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Constructing RKM-Method for Solving Fractional Ordinary Differential Equations of Fifth-Order with Applications

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Abstract

This paper sheds light on the vital role that fractional ordinary differential equations (FODEs) play in mathematical modeling and in real life, particularly in physical conditions. Furthermore, if the problem is handled directly using the numerical method, it is a far more powerful and efficient numerical method in terms of computational time, number of function evaluations, and precision. In this paper, we concentrate on the derivation of the direct numerical methods for solving fifth-order FODEs n one, two, and three stages. Additionally, it is important to note that the RKM-numerical methods with two- and three-stages for solving fifth-order ODEs are convenient. Numerical examples have been analyzed to demonstrate the efficacy of the new methods in comparison to the analytical method. Therefore, the numerical compression is carried out to confirm the efficiency and precision of the modified numerical methods. Significantly, the study demonstrates that the numerical outcomes of the proposed derived and modified numerically applied methods proved to be brilliant. Finally, based on the findings of the study, it could be said that the numerical outcomes of the test-problems using the proposed and modified methods agree well with the analytical solutions. Hence, we can conclude that the proposed numerical methods that are derived or modified in the analytic study of this paper are quite efficient.

Keywords RKM; Fractional Ordinary Differential Equations; Fifth- Order; DEs; FDEs.

1. Introduction

Undoubtedly, the fractional differential equation (FDE) is considered the most essential tool of mathematics, with applications in engineering and science. In addition to the significant part the FDE has concerning the various branches of applied mathematics, including physics, economics, chemistry, engineering, medicine, and biology, It is to be noted that the larger part of mathematical modeling in engineering and science is related to different kinds of FDEs. Moreover, solving the real problems that might occur in applied science and engineering is conducted by utilizing the mathematical methods and tools of FDEs. In the last thirty years, numerous significant studies in fractional calculus (FC) have been published in the science and engineering literature. The development of FC is detailed in different usages in fluid mechanics studies, electrochemistry, biological models, and viscoelastic models. Varied numerical or analytical approaches for solving differential equations (DEs), either classical or modernistic, have been the subject of studies over the years [3–7]. However, finding analytical or numerical solutions to many types of DEs challenged the ingenuity of mathematicians. At present, engineers and scientists can now apply a variety of powerful classical, modern numerical, and analytical approaches. The literature review regarding various recent strategies for solving mathematical models that incorporate DEs is listed as follows: Mechee, M. S. (2019) generalized RK integrators for solving a lass of sixth-order ODEs [8] while Mechee, M. S., & Senu, N. (2012), and Mechee, Mohammed S et al. (2019) studied the numerical solutions of FODEs [9-10]. The analytical methods for solving DEs are not always capable of solving all types of DEs, directly or indirectly. This proposition inspires us to study the derivation of direct numerical methods. Many researchers have derived one-step numerical methods for solving IVPs of ODEs of different orders, while others have derived multistep numerical ways for solving this problem. The FC is an efficient mathematical tool to solve different problems in engineering, sciences, and applied mathematics, and it is used to examine the problems of nonlinear dynamics of different. Principally, exact solutions cannot be obtained for the problems of FODEs, so it needs to find more semi-numerical and analytical methods and acknowledge the impacts of the nonlinear problems solutions. Recently, different approximate methods have been implemented to solve dynamical systems, both linear and nonlinear. Such Approximated methods are the variational iteration method (VIM), the Adomian decomposition method (ADM), the homotopy analysis method (HAM), the Homotopy perturbation method (HPM), the Laplace transform method, and the homotopy analysis transform method (HATM). In addition, utilizing the generalized Taylor series expansion, brought about a new approach in relation to the fractional Euler method and the modified trapezoidal rule. Lastly, Mechee & Senu (2012) solved the FODEs Lane-Emden type by the least square method [10], while Arshad et al. (2020) studied the numerical solutions of first order FODEs using the developed Euler method and they derived 2-stage fractional Runge-Kutta (FRK) method [2].

In this study, Euler and RKM numerical methods have been derived or developed to be consistent with classes of fifth-order FODEs. Firstly, the generalized Taylor series expansion was used to develop a one-stage Euler method as well as two- and three-stage RKM techniques to be suited for solving fifth-order FODEs. Secondly, the two and three-stage RKM methods are developed for solving fifth-order ODEs in harmony with solving two kinds of FODEs.

2. Preliminary

We recall some special types of equation, some of which will study some of them:

2.1 Kinds of Quasi-Linear FrODEs of Fifth-Order[8]:

In this subsection, some kinds of quasi-linear fifth-order FODEs have been introduced.

2.1.1 Kind of Quasi-Linear FODEs of Fifth-Order[8]:

The general kinds of quasi-linear FODEs of fifth-order are written as follows

$$D^{5\alpha}u(\tau) = f\left(\tau, u(\tau), u'(\tau), u''(t), u'''(\tau), u^{(4)}(\tau)\right), \tag{1}$$

$$0 < \alpha < 1, \tau \geq \tau_0$$

when the initial conditions

$$u^{(i\alpha)}(0) = \gamma_i, i = 0,1, \ldots, 4. \tag{2}$$

2.1.2 Kind-One of Quasi-Linear FrODEs of Fifth-Order [9]:

Consider the kind-one of quasi-linear FODEs fifth-order as follows:

$$D^{5\alpha}u(\tau) = f\left(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau), u^{(4)}(\tau)\right), \tag{3}$$

$$0 < \alpha < 1, \tau \geq 0$$

when the initial conditions are in Equation (2).

2.1.3 Kind-Two of Quasi-Linear FODEs of Fifth-Order [9]:

Consider the kind-two of quasi-linear FODEs of fifth-order as follows:

$$D^{5\alpha}u(\tau) = f\left(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)\right), 0 < \alpha < 1, \tau \geq \tau_0 \tag{4}$$

when the initial conditions are in Equation (2).

2.1.4 Kind-Three of Quasi-Linear FODEs of Fifth-Order [9]:

Consider the kind-three of quasi-linear FODEs of fifth-order as follows:

$$D^{5\alpha}u(\tau) = f\left(\tau, u(\tau), u'(\tau), u''(\tau)\right), 0 < \alpha < 1, \tau \geq \tau_0 \tag{5}$$

with the initial conditions are in Equation (2).

2.1.5 Kind-Four of Quasi-Linear FODEs of Fifth-Order[9]:

Consider the class-four of quasi-linear FODEs of fifth-order as follows:

$$D^{\ 5\alpha}u(\tau) = f\left(\tau, u(\tau), u'(\tau)\right), 0 < \alpha < 1, \tau \geq \tau_0 \tag{6}$$

when the initial conditions are in Equation (2).

2.1.6 Kind-Five of Quasi-Linear FODEs of Fifth-Order

Consider the kind-five of quasi-linear FODEs of fifth-order as follows:

$$D^{5\alpha}u(\tau) = f\left(\tau, u(\tau)\right), 0 < \alpha < 1, \tau \geq \tau_0 \tag{7}$$

when the initial conditions are in Equation (2).

2.2 The Solving of Class-Five, Fifth-Order ODEs by RKM Method

ODEs commands of the order more than two, many researchers used to solve by transmutation the ODE into a system of first-order ODEs where the dimension of the system are the same in that order.

2.2.1 The Solving of Class-Five Fifth-Order ODEs by RKM Method The formula of s-stage RKM method to solve kind-five of fifth-order ODEs is proposed as follows

Table 1 The Butcher Tableau RKM Method of Fifth-Order

0	0		
$\frac{3}{5}+\frac{\sqrt{6}}{10}$	$\frac{1}{2}$	0	
$\frac{3}{5}-\frac{\sqrt{6}}{10}$	$\frac{1}{2}$	$\frac{1}{2}$	0
	1	0	$-\frac{119}{120}$
	$-\frac{1}{40}-\frac{\sqrt{6}}{360}$	$\frac{1}{60}-\frac{\sqrt{6}}{360}$	0
	$\frac{1}{18}$	$\frac{1}{8}+\frac{\sqrt{6}}{48}$	$\frac{1}{8}+\frac{\sqrt{6}}{48}$
	$\frac{1}{9}$	$\frac{7}{36}-\frac{\sqrt{6}}{48}$	$\frac{7}{18}+\frac{\sqrt{6}}{18}$
	$\frac{1}{9}$	$\frac{7}{36}-\frac{\sqrt{6}}{48}$	$\frac{7}{18}+\frac{\sqrt{6}}{48}$

The coefficients of the numerical RKM methods are $a_{ij}; c_i;\ b_i;\ b_i^{\prime};\ b_i^{\prime\prime};\ b_i^{\prime\prime\prime};$

and $b_i^{(4)}$ assumed to be real for $i; j = 1,2, \dots, s$. However, Mechee & Kadhim derived direct numerical three-stage RKM methods of fifth-order for solving fifth-order ODEs [10], which expressed in the following.

3. Proposed Numerical Methods for Solving FODEs of Fifth-Order

In this part, we have established two numerical methods and modified

another two numerical methods for solving fifth-order FODEs which belong to a kind of quasi linear in Equation (7) with the ICs in Equation (2). Also, we give derivation of the generalized Euler method as the generalized Taylor expansion of $u(t+h)$ is:

$$u(\tau+h) = u(\tau) + \frac{h^{\alpha}}{\Gamma((\alpha+1)} D^{\alpha}u(\tau) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} D^{2\alpha}u(\tau) + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} D^{3\alpha}u(\tau) + \\ + \frac{h^{4\alpha}}{\Gamma(4\alpha+1)} D^{4\alpha}u(\tau) + \frac{h^{5\alpha}}{\Gamma(5\alpha+1)} D^{5\alpha}u(\tau) + + \frac{h^{6\alpha}}{\Gamma(6\alpha+1)} D^{6\alpha}u(\tau) + \cdots \qquad (8)$$

We ignore the higher terms involving $D^{5\alpha}u(\tau)$ in Equation (8) for the very small step size. Substitute the value of $D^{5\alpha}u(\tau)$ from Equation (8) to obtain the following formula:

$$u_{n+1}(\tau) = u_n(\tau) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} u_n^{\alpha}(\tau) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} u_n^{2\alpha}(\tau) + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} u_n^{3\alpha}(\tau) \\ + \frac{h^{4\alpha}}{\Gamma(4\alpha+1)} u_n^{4\alpha}(\tau) + \frac{h^{5\alpha}}{\Gamma(5\alpha+1)} f(\tau_n, u_n)\,, \qquad (9)$$

By derivation Equation (16) once, twice, three and four, we obtain the following:

$$u_{n+1}^{\alpha}(\tau) = u_n^{\alpha}(\tau) + \frac{h^{\alpha}}{\Gamma\alpha+1)} u_n^{2\alpha}(\tau) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} u_n^{3\alpha}(\tau) + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} u_n^{4\alpha}(\tau) \\ + \frac{h^{4\alpha}}{\Gamma(4\alpha+1)} f(\tau_n, u_n)\,, \qquad (10)$$

$$u\,{}_{n+1}^{2\alpha}(\tau) = u_n^{2\alpha}(\tau) + \frac{h^{\alpha}}{\Gamma\alpha+1)} u_n^{3\alpha}(\tau) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} u_n^{4\alpha}(\tau) + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} \\ + f(\tau_n, u_n) \qquad (11)$$

$$u_{n+1}^{3\alpha}(\tau) = u_n^{3\alpha}(\tau) + \frac{h^{\alpha}}{\Gamma\alpha+1)} u_n^{4\alpha}(\tau) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} f(\tau_n, u_n)\,, \qquad (12)$$

$$u_{n+1}^{4\alpha}(\tau) = u_n^{4\alpha}(\tau) + \frac{h^{\alpha}}{\Gamma\alpha+1)} f(\tau_n, u_n) \quad . \qquad (13)$$

In addition, by introducing the derivation of developed RKM method of two-stages using the chain rule, we get the following:

$$D^{6\alpha}u(\tau) = D^{\alpha}\left(D^{5\alpha}u(\tau)\right) = D^{\alpha}\left(f\left(\tau, u(\tau)\right)\right)$$
$$= D^{\alpha}\left(f\left(\tau, u(\tau)\right)\right) + f\left(\tau, u(\tau)\right)D_{u}^{\alpha}f\left(\tau, u(\tau)\right), \qquad (14)$$

We ignore the higher terms involving $D^{6\alpha}u(t)$ in Equation (8) for the very small step size and substitute the value of $D^{5\alpha}u(t)$ from Equation (7) to obtain the following formula:

$$u_{n+1} = u_n(\tau) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}u_n^{\alpha}(\tau) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)}u_n^{2\alpha}(\tau) + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)}u_n^{3\alpha}(\tau)$$
$$+ \frac{h^{4\alpha}}{\Gamma(4\alpha+1)}u_n^{4\alpha}(\tau) + \frac{h^{5\alpha}}{\Gamma(5\alpha+1)}f\left(\tau_n, u_n(\tau)\right)$$
$$+ \frac{h^{6\alpha}}{\Gamma(6\alpha+1)}u_n^{6\alpha}(\tau)$$
$$+ \frac{h^{6\alpha}}{\Gamma(6\alpha+1)}\left(D^{\alpha}\left(f\left(\tau, u(\tau)\right)\right) + f\left(\tau, u(\tau)\right)D_{u}^{\alpha}f\left(\tau, u(\tau)\right)\right)$$

$$= u_n(\tau) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}u_n^{\alpha}(\tau) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)}u_n^{2\alpha}(\tau) + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)}u_n^{3\alpha}(\tau)$$
$$+ \frac{h^{4\alpha}}{\Gamma(4\alpha+1)}u_n^{4\alpha}(\tau) + \frac{h^{5\alpha}}{\Gamma(5\alpha+1)}f(\tau_n, u_n)$$
$$+ \frac{h^{5\alpha}}{3!\,\Gamma(5\alpha+1)}(f\left(\tau_n + \frac{6h^{5\alpha}\Gamma(5\alpha+1)}{\Gamma(6\alpha+1)}, u_n\right) +$$

$$+ \frac{6h^{5\alpha}\Gamma(5a+1)}{\Gamma(6\alpha+1)}f\left(\tau_n, u_n(\tau_n)\right)), \qquad (15)$$

Taking the first and second derivatives of the above equation, we get:

$$u_{n+1}^{\alpha}(\tau) = u_n^{\alpha}(\tau) + \frac{h^{\alpha}}{\Gamma\alpha+1)}u_n^{2\alpha}(\tau) + \frac{h^{2\alpha}}{\Gamma(\alpha+2)}u_n^{3\alpha}(\tau) +$$

$$\frac{h^{3\alpha}}{\Gamma(\alpha+3)}u_n^{4\alpha}(t\tau) + \frac{h^{4\alpha}}{\Gamma(\alpha+4)}f(\tau_n, u_n), \qquad (16)$$

$$u_{n+1}^{2\alpha}(\tau) = u_n^{2\alpha}(\tau) + \frac{h^{\alpha}}{\Gamma\alpha+1)}u_n^{3\alpha}(\tau) + \frac{h^{2\alpha}}{\Gamma(\alpha+2)}u_n^{4\alpha}(\tau)$$
$$+ \frac{h^{3\alpha}}{\Gamma(\alpha+3)}f(\tau_n, u_n), \qquad (17)$$

$$u_{n+1}^{3\alpha}(\tau) = u_n^{3\alpha}(\tau) + \frac{h^{\alpha}}{\Gamma\alpha+1)}u_n^{4\alpha}(\tau) + \frac{h^{2\alpha}}{\Gamma(\alpha+2)}f(\tau_n, u_n), \quad (18)$$

$$u_{n+1}^{4\alpha}(\tau) = u_n^{4\alpha}(\tau) + \frac{h^\alpha}{\Gamma\alpha+1)} f(\tau_n, u_n)\,. \tag{19}$$

Hence, the formulas in equations $(15)-(19)$ are used to produce convergent sequence for solving Equation (7) with the ICs (2). Finally, we develop the RKM method of s-stage, and improve Equations (9)-(12) for solving 5^{th} -order ODEs to be convenient for solving 5^{th} - order FODEs; the following formula has been supposed for finding the numerical:

$$u_{n+1}(\tau) = u_n(\tau) + \frac{h^\alpha}{\Gamma(\alpha+1)} u_n^\alpha(\tau) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} u_n^{2\alpha}(\tau) + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} u_n^{3\alpha}(\tau) + \frac{h^{4\alpha}}{\Gamma(4\alpha+1)} u_n^{4\alpha}(\tau) + h^{5\alpha}\sum_{i=1}^{s} b_i k_i\,, \tag{20}$$

Taking the first, second, third, and fourth derivatives of the above equation, we get:

$$u_{n+1}^{\alpha}(\tau) = u_n^{\alpha}(\tau) + \frac{h^\alpha}{\Gamma(\alpha+1)} u_n^{2\alpha}(\tau) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} u_n^{3\alpha}(\tau) + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} u_n^{4\alpha}(\tau) + h^{4\alpha}\sum_{i=1}^{s} b_i' k_i\ , \tag{21}$$

$$u_{n+1}^{2\alpha}(\tau) = u_n^{2\alpha}(\tau) + \frac{h^\alpha}{\Gamma\alpha+1)} u_n^{3\alpha}(\tau) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} u_n^{4\alpha}(\tau) + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} u_n^{4\alpha}(\tau) + h^{3\alpha}\sum_{i=1}^{s} b_i'' k_i\,, \tag{22}$$

$$u_{n+1}^{3\alpha}(\tau) = u_n^{3\alpha}(\tau) + \frac{h^\alpha}{\Gamma\alpha+1)} u_n^{4\alpha}(\tau) + h^{2\alpha}\sum_{i=1}^{s} b_i''' k_i, \tag{23}$$

$$u_{n+1}^{4\alpha}(\tau) = u_n^{3\alpha}(\tau) + h^{\alpha}\sum_{i=1}^{s} b_i^{(4)} k_i z\ , \tag{24}$$

$$k_1 = f(\tau_n, u_n), \tag{25}$$

$$k_2 = f(\tau_n + hc_2, u_n + ha_{21}k_1)\,, \tag{26}$$

and

$$k_3 = f(\tau_n + hc_3, u_n + ha_{31}k_1 + ha_{32}k_2)\ . \tag{27}$$

4. Applications and Results:

In this section, we demonstrate the effectiveness of the suggested method using various numerical examples.

Example 4.1

Let following linear FODE of fifth-order:

$$D_x^{5\alpha} y(\tau) = 2^{5\alpha} y(\tau), \qquad 0 < \alpha \leq 1 \quad , \tau > 0 . \qquad (28)$$

When initial conditions

$$y(0) = 1, D_\tau^{\alpha} y(0) = 2^{\alpha}, D_\tau^{2\alpha} y(0) = 2^{2\alpha}, D_\tau^{3\alpha} y(0) = 2^{3\alpha},$$

$$D_\tau^{4\alpha} y(0) = 2^{4\alpha},$$

The exact solution:

$$y(\tau) = e^{2\tau}.$$

Example 4.2

Let following linear FODE of fifth-order:

$$D_x^{5\alpha} y(\tau) + 2y(\tau) + y''(\tau) = 2^{5\alpha} \, sin \, sin \, (2\tau + \frac{5\pi}{2} \alpha) \, , 0 < \alpha \leq 1 \, , \tau > 0 \qquad (29)$$

When initial conditions

$$y(0) = 0, D_\tau^{\alpha} y(0) = 2^{\alpha} \, sin \, sin \, (\frac{\pi}{2} \alpha) \, , D_\tau^{2\alpha} y(0) = 2^{2\alpha} sin(\Pi\alpha),$$

$$D_\tau^{3\alpha} y(0) \quad = 2^{3\alpha} \, sin \, sin \, (\frac{3\pi}{2} \alpha) \, , D_\tau^{4\alpha} y(0) = 2^{4\alpha} sin(2\Pi\alpha).$$

Example 4.3

Let following linear FODE of fifth-order:

$$D^{5\alpha} \, y(\tau) + 2y' + y'' + y^{(4)} = 2^{5\alpha} \, cos \, cos \, (2\tau + \tfrac{5\pi}{2} \alpha) + 16 \, cos \, cos \, 2\tau \, , 0 < \alpha \leq 1, \tau > 0 \quad (30)$$

when initial conditions:

$$y(0) = 1 \quad , D^{\alpha} \, y(0) = 2^{\alpha} \, cos \, cos \, (\frac{\pi}{2} \alpha) \quad , D^{2\alpha} \, y(0) = 2^{2\alpha} \, cos \, cos \, (\pi \, \alpha),$$

$$D^{3\alpha} \, y(0) = 2^{3\alpha} \, cos \, cos \, (\frac{3\pi}{2} \alpha) \quad , D^{4\alpha} \, y(0) = 2^{4\alpha} \, cos \, cos \, (\, 2\pi\alpha)$$

Example 4.4

Let following linear FODE of fifth-order:

$D^{5\alpha}\, y(\tau) + y^{'} + y^{''} \; = sin\, sin\,(2\tau + \frac{5\pi}{2}) + cos\, cos\,(2\tau + \frac{5\pi}{2}\,\alpha) - 2\, sin\, sin\,(\tau)\,, 0\; < \alpha \leq 1\;, \tau >$ 0 (31)

when initial conditions:

$$y(0) = 1\;\;, D^{\alpha}\, y(0) = \frac{\pi}{2}\alpha) + cos\, cos\,(\frac{\pi}{2}\,\alpha)\;\;, \qquad D^{2\alpha}\, y(0)\; = sin\, sin\,(\pi\alpha)\; + cos\, cos\,(\pi\,\alpha),$$
$$D^{3\alpha}\, y(0) = sin\, sin\,(\frac{3\pi}{2}\,\alpha)(\frac{3\pi}{2}\,\alpha)\;\, , D^{4\alpha}\, y(0) = \; sin(2\pi\alpha) + cos\, cos\,(\, 2\pi\alpha)\,.$$

Example 4.5

Let following linear FODE of fifth-order:

$$D^{5\alpha}\, y(\tau) + 9y^{'} + y^{''} \; = 3^{5\alpha}\;\; cos\, cos\,(2\tau + \frac{5\pi}{2}\,\alpha)\;,\;\; 0\; < \;\alpha \quad \leq\; 1 \quad , \tau > 0 \qquad (32)$$

when initial conditions

$$y(0) = 1\;\;, D^{\alpha}\, y(0) = 3^{\alpha}\;\; cos\, cos\,(\frac{\pi}{2}\,\alpha)\;\;, D^{2\alpha}\, y(0) = \; 3^{2\alpha}\;\; cos\, cos\,(\pi\,\alpha),$$
$$D^{3\alpha}\, y(0) = 3^{3\alpha}\;\; cos\, cos\,(\frac{3\pi}{2}\,\alpha)\quad , D^{4\alpha}\, y(0) = 3^{4\alpha}\;\; cos\, cos\,(\, 2\pi\alpha).$$

Example 4.6

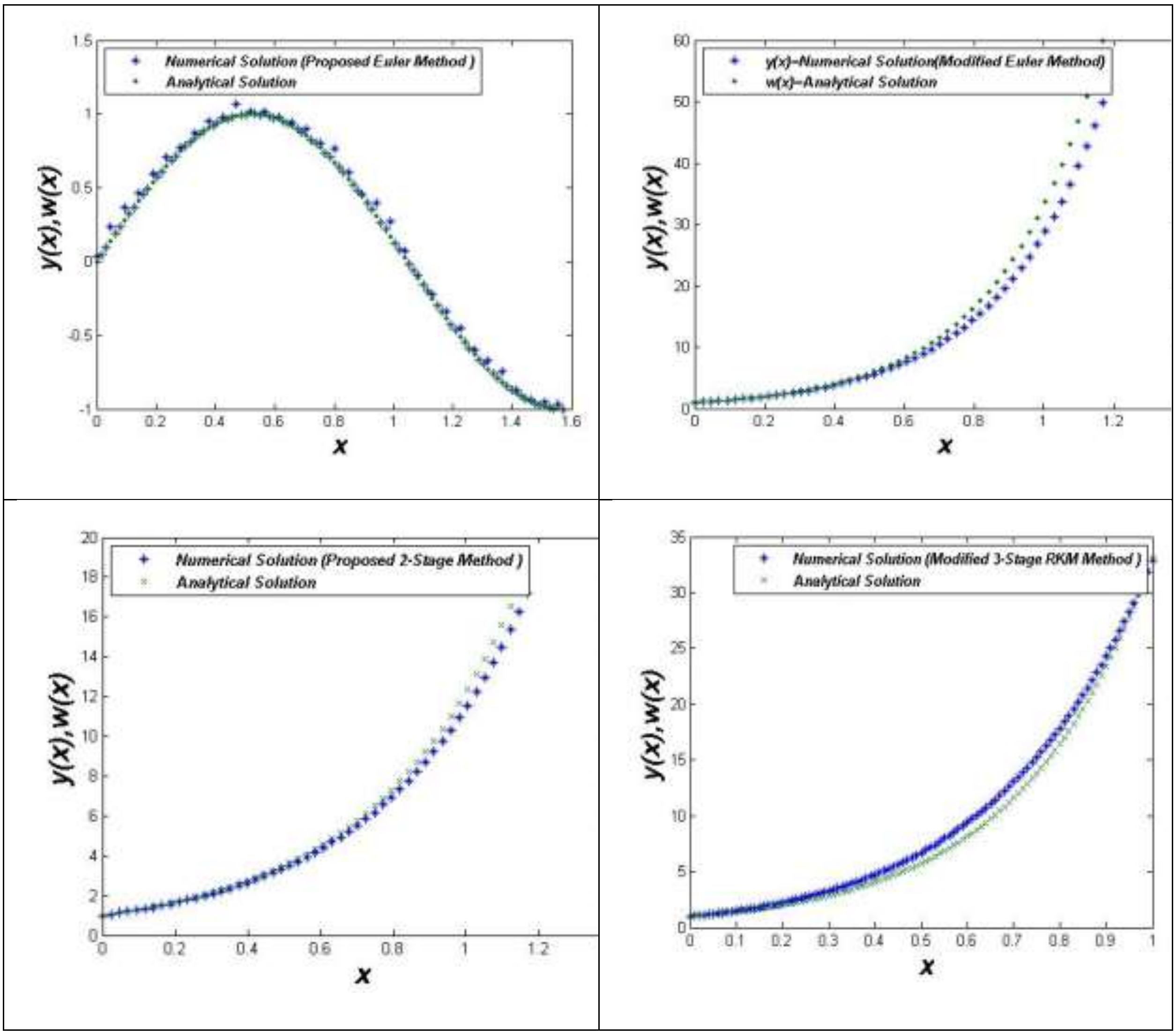
Let following linear FODE of fifth-order:

$$D_x^{5\alpha} y(\tau) + 9y(\tau) + y''(\tau) = 3^{5\alpha}\frac{\pi}{2}\alpha)\;, \qquad 0\; < \alpha \leq\; 1, \tau > 0 \qquad (33)$$

when initial conditions

$y(0) = 0, D_\tau^{\alpha} y(0) = \frac{\pi}{2}\alpha)\,, D_\tau^{2\alpha} y(0) = sin\, sin\, 3^{2\alpha}(\Pi\alpha)\;\, ,\;\, D_\tau^{3\alpha} y(0) =$
$3^{3\alpha}\,\frac{3\pi}{2}\alpha)\;\;\, , D_\tau^{4\alpha} y(0) = 3^{\alpha} sin(2\Pi\alpha)\,.$

Table 1 Numerical-Comparisons on the Approximated-Solutions versus the Analytical Solutions for Solving the examples 4.1 and 4.2 Using (1) Modified-Euler-Method, (2) Proposed two-stage RKM-method and (3) Modified three-stage RKM-Method for N=100, α = 0.96 .



5. Conclusion

The fundamental objective is to construct the numerical method RKM for solving a kind of fifth-order fractional ordinary differential equation. For this purpose, we have derived the three-stage RKM method. We introduced the object of the method by utilizing various examples of fifth-order FODEs to inquire about the activity of the new method and examine the new method's effectiveness. The comparison of numerical results with exact solutions in Table 1 shows that the numerical finding of Example 4.1 is more efficacious and precise than well-known present methods.

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