively equivalent of arcs, we have only two projectively distinct (16,3)-arcs which are:

R_1 = {1,2,11,21,3,12,20,42,43,33,32,49,86,64,70,67},
R_2 = {1,2,11,21,3,32,40,54,37,15,72,50,79,22,56,64}.

R_1 and R_2 are complete (16,3)-arcs since $C_0 = 0$ for R_1 and R_2.

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(GF(9),*)

*	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	6	8	7	3	5	4
3	3	6	4	7	1	8	2	5
4	4	8	7	2	3	5	6	1
5	5	7	1	3	8	2	4	6
6	6	3	8	5	2	4	1	7
7	7	5	2	6	4	1	8	3
8	8	4	5	1	6	7	3	2

(GF(9),+)

+	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	2	0	4	5	3	7	8	6
2	2	0	1	5	3	4	8	6	7
3	3	4	5	6	7	8	0	1	2
4	4	5	3	7	8	6	1	2	0
5	5	3	4	8	6	7	2	0	1
6	6	7	8	0	1	2	3	4	5
7	7	8	6	1	2	0	4	5	3
8	8	6	7	2	0	1	5	3	4

Points and Lines of PG(2,9)

i	P_i	ℓ_i									
1	(1,0,0)	2	11	20	29	38	47	56	65	74	83
2	(0,1,0)	1	11	12	13	14	15	16	17	18	19
3	(1,1,0)	4	11	22	30	44	55	63	68	9	87
4	(2,1,0)	3	11	21	31	41	51	61	71	81	91
5	(3,1,0)	9	11	27	34	40	53	60	66	82	86
6	(4,1,0)	6	11	24	37	45	49	59	70	80	84
7	(5,1,0)	8	11	26	32	46	52	58	69	75	90
8	(6,1,0)	7	11	25	36	39	50	64	67	78	89
9	(7,1,0)	5	11	23	35	42	54	57	73	76	88
10	(8,1,0)	10	11	28	33	43	48	62	72	77	85
11	(0,0,1)	1	2	3	4	5	6	7	8	9	10
12	(1,0,1)	2	13	22	31	40	49	58	67	76	85
13	(2,0,1)	2	12	21	30	39	48	57	66	75	84
14	(3,0,1)	2	18	27	36	45	54	63	72	81	90
15	(4,0,1)	2	15	24	33	42	51	60	69	78	87
16	(5,0,1)	2	17	26	35	44	53	62	71	80	89
17	(6,0,1)	2	16	25	34	43	52	61	70	79	88
18	(7,0,1)	2	14	23	32	41	50	59	68	77	86
19	(8,0,1)	2	19	28	37	46	55	64	73	82	91
20	(0,1,1)	1	29	30	31	32	33	34	35	36	37
21	(1,1,1)	4	13	21	29	46	54	62	70	78	86
22	(2,1,1)	3	12	22	29	42	52	59	72	82	89
23	(3,1,1)	9	18	25	29	44	51	58	73	77	84

24	(4,1,1)	6	15	28	29	40	50	63	71	75	88
25	(5,1,1)	8	17	23	29	43	49	64	66	81	87
26	(6,1,1)	7	16	27	29	41	55	57	69	80	85
27	(7,1,1)	5	14	26	29	45	48	60	67	79	91
28	(8,1,1)	10	19	24	29	39	53	61	68	76	90
29	(0,2,1)	1	20	21	22	23	24	25	26	27	28
30	(1,2,1)	3	13	20	30	43	50	60	73	80	90
31	(2,2,1)	4	12	20	31	45	53	64	69	77	88
32	(3,2,1)	7	18	20	34	46	48	59	71	76	87
33	(4,2,1)	10	15	20	37	44	52	57	67	81	86
34	(5,2,1)	5	17	20	32	39	51	63	70	82	85
35	(6,2,1)	9	16	20	36	42	49	62	68	75	91
36	(7,2,1)	8	14	20	35	40	55	61	72	78	84
37	(8,2,1)	6	19	20	33	41	54	58	66	79	89
38	(0,3,1)	1	74	75	76	77	78	79	80	81	82
39	(1,3,1)	8	13	28	34	45	51	57	68	74	89
40	(2,3,1)	5	12	24	36	43	55	58	71	74	86
41	(3,3,1)	4	18	26	37	42	50	61	66	74	85
42	(4,3,1)	9	15	22	35	41	48	64	70	74	90
43	(5,3,1)	10	17	25	30	40	54	59	69	74	91
44	(6,3,1)	3	16	23	33	46	53	63	67	74	84
45	(7,3,1)	6	14	27	31	39	52	62	73	74	87
46	(8,3,1)	7	19	21	32	44	49	60	72	74	88
47	(0,4,1)	1	47	48	49	50	51	52	53	54	55
48	(1,4,1)	10	13	27	32	42	47	64	71	79	84
49	(2,4,1)	6	12	25	35	46	47	60	68	81	85
50	(3,4,1)	8	18	24	30	41	47	62	67	82	88
51	(4,4,1)	4	15	23	34	39	47	58	72	80	91
52	(5,4,1)	7	17	22	33	45	47	61	73	75	86
53	(6,4,1)	5	16	28	31	44	47	59	66	78	90
54	(7,4,1)	9	14	21	37	43	47	63	69	76	89
55	(8,4,1)	3	19	26	36	40	47	57	70	77	87
56	(0,5,1)	1	65	66	67	68	69	70	71	72	73
57	(1,5,1)	9	13	26	33	39	55	59	65	81	88
58	(2,5,1)	7	12	23	37	40	51	62	65	79	90
59	(3,5,1)	6	18	22	32	43	53	57	65	78	91
60	(4,5,1)	5	15	27	30	46	49	61	65	77	89
61	(5,5,1)	4	17	28	36	41	52	60	65	76	84
62	(6,5,1)	10	16	21	35	45	50	58	65	82	87
63	(7,5,1)	3	14	24	34	44	54	64	65	75	85
64	(8,5,1)	8	19	25	31	42	48	63	65	80	86
65	(0,6,1)	1	56	57	58	59	60	61	62	63	64
66	(1,6,1)	5	13	25	37	41	53	56	72	75	87
67	(2,6,1)	8	12	27	33	44	50	56	70	76	91
68	(3,6,1)	3	18	28	35	39	49	56	69	79	86
69	(4,6,1)	7	15	26	31	43	54	56	68	82	84
70	(5,6,1)	6	17	21	34	42	55	56	67	77	90
71	(6,6,1)	4	16	24	32	40	48	56	73	81	89
72	(7,6,1)	10	14	22	36	46	51	56	66	80	88

73	(8,6,1)	9	19	23	30	45	52	56	71	78	85
74	(0,7,1)	1	38	39	40	41	42	43	44	45	46
75	(1,7,1)	7	13	24	35	38	52	63	66	77	91
76	(2,7,1)	9	12	28	32	38	54	61	67	80	87
77	(3,7,1)	10	18	23	31	38	55	60	70	75	89
78	(4,7,1)	8	15	21	36	38	53	59	73	79	85
79	(5,7,1)	3	17	27	37	38	48	58	68	78	88
80	(6,7,1)	6	16	26	30	38	51	64	72	76	86
81	(7,7,1)	4	14	25	33	38	49	57	71	82	90
82	(8,7,1)	5	19	22	34	38	50	62	69	81	84
83	(0,8,1)	1	83	84	85	86	87	88	89	90	91
84	(1,8,1)	6	13	23	36	44	48	61	69	82	83
85	(2,8,1)	10	12	26	34	41	49	63	73	78	83
86	(3,8,1)	5	18	21	33	40	52	64	68	80	83
87	(4,8,1)	3	15	25	32	45	55	62	66	76	83
88	(5,8,1)	9	17	24	31	46	50	57	72	79	83
89	(6,8,1)	8	16	22	37	39	54	60	71	77	83
90	(7,8,1)	7	14	28	30	42	53	58	70	81	83
91	(8,8,1)	4	19	27	35	43	51	59	67	75	83

تصنيف وبناء الاقواس (k,3) في مستوي إسقاطي حول حقل كالوا GF(9)

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الخلاصة

يتم في هذا البحث، بناء وتصنيف الاقواس (k,3) الاسقاطية المختلفة في المستوي الاسقاطي PG(2,9) حيث أن $k \geq 5$، وبرهان ان الاقواس – (k,3) الكاملة غير موجودة، حيث أن $3 \leq k \leq 5$ وقد وجد أن اكبر قوس – (k,3) كامل في PG(2,9) هو القوس – (16,3) وان اصغر قوس – (k,3) كامل القوس –(14,3) .
علاوة على ذلك، تم ايجاد الاقواس – (k,3) الكاملة بينهما.

الكلمات المفتاحية : أقواس ،مستوي اسقاطي ،حقل كالوا