

Efficient Semi-Analytic Technique for Solving Nonlinear Singular Initial Value Problems

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Abstract

The aim of this paper is to present a semi - analytic technique for solving singular initial value problems of ordinary differential equations with a singularity of different kinds to construct polynomial solution using two point osculatory interpolation.

The efficiency and accuracy of suggested method is assessed by comparisons with exact and other approximate solutions for a wide classes of non-homogeneous, non-linear singular initial value problems.

A new, efficient estimate of the global error is used for adaptive mesh selection. Also, analyze some of the numerical aspects relevant for the implementation, describe measures to increase the efficiency of the code and compare its performance with the performance of established standard codes for singular initial value problems.

Many examples are presented to demonstrate the applicability and efficiency of the suggested method on one hand and to confirm the convergence order on the other hand.

Kay ward : ODE, Singular initial value problems .

1. Introduction

The modeliy of nonlinear phenomena in physics, engineering and other sciences, leads to singular initial value problems (SIVP) associated with nonlinear second order ordinary differential equations (ODE) [1].

The general form of the SIVP is :

$$x^m y'' + f(x, y, y') = 0 \quad , \quad 0 < x < 1 \quad , \quad (1)$$

with some initial conditions.

The object is to find a function y that satisfies the differential equation together with IC. We view f as a generally nonlinear function of y and y' , but here it wins, we as look $f = f(x)$ only. For such a problem to have a solution, it is generally necessary either that $f(x) \neq 0$ hold. When $f(x) \equiv 0$, and/or $A = 0$, $B = 0$ the IVP is said to be homogeneous and will in general have only the trivial solution, $y(x) \equiv 0$ [2].

2. Solution of Second Order SIVP

In this section, we suggest a semi analytic technique which is based on osculatory interpolating polynomials P_{2n+1} and Taylor series expansion to solve second order SIVP, we consider the problem (1), where f are in general nonlinear functions of their arguments .

Now, to solve the problem by suggested method we do the following steps:

Step one :-

Evaluate Taylor series of $y(x)$ about $x = 0$:

$$y = a_0 + a_1 x + \sum_{i=2}^{\infty} a_i x^i \quad (2)$$

where $y(0) = a_0$, $y'(0) = a_1$, $y''(0) / 2! = a_2, \dots, y^{(i)}(0) / i! = a_i$, $i = 3, 4, \dots$

And evaluate Taylor series of $y(x)$ about $x = 1$:

$$y = b_0 + b_1(x-1) + \sum_{i=2}^{\infty} b_i(x-1)^i \quad (3)$$

where $y(1) = b_0$, $y'(1) = b_1$, $y''(1) / 2! = b_2, \dots, y^{(i)}(1) / i! = b_i$, $i = 3, 4, \dots$

Step two :-

Insert the series form (3) into equation (1) and put $x = 1$, then equate the coefficients of powers of $(x-1)$ to obtain b_2 .

Step three :-

Derive equation(1) with respect to x , to get new form of equation say (4)

$$x^m y''' + m x^{m-1} y'' + df(x, y, y') / dx = 0 \quad (4)$$

then, insert the series form (2) into equation (4) and put $x = 0$ and equate the coefficients of powers of x to obtain a_2 , again insert the series form (3) into equation(4) and put $x = 1$, then equate the coefficients of powers of $(x-1)$ to obtain b_3 .

Step four :-

Iterate the above process many times to obtain a_3, b_4 , then a_4, b_5 and so on, that is, to get a_i and b_i for all $i \geq 2$, the resulting equations can be solved using MATLAB version 7.10, to obtain a_i and b_i for all $i \geq 2$.

Step five :-

The notation implies that the coefficients depend only on the indicated unknowns a_0, a_1, b_0, b_1 , and use the IC to get a_0, a_1 , therefore we have only two unknown coefficients b_0 and b_1 . Now, we can construct two point osculatory interpolating polynomials $P_{2n+1}(x)$ from these coefficients (a_i 's and b_i 's) by the following :

$$P_{2n+1} = \sum_{i=0}^n \{ a_i Q_i(x) + (-1)^i b_i Q_i(1-x) \} \quad (5)$$

$$\text{where } Q_j(x) / j! = (x^j / j!) (1-x)^{n+1} \sum_{s=0}^{n-j} \binom{n+s}{s} x^s$$

Step six :-

To find the unknowns coefficients b_0 and b_1 , integrate equation (1) on $[0, x]$ to obtain :

$$x^m y'(x) - mx^{m-1}y(x) + m(m-1) \int_0^x x^{m-2} y(x) dx + \int_0^x f(x, y, y') dx = 0 \quad (6a)$$

and again integrate equation (6a) on $[0, x]$ to obtain :

$$x^m y(x) - 2m \int_0^x x^{m-1} y(x) dx + m(m-1) \int_0^x (1-x)x^{m-2} y(x) dx + \int_0^x (1-x)f(x, y, y') dx = 0 \quad (6b)$$

Step seven :-

Putting $x = 1$ in equations (6) to get :

$$b_1 - mb_0 + m(m-1) \int_0^1 x^{m-2} y(x) dx + \int_0^1 f(x, y, y') dx = 0 \quad (7a)$$

and

$$b_0 - 2m \int_0^1 x^{m-1} y(x) dx + m(m-1) \int_0^1 (1-x)x^{m-2} y(x) dx + \int_0^1 (1-x)f(x, y, y') dx = 0 \quad (7b)$$

Step eight :-

Use $P_{2n+1}(x)$ which defined in equation (5) as a replacement of $y(x)$, we see that equations (7) have only two unknown coefficients b_0 and b_1 . So, we can find b_0 and b_1 by solving the system of algebraic equations (7) using MATLAB, so insert the value of b_0 and b_1 into equation (5), thus equation (5) represents the solution of the problem.

Note

Extensive computations have shown that this generally provides a more accurate polynomial representation for a given n .

3. Examples

In this section, many examples will be given to illustrate the efficiency, accuracy, implementation and utility of the suggested method. The `bvp4c` solver of MATLAB has been modified accordingly so that it can solve some class of SIVP as effectively as it previously solved nonsingular IVP.

Example 1

Consider the following 2nd order nonlinear SIVP :

$$y'' + (4/x) y' + y^2 = x^6 + 4x^3 + 18x + 4$$

with IC : $y(0) = 2$, $y'(0) = 0$

It is clear that $x = 0$, is singular point and it is singularity of first kind and the exact solution is $y(x) = x^3 + 2$. Now, this example is solved by using semi-analytic technique, from equation (5) we have : $P_3 = x^3 + 2$

Example 2

Consider the following 2nd order nonlinear SIVP :

$$y'' + (6/x) y' + (6/x^2) y + y^2 = 20 + x^4$$

with IC : $y(0) = 0$, $y'(0) = 0$

It is clear that $x = 0$, is singular point and the exact solution is $y(x) = x^2$. Now, this example is solved by using semi-analytic technique, from equation(5) we have : $P_3 = x^2$

Now, the suggested method is used to investigate the solution of Emden–Fowler equations as a class of nonlinear SIVP, which plays very important role because of the singularity behaviour at the origin. We study the Emden–Fowler equations of the form :

$$y'' + (a/x) y' + b x^{m-1} y^n = g(x) ; \quad 0 \leq x \leq 1 , a \geq 0 , b \in \mathbb{R} \quad (8)$$

under the following initial conditions : $y(0) = A$, $y'(0) = 0$

where A, a, b, m are constants, $n \neq 0, n \neq 1$. When $m = 1, a = 2$ and $b = 1$ equation (8) reduces to the Lane – Emden equation [3].

Example 3

Consider the following Emden – Fowler equation as a SIVP :

$$y'' + (8/x) y' + x y^2 = x^5 + x^4 ; \quad 0 \leq x \leq 1$$

with IC : $y(0) = 1$, $y'(0) = 0$

It is clear that $x = 0$, is nonlinear singular point of the first kind, we solve this example by using semi-analytic technique , from equation (5), we have :

$$P_9 = -0.00078826882 x^9 + 0.0012952791 x^8 + 0.00879760172 x^7 + 0.014414828109 x^6 - 0.000150192x^5 - 0.03333333333333337578 x^3 + 1.0$$

higher accuracy can be obtained by evaluating higher n , now, take $n = 5$, i.e.,

$$P_{11} = -0.00021822695x^{11} + 0.00086427359x^{10} - 0.00189361335x^9 + 0.0014538812 x + 0.009581210401 x^7 + 0.0137816833 x^6 - 0.03333333333333337578 x^3 + 1.0$$

Now, increase n , to get higher accuracy , let $n = 6$, i.e. ,

$$P_{13} = -0.0000996316x^{13} + 0.0004488478x^{12} - 0.000849659317x^{11} + 0.0007848859x^{10} - 0.0007538654x^9 + 0.0001852901x^8 + 0.01017813169x^7 + 0.01367521368x^6 - 0.033333333333333375776x^3 + 1.0$$

Demir[3] solved this example by DTM and gave the following solution:

$$y_{20}(x) = 1 - \frac{1}{30}x^3 + \frac{8}{585}x^6 + \frac{1}{98}x^7 - \frac{37}{187200}x^9 - \frac{1}{8330}x^{10} + \frac{367}{64022400}x^{12} + \frac{23}{6497400}x^{13} - \frac{55049}{85830030000}x^{15} - \frac{2867}{3586564800}x^{16} - \frac{1}{3918432}x^{17} + \frac{1618343}{102996036000000}x^{18} + \frac{164117}{8858815056000}x^{19} + \frac{29}{5289883200}x^{20}$$

Also, this example is solved by numerical method by using the Maple11 as given in [3] and the results are compared with the suggested method and its results are given in table (1a) and (1b). Figure (1) give a comparison between the numerical solution in[3] and P_{13} ; table (2) gave the results obtained with DTM for different values of n which are given [3].

4. Error / Defect Weights

Every known IVP software package reports an estimate of either the relative error or the maximum relative defect. The weights used to scale either the error or the maximum defect differ among IVP software. Therefore, the IVP component of pythODE [4] allows users to select the weights they wish to use. The default weights depend on whether an estimate of the error or maximum defect is being used. If the error is being estimated, then the IVP component of pythODE uses, in this thesis we modify this package to consist SIVP and named "pythSODE", defined as:

$$\|y(x) - P(x)\|_{\infty} / (1 + \|P(x)\|_{\infty}) \quad ; \quad 0 \leq x \leq 1 \quad , \quad (9)$$

where $y(x)$ is exact solution and $P(x)$ is suggested solution of SIVP (1).

If the maximum defect is being estimated, then the SIVP component of pythSODE uses:

$$\|P''(x) - f(x, P(x), P'(x))\|_{\infty} / (1 + \|f(x, P(x), P'(x))\|_{\infty}) \quad ; \quad 0 \leq x \leq 1 \quad , \quad (10)$$

The relative estimate of both the error and the maximum defect are slightly modified from the one used in SIVP SOLVER.

Again applied equation (10) for example 3 is as following:

Consider the nonlinear SIVP: $y'' + (8/x)y' + xy^2 = x^5 + x^4$
with IC: $y(0) = 1$, $y'(0) = 0$, apply equation (10) as the following:

Then equation (10) became: (for more detail see table 3)

$$\|P_{13}''(x) - f(x, P_{13}(x), P_{13}'(x))\|_{\infty} / (1 + \|f(x, P_{13}(x), P_{13}'(x))\|_{\infty}) =$$

$$1.683960224221478e-007 / 1.999966681360863 = 8.419941391601780e-008$$

5. Accuracy of The Computed Values

The convergence orders of the approximations appear to be very stable, indicating that our results are reliable and accurate. In this section, we try to quantify the accuracy we can obtain. To examine the error of the approximation, let

$$E(n) = \max_{1 \leq i \leq N} |y(x_i) - y_i| \quad (11)$$

where $y(x_i)$ is the exact solution at the x_i and y_i suggested solution at x_i . It is known that for a large N , if M is the order of convergence of the method, then :

$$E(n) \approx K |n|^M \quad (12)$$

where K is the error constant and $E(n)$ is the max error . Following the generalized algorithm for the order verification of methods given in [5], we can find M and K from the line :

$$y = M x + \log K \quad (13)$$

that best fits the equation :

$$\log E(n) = \log K + M \log n \quad (14)$$

where y represent $\log E(n)$ and x represent $\log n$.

Applying this algorithm for the following example :

$$y'' + (2/x) y' - 6y = -4y \ln y, \text{ with the IC: } y(0) = 1, y'(0) = 0$$

the exact solution is $y(x) = \exp(x-2)$, we have from table (4) the best fit line whose graph is shown in figure (2) ,i.e.,

$$y = -6.6214 x - 1.3330$$

From equation (13), we conclude that the order of convergence $|M| \approx 6$ with error constant : $K = 10^{1.3330}$.

6. Conclusions

Our decision to use semi-analytic technique for the solution of the underlying initial value problem was motivated by its advantageous convergence properties , whereas in presence of a singularity other direct higher order methods (finite difference) show order reductions and become inefficient. Moreover, the standard acceleration techniques like Iterated .

Our reason for choosing the *global* error estimate instead of monitoring the local error is the non-smoothness of the latter near the singular point and the order reductions it suffers from.

The results show that the convergence order estimates are increased, if we increase n . This is not surprising.

References

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Table No.(1a):The result of suggested method for n =2, 3, 4& method in[3] for example 3

b_0		0.990235914910346	0.990235874852203	0.990235879509409
b_1		0.0505890348752984	0.050589341476757	0.050589305215627
x_i	Numerical solution (y)	P_9	P_{11}	P_{13}
0	1	1	1	1
0.1	0.999966680000000	0.999966680471501	.999966681419201	0.999966681360863.
0.2	0.999734330000000	0.999734323342549	0.999734340836996	0.999734338980186
0.3	0.999112190000000	0.999112136946651	0.999112205091858	0.999112195858900
0.4	0.997939330000000	0.997939228062444	0.997939352185325	0.997939333529043
0.5	0.996126220000000	0.996126122885822	0.996126243561995	0.996126225634366
0.6	0.993720970000000	0.993720947623996	0.993720981359821	0.993720978536793
0.7	0.991004630000000	0.991004695132483	0.991004619559273	0.991004633063186
0.8	0.988619280000000	0.988619382084590	0.988619263799973	0.988619280474356
0.9	0.987723192000000	0.987731993184585	0.987731909996439	0.987731920033536
1	0.990235880000000	0.990235914910346	0.990235874852203	0.990235879509409

Table No.(1b):Comparison between suggested method with method in[3] for example 3

x_i	Error $ y-P_9 $	Error $ y-P_{11} $	Error $ y-P_{13} $
0	0	0	0
0.1	4.715006163279999e-010	1.419200645891300e-009	1.360863421950850e-009
0.2	6.65745103487000e-009	1.083699618931180e-008	8.980186372298251e-009
0.3	5.305334893623799e-008	1.509185754144940e-008	5.858900142818871e-009
0.4	1.019375558408700e-007	2.218532491049530e-008	3.529043057781680e-009
0.5	9.711417825464299e-008	2.356199502173690e-008	5.634365973783420e-009
0.6	2.237600371657300e-008	1.135982052868910e-008	8.536792939217721e-009
0.7	6.513248329564900e-008	1.044072739375680e-008	3.063186371221600e-009
0.8	1.020845900034930e-007	1.620002665347900e-008	4.743559989250900e-010
0.9	7.318458528082800e-008	1.000356053548760e-008	3.353606281564000e-011
1	3.491034628044800e-008	5.147797188520500e-009	4.905914563480000e-010
SSE	4.442053594825927-014	2.021641245529962e-015	2.437493949383256e-016

Table No.(2): Comparison of numerical results with the Maple11&DTM solution for different values of n

x	Maple11 Solution	$n = 35$	$n = 30$	$n = 20$	$n = 10$
0.1	0.99996668	0.99996668	0.99996668	0.99996668	0.99996668
0.2	0.99973433	0.99973433	0.99973433	0.99973433	0.99973433
0.3	0.99911219	0.99911219	0.99911219	0.99911219	0.99911219
0.4	0.99793933	0.99793933	0.99793933	0.99793933	0.99793933
0.5	0.99612622	0.99612622	0.99612622	0.99612622	0.99612622
0.6	0.99372097	0.99372097	0.99372097	0.99372097	0.99372096
0.7	0.99100463	0.99100463	0.99100463	0.99100463	0.99100452
0.8	0.98861928	0.98861927	0.98861927	0.98861927	0.98861874
0.9	0.98773192	0.98773191	0.98773191	0.98773191	0.98772971
1.0	0.99023588	0.99023587	0.99023587	0.99023588	0.99022826

Table No. (3) : Max. error of example 3

b_0	0.990235879509409	
b_1	0.050589305215627	
x_i	P_{13}''	Error $P_{13}'' - f(x, P_{13}(x), P_{13}'(x))$
0.1	-0.019954693976617	9.826181241272956e-009
0.2	-0.039206687602793	9.851733395474671e-008
0.3	-0.055639331969225	1.564345449579437e-007
0.4	-0.065139098987678	6.350270132559042e-009
0.5	-0.061118434331738	1.683960224221478e-007
0.6	-0.034078685565394	1.159391418267258e-007
0.7	0.028767057993287	4.083249415301991e-008
0.8	0.143788913036560	6.01923548237699e-008
0.9	0.331125863858524	7.582132859063062e-009
1	0.614718461206487	4.761522773288895e-012
	Max. Error	1.683960224221478e-007

Table No. (4) : The log of max. error of the suggested solution of example

P_{2n+1}	n Log	Max error	Log of max error
P_5	0.301029995663981	3.904271375693158e-004	-3.408460003282497
P_7	0.477121254719662	5.080546580960021e-005	-4.294089562455153
P_9	0.602059991327962	3.669790521687588e-006	-5.435358725358707

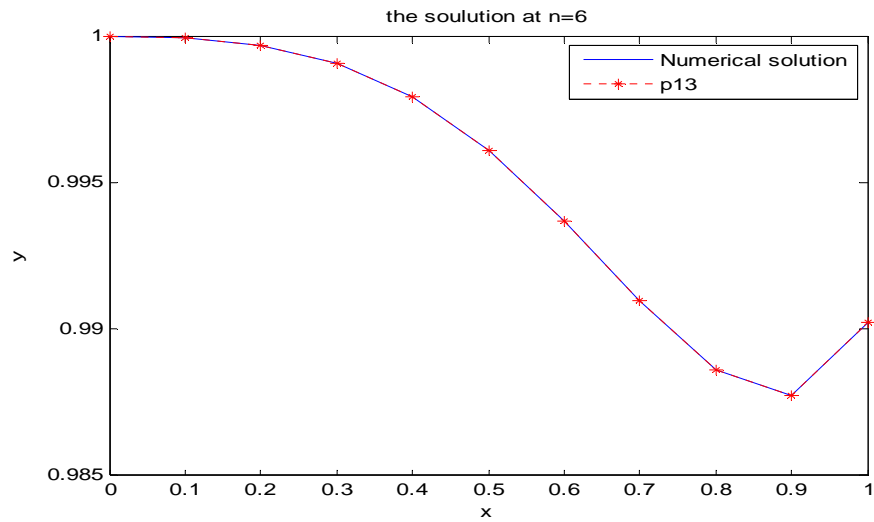


Figure No.(1): Comparison between the numerical solution in[3] and P_{13} for example 3

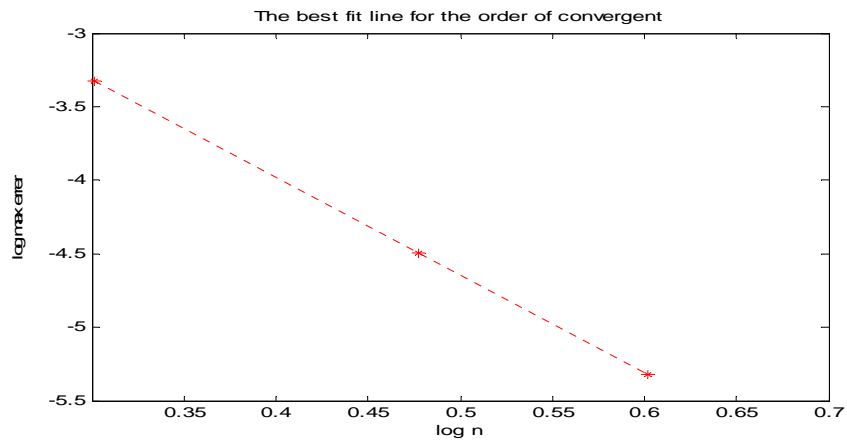


Figure No.(2) : The best fit line for the order of convergence

كفاءة التقنية شبه التحليلية في حل مسائل القيم الابتدائية غير الخطية الشاذة

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الخلاصة

الهدف من هذا البحث اقتراح تقنية شبه التحليلية لحل مسائل القيم الابتدائية الشاذة للمعادلات التفاضلية الاعتيادية وبأنواع مختلفة للحصول على الحل بوصفها متعددة حدود باستخدام الاندراج التماسي ذي النقطتين أوجدنا كفاءة الطريقة المقترحة ودقتها من خلال المقارنات مع الحل المضبوط و حلول تقريبية أخرى لأصناف واسعة متجانسة وغير متجانسة خطية و غير خطية لمسائل القيم الابتدائية الشاذة . استخدم تخمين جديد و كفوء للخطأ الرئيس لاختيار الشبكة المناسبة أيضا ناقشنا بعض الصفات العددية ذي العلاقة لتطبيق المقاييس ومناقشتها لزيادة كفاءة الشفرة (code) لمسائل القيم الابتدائية الشاذة . كذلك عدد من الأمثلة نوقشت لتوضيح الدقة و الكفاءة وسهولة أداء الطريقة المقترحة من جهة ولتعزيز و تأكيد رتبة التقارب من جهة أخرى.

الكلمات المفتاحية : معادلات تفاضلية اعتيادية , مسائل القيم الابتدائية الشاذة