On Solution of Nonlinear Singular Boundary Value Problem

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Received in : 30 September 2012 , Accepted in : 3 February 2013

Abstract

This paper is devoted to the analysis of nonlinear singular boundary value problems for ordinary differential equations with a singularity of the different kind. We propose semi - analytic technique using two point osculatory interpolation to construct polynomial solution, and discussion behavior of the solution in the neighborhood of the singular points and its numerical approximation. Two examples are presented to demonstrate the applicability and efficiency of the methods**.** Finally, we discuss behavior of the solution in the neighborhood of the singularity point which appears to perform satisfactorily for singular problems.
Kay ward : Singular boundary value problems, ODE, BVP.

1. Introduction

In the study of nonlinear phenomena in physics, engineering and other sciences, many mathematical models lead to singular two-point boundary value problems (SBVP) associated with nonlinear second order ordinary differential equations (ODE).

In mathematics, a singularity is in general a point at which a given mathematical object is not defined, or a point of an exceptional set where it fails to be well-behaved in some particular way, such as many problems in varied fields as thermodynamics, electrostatics, physics, and statistics give rise to ordinary differential equations of the form :

$$y'' = f(x, y, y') \quad , \quad a < x < b \quad , \qquad (1)$$

On some interval of the real line with some boundary conditions(BC).

A two-point BVP associated to the second order differential equation (1) is singular if one of the following situations occurs:

a and/or b are infinite; f is unbounded at some $x_0 \in [0,1]$ or f is unbounded at some particular value of y or y′ [1] .

There are two types of a point $x_0 \in [0,1]$: Ordinary Point and Singular Point. Also, there are two types of Singular Point : Regular and Irregular Points, A function y(x) is analytic at x_0 if it has a power series expansion at x_0 that converges to y(x) on an open interval containing x_0. A point x_0 is an ordinary point of the ODE (2), if the functions P(x) and Q(x) are analytic at x_0. Otherwise x_0 is a singular point of the ODE, i.e.

$$y'' + P(x)y' + Q(x)y = 0 \quad , \qquad (2)$$

$$P(x) = P_0 + P_1(x-x_0) + P_2(x-x_0)^2 + \ldots\ldots = \sum_{i=0}^{\infty} P_i (x - x_0)^i \quad , \qquad (3)$$

$$Q(x) = Q_0 + Q_1(x-x_0) + Q_2(x-x_0)^2 + \ldots\ldots = \sum_{i=0}^{\infty} Q_i (x - x_0)^i \quad , \qquad (4)$$

On the other hand, if P(x) or Q(x) are not analytic at x_0 then x_0 is said to be a singular [2].

There are four kinds of singularities :

- The first kind is the singularity at one of the ends of the interval [0,1] ;
- The second kind is the singularity at both ends of the interval [0,1]
- The third kind is the case of a singularity in the interior of the interval;
- The forth and final kind is simply treating the case of a regular differential equation on an infinite interval.

In this paper, we focus on the first kind.

2. Solution of Second Order Nonlinear SBVP

In this section , we suggest semi analytic technique to solve second order nonlinear SBVP as following, we consider the SBVP :

$$x^m y'' + f(x, y, y') = 0 \quad , \qquad (5a)$$

$$g_i(y(0), y(1), y'(0), y'(1)) = 0 \,, \quad i = 1, 2, m \in N \quad , \qquad (5b)$$

where f , g_1, g_2 are in general nonlinear functions of their arguments .

The simple idea behind the use of two-point polynomials is to replace y(x) in problem (5), or an alternative formulation of it, by a P_{2n+1} which enables any unknown boundary values or derivatives of y(x) to be computed .

Therefore, the first step is to construct the P_{2n+1} and to do so, we need the Taylor coefficients of y (x) at x = 0 :

$$y = a_0 + a_1 x + \sum_{i=2}^{\infty} a_i x^i \quad , \qquad (6)$$

where $y(0)= a_0$, $y'(0)= a_1$, $y''(0) / 2! = a_2$,…, $y^{(i)}(0) / i! = a_i$, i= 3, 4,…
then insert the series forms (6) into (5a) and equate coefficients of powers of x to obtain a_2 . Also we need Taylor coefficient of y(x) about x = 1:

$$y = b_0 + b_1 (x-1) + \sum_{i=2}^{\infty} b_i (x-1)^i \quad , \qquad (7)$$

where $y(1) = b_0$, $y'(1) = b_1$, $y''(1) / 2! = b_2$, … , $y^{(i)}(1) / i! = b_i$, i = 3,4,…
then insert the series form (7) into (5a) and equate coefficients of powers of (x-1) to obtain b_2, then derive equation (5a) with respect to x and iterate the above process to obtain a_3 and b_3 , now iterate the above process many times to obtain a_4, b_4, then a_5, b_5 and so on, that is, we can get a_i and b_i for all $i \geq 2$(the resulting equations can be solved using MATLAB to obtain a_i and b_i for all $i \geq 2$), the notation implies that the coefficients depend only on the indicated unknowns a_0, a_1, b_0, b_1, we get two of these four unknown by the boundary condition. Now, we can construct a $P_{2n+1}(x)$ from these coefficients (a_is and b_is) by the following :

$$P_{2n+1} = \sum_{i=0}^{n} \{ a_i Q_i(x) + (-1)^i b_i Q_i(1-x) \} \quad , \qquad (8)$$

where $(x^j / j!)(1-x)^{n+1} \sum_{s=0}^{n-j} \binom{n+s}{s} x^s = Q_j(x) / j!$

it can be seen that (8) have only two unknowns from a_0, b_0, a_1 and b_1 to find this, we integrate equation (5a) on [0, x] to obtain :

$$x^m y'(x) - m x^{m-1} y(x) + m(m-1)\int_0^x x^{m-2} y(x)\,dx + \int_0^x f(x, y, y')\,dx = 0 \quad , \quad (9a)$$

and again integrate equation (9a) on [0, x] to obtain:

$$x^m y(x) - 2m\int_0^x x^{m-1} y(x)\,dx + m(m-1)\int_0^x (1-x)x^{m-2} y(x)dx + \int_0^x (1-x)f(x,y,y') = 0 \, , \quad (9b)$$

Putting x = 1 in (9) then gives :

$$b_1 - m b_0 + m(m-1)\int_0^1 x^{m-2} y(x)\,dx + \int_0^1 f(x, y, y')\,dx = 0 \quad , \qquad (10a)$$

and

$$b_0 - 2m\int_0^1 x^{m-1} y(x)dx + m(m-1)\int_0^1 (1-x)x^{m-2} y(x)\,dx + \int_0^1 (1-x)f(x, y, y')dx = 0 \, , \quad (10b)$$

Use P_{2n+1} as a replacement of y(x) in (10) and substitute the boundary conditions (5b) in (10) then, we have only two unknown coefficients b_1, b_0 and two equations (10) so, we can find b_1, b_0 for any n by solving this system of algebraic equations using MATLAB, so insert b_0 and b_1 into (8), thus (8) represents the solution of (5).

Extensive computations have shown that this generally provides a more accurate polynomial representation for a given n .

3. Examples

In this section, we introduce two examples of second order SBVP, non homogenous, nonlinear ordinary differential equations with two point BC to assess the performance of the proposed method.

Example 1 : Emden's equation

Emden's equation arises in modeling a spherical body of gas. The PDE of the model is reduced by symmetry to the ODE :

$$y'' + \frac{2}{x}y' + y^5 = 0$$

$x \in [0, 1]$, the coefficient 2/x is singular at x = 0, but symmetry implies the BC: y'(0) = 0. With this boundary condition, the term (2/x)y'(0) is well-defined as x approaches 0. For the boundary condition , $y(1) = \frac{\sqrt{3}}{2}$, this nonlinear SBVP has the analytical solution :

$$y(x) = \left(1 + \frac{x^2}{3}\right)^{-1/2}$$

We solve this problem by using semi-analytic technique by the following; from equation (8), we have:

$P_9 = -0.0005654571561x^9 + 0.001454732112x^8 + 0.00321137204x^7 - 0.01349180792\ x^6 + 0.0004165633765x^5 + 0.04166666759x^4 - 0.1666666687x^2 + 1.000000002$

Now, increase n, to get higher accuracy ,let n = 5, 6,respectively ,i.e.,

$P_{11} = 0.000030043522x^{11} + 0.0002663457696x^{10} - 0.00234783883x^9 + 0.00531993067\ x^8 - 0.0008040441181x^7 - 0.01143903456x^6 + 0.04166666759x^4 - 0.1666666687x^2 + 1.000000002$

$P_{13} = 0.00002406664878\ x^{13} - 0.0002794397776\ x^{12} + 0.001201282562\ x^{11} - 0.00224058548x^{10} + 0.0007139586051x^9 + 0.003149932497x^8 + 0.00003026184015x^7 - 0.01157407444x^6 + 0.04166666759x^4 - 0.1666666687x^2 + 1.000000002$

The standard bvp4c syntax was implemented in [3] to solve the current problem, the example evaluates the analytical solution at 100 equally-spaced points and plots it along with the numerical solution computed using bvp4c and the results given in Table 1 with the results of suggested method for n = 4, 5, 6, i.e., P_5, P_7, P_9 and the results of [3] . Also Table 2 gives the accuracy of suggested method and result of [3]. Figures 1, 2, 3, illustrate Emden problem, and suggested method for n = 4, 6 respectively .

It can be seen from Table2 that the maximum error of method in [3] is 0.0000018522, while the maximum error of suggested method is $2.763537509942182e^{-009}$

Example 2

Consider the following nonlinear SBVP :

$$y'' + \left(\frac{1}{x}+1\right) y' = \frac{5x^3(5x^5e^y - x - b - 4)}{4 + x^5} \quad , \ b \in R , \quad 0 \le x \le 1$$

with BC : y'(0) = 0 , y'(1) = -1. The exact solution for this problem is:

$y = -\ln(x^5 + 4)$

This problem is an application of oxygen diffusion , we solve this problem by suggested method and we take b=1, by applying equation (8) we have (for n = 7) :

$P_{15} = -0.1142318896\ x^{15} + 0.7852891723\ x^{14} - 2.23317271\ x^{13} + 3.413936838\ x^{12} - 3.087794733x^{11} + 1.697854523x^{10} - 0.4991720457x^9 + 0.06414729306x^8 - 0.25x^5 - 1.386294361.$

For more details ,table(3) gives the results for different nodes in the domain, for n = 7 and figures (4) illustrate suggested method for n = 7.

Abukhaled et al in [4] applying L'Hopital's rule to overcome the singularity at x = 0 and then the modified spline approach is used and got maximum error $7.79e^{-4}$ and resolution this problem using finite difference method then gives maximum error $1.46e^{-3}$, from Table 3 we have maximum error $9.399395723974635e^{-007}$ Therefore, the proposed method provides superiority results.

4. Behavior of the solution in the neighborhood of the singularity x= 0

Our main concern in this section will be to study the behavior of the solution in the neighborhood of singular point .

Consider the following SIVP :

$$y''(x) + ((N-1)/x)\, y'(x) = f(y) \ , N \ge 1 , \quad 0 < x < 1 , \qquad (11)$$

$$y(0) = y_0 \ , \quad \lim_{x\to 0^+} x\, y'(x) = 0 \qquad , \qquad (12)$$

where f(y) is continuous function .

As the same manner in [5], let us look for a solution of this problem in the form :

$$y(x) = y_0 - C x^k (1 + o(1)) \quad , \quad (13)$$
$$y'(x) = - C k x^{k-1}(1 + o(1)) \quad ,$$
$$y''(x) = - C k (k - 1) x^{k-2}(1 + o(1)) , \quad x \to 0^+$$

where C is a positive constant and k > 1. If we substitute (13) in (11) we obtain :

$$C = (1/ k) (f(y_0) /N)^{k-1} \quad , \quad (14)$$

In order to improve representation (13) we perform the variable substitution :

$$y(x) = y_0 - C x^k (1 + g(x)) \quad , \quad (15)$$

we easily obtain the following result which is similar to the results in [5].

Theorem 2

For each $y_0 > 0$, problem (11), (12) has a unique solution in the neighborhood of x = 0 that can be represented by:

$$y(x, y_0) = y_0 - C x^k (1 + g x^k + o(x^k)) \ ,$$

where k, C and g are given by (14) and (15), respectively.

We see that these results are in good agreement with the ones obtained by the method in [5], they are also consistent with the results presented in [4]. In order to estimate the convergence order of the suggested method at x = 0, we have carried out several experiments with different values of n and used the formula :

$$c_{y0} = - \log_2 (|y_0^{n3} - y_0^{n2}| / |y_0^{n2} - y_0^{n1} |) \quad , \quad (16)$$

where y_0^{ni} is the approximate value of y_0 obtained with n_i = 1,2, 3, 4,…

Now, we apply the formula in equation (16) to the example 1, as following :

Let y_{0i} is the approximate value of y_0 evaluated by suggested method with n = i, i = 2, 3, 4, 5 ,6 .

P_{2n+1}	y_{0i}
P_5	1.000048243794667
P_7	1.000009487495905
P_9	1.000000801904571
P_{11}	1.0000000507769192
P_{13}	1.0000000024600459

First ,we take i= 2, 3, 4, i.e., $C_{y0} = -\log_2 \frac{| y_{04} - y_{03} |}{| y_{03} - y_{02} |}$

$$C_{y0} = -\log_2 \frac{8.68551333934539e^{-006}}{3.875629876204378e^{-005}} = 2.157734820323959 .$$

The value of C_{y0} illustrates that the convergence order estimate of this case is close to 2. Now, if we take i= 3, 4, 5, i.e.,

$$C_{y0} = -\log_2 \frac{| y_{05} - y_{04} |}{| y_{04} - y_{03} |} = -\log_2 \frac{7.511276518545884e^{-007}}{8.68551333934539e^{-006}} = 3.531494058780649.$$

The value of C_{y0} illustrates that the convergence order estimate of this case is close to 4. Now if we take i= 4, 5, 6 , i.e.,

$$C_{y0} = -\log_2 \frac{| y_{06} - y_{05} |}{| y_{05} - y_{04} |} = -\log_2 \frac{4.831687316908528e^{-008}}{7.5112765185645884e^{-007}}, \ =3.958459111546360$$

The value of C_{y0} illustrate that the convergence order estimate of this case is close to 4, and so on, if we increase n, we see that the order of convergence also increases.

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Table 1:The result of the method for n = 4,5,6, and result in [3] for Example1

x_i	exact solution y(x)	$y_1(x)$ using numerical solution	Osculatory interpolation P_9	Osculatory interpolation P_{11}	Osculatory interpolation P_{13}
0.000	1.000000000000000	1.0000005109	1.000000801904571	1.000000050776919	1.000000002460046
0.125	0.997405961908059	0.9974064705	0.997406758716985	0.997406012221857	0.997405964341573
0.250	0.989743318610787	0.9897438277	0.989744154146833	0.989743371348405	0.989743321151803
0.500	0.960768922830523	0.9607707751	0.960769739668355	0.960768977298348	0.960768925594060
0.750	0.917662935482247	0.9176644937	0.917663076945894	0.917662942247223	0.917662935723868
1.000	0.866025403784439	0.8660254037	0.866025403784439	0.866025403784439	0.866025403784439

Table 2:The accuracy of the method for n= 6 ,i.e, P_{13} and result in [3] for Example1

x_i	exact solution y(x)	Error $\|y(x) - y_1(x)\|$	Error $\|y(x) - P_{13}\|$
0.000	1.000000000000000	0.0000005109	2.460045944729927e-009
0.125	0.997405961908059	0.0000005086	2.433513945909738e-009
0.250	0.989743318610787	0.0000005091	2.541016064228074e-009
0.500	0.960768922830523	0.0000018522	2.763537509942182e-009
0.750	0.917662935482247	0.0000015582	2.416213895628516e-010
1.000	0.866025403784439	0.0000000000	0.0000000000
			S.S.E=2.612609927853621e-017

Table 3 :The exact and suggested solution for n = 7 of Example 2

		P_{15}	
a_o		-1.386294361119891	
b_o		-1.6094379124341	
	exact solution y(x)	P_{15}	Errors $\|y(x) - P_{15}\|$
0	-1.386294361119891	-1.386294361119891	4.440892098500626e-016
0.1	-1.386296861116765	-1.386296860835485	2.812807764485115e-010
0.2	-1.386374357920061	-1.386374329577653	2.834240797611187e-008

0.3	-1.386901676666466	-1.386901427706747	$2.489597183963355e^{-007}$
0.4	-1.388851089901581	-1.388850379927776	$7.099738041915771e^{-007}$
0.5	-1.394076501561946	-1.394075561622373	$9.399395723974635e^{-007}$
0.6	-1.405547818041777	-1.405547194355350	$6.23686427614345e^{-007}$
0.7	-1.427453098935757	-1.427452912675769	$1.862599887658689e^{-007}$
0.8	-1.465031601657275	-1.465031584864205	$1.679306937951708e^{-008}$
0.9	-1.523986772187307	-1.523986772073695	$1.136124527789661e^{-010}$
1	-1.609437912434100	-1.609437912434100	$2.220446049250313e^{-016}$
S.S.E = 1.874293078482109e-012			

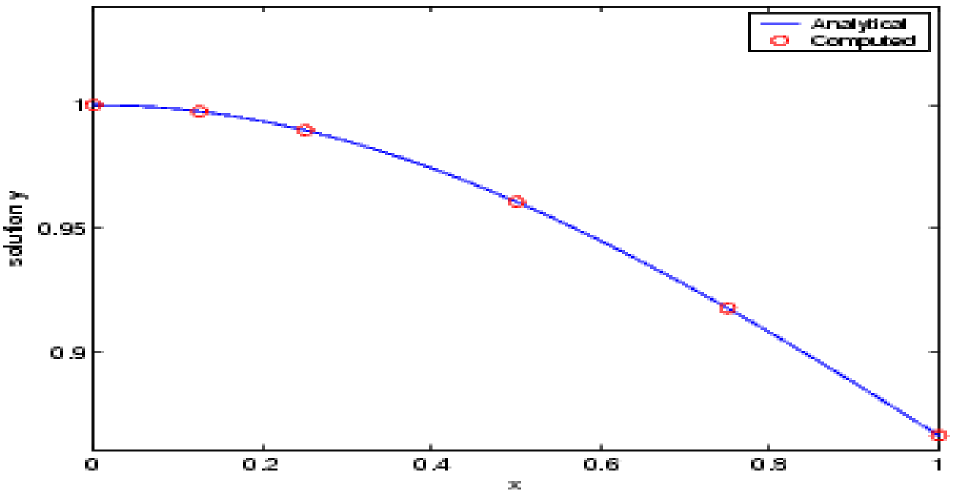

Figure 1: Emden problem – SBVP

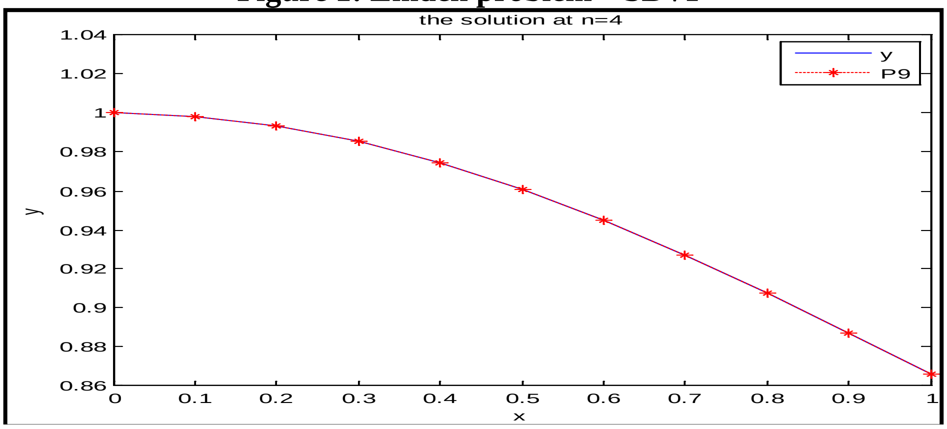

Figure 2: Comparison between the exact and semi-analytic solution P_9 of example1

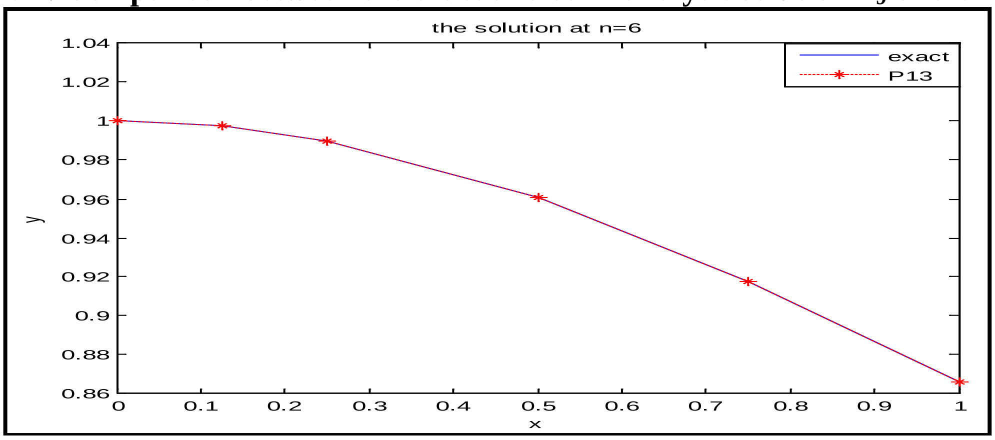

Figure 3 :Comparison between the exact and semi-analytic solution P_{13} of example1

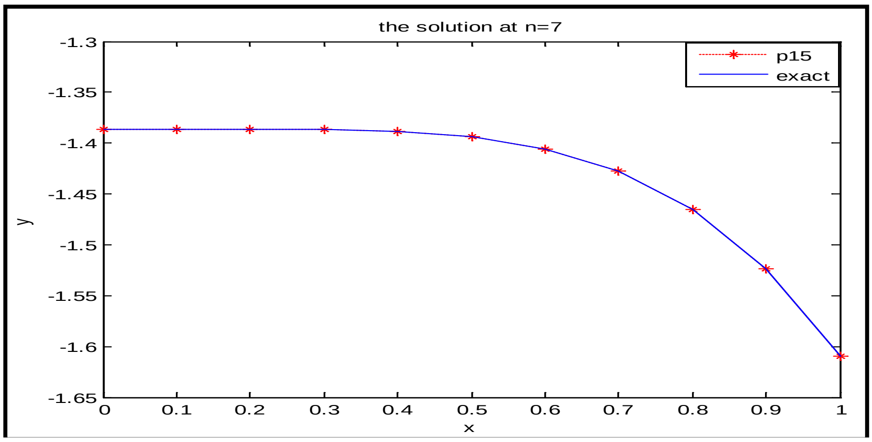

Figure 4:Comparison between the exact and suggested solution P_{15} of example2

حول حل مسائل القيم الحدودية الشاذة غير الخطية

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أستلم البحث في : 30 أيلول 2012 ، قبل في 3 شباط 2013

المستخلص

الهدف من هذا البحث عرض دراسة تحليلية لمسائل القيم الحدودية الشاذة غير الخطية للمعادلات التفاضلية الاعتيادية وبأنواع مختلفة إذ إننا نقترح التقنية شبه التحليلية باستخدام الاندراج التماسي ذي النقطتين للحصول على الحل كمتعددة حدود، كذلك ناقشنا مثالين لتوضيح الدقة ،والكفاية وسهولة أداء الطريقة المقترحة و أخيرا ناقشنا سلوك الحل في جوار النقاط الشاذة، وإيجاد الحل التقريبي لها بشكل مرضي فيما يخص المسائل الشاذة .