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⊕-s-extending modules

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Abstract

The concept ⊕-s-extending modules will be purpose of this paper, a module M is ⊕-s-extending if each submodule in M is essential in submodule has a supplement that is direct summand. Initially, we give relation between this concept with weakly supplement extending modules and ⊕-supplemented modules. In fact, we gives the following implications:

Lifting modules ⟹ ⊕-supplemented modules ⟹ ⊕-s-extending modules ⟹ weakly supplement extending modules.

It is also we give examples show that, the converse of this result is not true. Moreover, we study when the converse of this result is true.

Keyword: Extending modules, weakly supplement extending modules, ⊕-supplemented modules, ⊕-s-extending modules.

1. Introduction

In this paper, motivated by the concept of closed ⊕-supplemented that is, a module M is called closed ⊕-supplemented, if each closed submodule in M has a supplement which is direct summand, and weakly supplement extending module we give concept that is ⊕-s-extending module, a module M is ⊕-s-extending if each submodule in M is essential in submodule has a supplement which is direct summand. We can answer the question: When is the ⊕-s-extending module inherited by direct summand.

Following [1]. Studied a weakly supplement extending modules, a module M is called weakly supplement extending if each submodule in M is essential in weakly supplement submodule in M. Also, ⊕-supplemented module were studied by [2]. A module M is ⊕-supplemented, if for any submodule N in M has a supplement which is

Direct summand. In this work, R is associative rings with identity and M is left R-module. A submodule W of M is said to be essential, if W∩V=0 then V=0[2]. A submodule W of M is called small (denoted by W≪M) if for each submodule L in M such that M≠W+L implies L≠M [2]. A submodule N of M is called closed, if N has no proper essential extension in M [2]. A submodule N of M is weakly supplement, if there exists a submodule W in M such that M=W+N and N∩ W≪M. A module M is called weakly supplemented if each submodule W in M is weakly supplement of M [2]. A submodule V of M is supplement, if there is a submodule W of M such that M=V+W and V∩ W≪V [2]. A module M is called supplemented if each submodule V of M has a supplement submodule in M [2]. A module M is said to be uniform if each nonzero submodule in M is essential submodule in M [3]. A module M is called non-singular if Z(M)=0 where Z(M)={m∈M | Xm=0 for some essential left ideal X of R} , and M is singular if Z(M)=M [3]. A module *M* is called lifting, if each submodule V in M there is a direct summand W in M with W⊆ V such that M=W⊕W' and $W' \cap V$ ≪W' [2]. If *f*(N)⊆(N) for each R-endomorphism *f* of M, then a submodule N of a module M is said to be fully invariant [4]. A module M is semi-simple, if every submodule is direct summand [3]. A module M is called extending if each submodule in M is essential in direct summand of M [3]. The extending property and their generalization are studied by different authors such as [1, 5,6].

2. ⊕-s-extending modules

In this work, we will present the following concept that is stronger than of weakly supplement extending modules:

Definition (1)

A module M is called ⊕-s-extending if each submodule in M is essential in submodule has a supplement which is direct summand.

Proposition (2)

A module M is called ⊕-s-extending if and only if each submodule in M is essential in submodule has a weakly supplement which is direct summand.

Proof

Let N be a submodule of ⊕-s-extending module, then N is essential in submodule K in M has a supplement which is direct summand. Since every supplement submodule is weakly supplement, then we have K has a weakly supplement which is direct summand. Conversely, let N be a submodule of a module M. By hypothesis, N is essential in submodule K has a weakly supplement L which is direct summand (i.e) M=K+L and K∩L≪M where L is direct summand in M. Since K∩L⊆L⊆M and L is direct summand in M, thus we have K∩L≪L. Then N is essential in submodule K has a supplement L which is direct summand. Hence M is ⊕-s-extending.

Proposition (3)

A module M is ⊕-s-extending if and only if each closed submodule in M has a (weakly) supplement which is direct summand.

Proof

(⟹) Let B be a closed submodule in M. Since M is ⊕-s-extending, so B is essential in submodule K has a supplement which is direct summand. But B is closed, so we have B has a supplement which is direct summand.

(⟸) Let B be submodule in M. Then by using Zorn's lemma, so there is a closed submodule D in M such that B is essential in D. By hypothesis, D has a supplement which is direct summand.

Following [1]. A module M is closed ⊕-supplemented if each closed submodule in M has a weakly supplement which is direct summand. The next result is directly by proposition (2).

Corollary (4)

A module M is ⊕-s-extending if and only if M is closed ⊕-supplemented.

Remarks and Examples (5)

1. Each extending module is ⊕-s-extending**,** while it is not conversely. For example, M=$Z_8⊕Z_2$ as Z-module is ⊕-s-extending which it is not extending [1].
2. Every ⊕-s-extending module is weakly supplement extending, while the other direction is not true in general.
3. Every ⊕-supplemented module is ⊕-s-extending, while it is not conversely. In fact, Z is ⊕-s-extending Z-module which it is not ⊕-supplemented (because a submodule 2Z in Z has no a supplement submodule in Z).
4. Every uniform module is ⊕-s-extending, while the converse is not true. For example, Z_{10} is ⊕-s-extending Z-module which it is not uniform.
5. In [1]. every lifting module is weakly supplement extending. This result can be generalized to ⊕-s-extending (i.e), every lifting module is ⊕-s-extending (since every lifting module is ⊕-supplemented), while the converse is not true. In fact, Z is ⊕-s-extending Z-module which is not lifting.
6. Every weakly (supplemented) module is weakly supplement extending [1]. This result is not still valid for ⊕-s-extending. Not every weakly supplemented module is ⊕-s-extending. Moreover, not every ⊕-s-extending module is weakly supplemented, Z is ⊕-s-extending Z-module which it is not weakly supplemented.
7. Following [7]. Every semi-simple module is ⊕-supplemented. So every semi-simple is ⊕-s-extending, while the converse is not true. Q is ⊕-s-extending Z-module which it is not semi-simple.

Recall that, a module M is called refinable if for every submodule W, V in M with W+V=M, then there is a direct summand W' in M such that W'⊆W and W'+V=M [2].

Following [1]. Every closed ⊕-supplemented module is weakly supplement extending. Also, from [1]. Studied when the converse is true. Moreover, we have this corollary:

Corollary (6)

Let M be a refinable module. Then the following are equivalent:

1. M is weakly supplement extending module.
2. M is ⊕-s-extending module.

Recall that, a module M is said to be wd-module if each weakly supplement submodule is direct summand [1].

Proposition (7)

Let M be a wd-module. Then the following statement are equivalent:

1. M is ⊕-s-extending module.

2. M is weakly supplement extending module.

Proof

(1⟹2) Directly by (Remarks and Examples (5)).

(2⟹1) Let W be a closed submodule of weakly supplement extending module M, so W is a weakly supplement submodule of V in M. Also V is weakly supplements of W in M. But M is wd-module. Hence we have V is direct summand of M and so M is ⊕-s-extending module.

Proposition (8)

Let M be a wd-module. Then the following statement are equivalent:

1. M is extending module.
2. M is ⊕-s-extending module.

Proof

(1⟹2) Directly by (Remarks and Examples (5)).

(2⟹1) Let A be a closed submodule of ⊕-s-extending module M, so A has a weakly supplement submodule K in M which is direct summand. Also K is weakly supplement of A in M. But M is wd-module. Then we have A is direct summand of M. Hence M is ⊕-s-extending module. The following lemma helps us in the next results

Lemma (9)

Let M be a ⊕-s-extending module. Suppose that W be a closed submodule in M and B is small in M. Then there is a submodule C in M such that M=W+C=W+C+B and W∩C≪M, so C∩(W+B) ≪M.

Proof

Let W be a closed submodule of ⊕-s-extending module M, then W has a weakly supplements C in M that is direct summand (i.e) M=W+C and W∩C≪M. Now let h: M→(M/W)⊕(M/C) is defined by h(d)=(d+W, d+C) and let j: (M/W)⊕(M/C) → (M/$W+B$)⊕(M/C) is defined by j(d+W, m'+C)=(d+W+B, m'+C). Since W∩C≪M, then h is epimorphism and Kerh= W∩C≪M, and since Kerj=((W+B/W)⊕0 and (W+B)/W=(B)≪M/W when the canonical epimorphism ν: M→M/W. So we have j is a small epimorphism and jh is small epimorphism since Kerjh= C∩(B+W)≪M.

We noticed that, every ⊕-supplemented module is ⊕-s-extending see (Remarks and Example (5)). Next we explained when the convers is true.

Proposition (10)

Let M be a module in which each submodule L in M there exists a closed submodule W (depending on L) of M such that L=W+C or W=L+C for some C small in M. Then M is ⊕-s-extending module if and only if M ⊕-supplemented.

Proof

Let L be a submodule in ⊕-s-extending module M, so there exists a closed submodule W in M such that L=W+C where C≪M. But M is ⊕-s-extending, thus W has a supplements D in M that is direct summand (i.e) M=W+D=W+C+D=L+D where C≪M and W∩D≪D, then L∩D ⊆ (W+C)∩D. Since W∩D≪D, then W∩D≪M So by lemma (9) (W+C)∩D≪M. Thus L∩D≪M. Now since L∩D ⊆D⊆M and D direct summand, so we have L∩D≪D and hence M is ⊕-supplemented. Or, W=L+C where C≪M. Since M is ⊕-s-extending, thus W has a supplement D in M that is direct summand (i.e) M=W+D=L+C+D=L+D where C ≪M and L∩D⊆W∩D≪D, so L∩D≪D. thus M is ⊕-supplemented. Conversely see (Remarks and Examples (5)).

Recall that, a module M is called injective hull of a module Q if it is both an essential extension of Q and an injective module [3].

The following proposition gives another characterization of ⊕-s-extending module.

Proposition (11)

For any a module M. The following statement are equivalent:

1. M is ⊕-s-extending module.
2. The intersection of M with any direct summand of injective hull of M, has a weakly supplements submodule in M that is direct summand.

Proof

(1) ⟹ (2) Let W be a direct summand of injective hull of M, i.e E(M)=W⊕V, where V is a submodule of injective hull of M. It is easy to show that W∩M is closed of M. So by proposition (3) W∩M has a weakly supplement submodule of M which is direct summand.

(2) ⟹ (1) Let W be a submodule of M. and let V be a relative complement of W in M, then W⊕V is essential in M, but M is essential in injective hull of M, Therefore W⊕V is essential in injective hull of M, Then E(M) =E(W⊕V) =E(W) ⊕E(V). Since E(W) is direct summand of E(M) then E(W)∩M has a weakly supplement submodule in M that is direct summand. But W is essential in E(W) and M is essential in M, So W=W∩M is essential in E(W)∩M which has a weakly supplement in M that is direct summand. Hence M is ⊕-s-extending module.

Recall that, if each submodule A of M, there is an ideal J of R such that A=JM, then a module M is said to be multiplication [8].

Following [4]. let K be a submodule of a module M and R be a ring. The ideal {X∈R| XM⊆K}will be denoted by [K:M] and the annihilater of M denoted by ann_R (M) is ann_R(M)=[0:M]. Also, a module M is called faithful if ann_R(M)=0

Proposition (12)

Let R be a commutative ring and M a finitely generated faithful multiplication module. Then R is ⊕-s-extending if and only if M is ⊕-s-extending.

Proof

Let N be a closed submodule in M, then by [9]. There exists a closed ideal I in R such that IM=N. Since R is ⊕-s-extending, so I has a weakly supplement J which is direct summand in R (i.e) R= I+J and I∩J≪R. Let K=MJ where K is submodule in M. Now since M is multiplication then M=RM= (I+J)M=IM+JM=N+K and by [10, lemma (4.11)]. N∩K≪M. Also, since J is direct summand, so we have J+S =R and J∩S=0 where S is ideal in R and such that SM=F. Now since M is multiplication, then M=RM=(J+S)M=JM+SM=K+F and K∩F=JM∩SM=(J∩S)M=0. Thus we have K is direct summand in M. Hence M is ⊕-s-extending. Conversely, let I be a closed ideal in R, then by [10, lemma (4.10)]. There is a closed submodule N be in M such that N=IM. Since M is ⊕-s-extending, so N has a weakly supplement K in M which is direct summand (i.e) M=K+N and K∩N≪M. Let K=JM where J is ideal in R. Since M is multiplication then M=N+K=IM+JM=(I+J)M=RM Thus we have R=I+J and by [10, lemma (4.11)]. I∩J≪R and J is direct summand in R. Then R is ⊕-s-extending.

Recall that, let f :R⟶T be a ring hommorphism and M a right T-module. On can be defined to as a right R-module by mr=m f(r) for all m∈M and r∈R. Moreover, if f is an epimorphism and M is a right R-module such that ker$f$ ⊆ r(M), so also can also be define to be a right T-module by mt=mr, where f(r)=t. We denote by M_T , M_R then M is a right T-module, right R-module [10].

Proposition (13)

Let f: R⟶T be a ring epimorphism and M a right R-module with ker$f$ ⊆ r(M). Then M_R is ⊕-s-extending if and only if M_T is ⊕-s-extending.

Proof

Let A_T be a closed submodule of M_T, then A_R is closed submodule in M_R, since M_R is ⊕-s-extending, so A_R has a weakly supplement B_R that is direct summand. So B_R we can define to be T-module by mt=mr, where f (r)=t. Thus A_T has a weakly supplement $B_T$which is direct summand in M_T. Then M_T is ⊕-s-extending. Conversely, Let A_R be a closed submodule of M_R, then A_T is closed submodule in M_T, since M_T is ⊕-s-extending, so A_T has a weakly suplement B_T that is direct summand. So B_T we can define to be R-module by mr=mf(r) for each r∈R and m∈M. Thus A_R has a weakly supplement $B_R$which is direct summand in M_R. Hence M_R is ⊕-s-extending.

It is known that, a factor module of ⊕-supplemented need not necessary ⊕-supplemented [7]. Also, in ⊕-s-extending module is not verified. Thus in the following result we obtain a condition under it a factor module of ⊕-s-extending module is ⊕-s-extending.

Proposition (14)

Let M be a ⊕-s-extending module. Then any nonsingular (epimorphic) image of M is ⊕-s-extending module.

Proof

Let f: M→N be an epimorphism mapping and let K be a closed submodule of N, so H=f^{-1}(K) is a closed submodule of M. since M is ⊕-s-extending module, so H has a weakly supplement submodule W of M which is direct summand. Then M=H+W and H∩W≪M. Now N= f (M)=f (H+W)=f (H)+f (W)=K+ f (W) and (since f is epimorphism and ker f⊆H) f(H∩W)= f(H) ∩f (W) ≪f (M), so we have K∩ f (W) ≪ f (M) =N. Then f (W) is weakly supplement of K in N. Now to show that f(W) is direct summand of N. Let M =W+F and W∩F=0 where F is submodule of M, so N= f(M)= f(F+W)=f(F)+f(W) and

f(F∩W)= f(F) ∩f(W)=0. Then f(W) is direct summand of N and thus N is ⨁-s-extending module.

Proposition (15)

If a module M_1 is ⨁-s-extending module and $M_1 \cong M_2$, then M_2 is ⨁-s-extending module.

Proof

Let f: $M_1 \to M_2$ be an isomorphic image and $M_1$is ⨁-s-extending module. Let W is a submodule of M_2 then we have f^{-1}(W) is a submodule of M_1. Since M_1 is ⨁-s-extending module, so f^{-1} (W) is essential in V where V has a weakly supplement submodule of M_1 which is direct summand, so W= f(f^{-1} (W)) is essential in f(V). Since V has a weakly supplement submodule in M_1 then there exists H is a submodule of M_1 such that M_1=V+H and V∩H $\ll M_1$. Then $f(M_1)$= f(V)+ f(H), so M_2= f(V)+ f(H) and (since f is monomorphism then ker$f$=0) so we have f(V∩H)= f(V)∩ f(H) $\ll$ M_2. Then f(V) has a weakly supplement submodule in M_2, since H is direct summand so, we have M_1=H+L and H∩L=0 where L is submodule of M_1 then f (M_1)= f(H)+ f(L) and f(H) ∩ f(L)=0. Then M_2 ⨁-s-extending module.

The next result gives a condition under which a direct summand of ⨁-s-extending module is ⨁-s-extending.

Proposition (16)

Every direct summand A of ⨁-s-extending module such that the intersection of two direct summand in M is direct summand in N is ⨁-s-extending module.

Proof

Let A be a direct summand of ⨁-s-extending module M and let W be a closed submodule in A, so W is a submodule of M. Since A is a direct summand of M, so A is closed submodule of M and we have W is closed submodule of M, but M is ⨁-s-extending module. Thus by using proposition (2.3), W has a weakly supplement submodule of M which is direct summand then M=W+H and W∩H << M, where H is a direct summand of M. So A∩M=A∩(W+H) then (by modular law) A=W+(A∩H) and W∩(A ∩M)=(W∩H)∩A. Since (W∩H)∩A be a submodule of W∩H and W∩H≪ M. So, W∩(A∩H) ≪ M and since W∩(A∩H) be a submodule in A and A is a submodule in M. Hence W∩(A∩H) ≪ A, then W has a weakly supplement submodule in A, then by hypothesis (A∩H) is direct summand in A, then by proposition (3), A is ⨁-s-extending module.

Proposition (17)

Every fully invariant direct summand A of ⨁-s-extending module M is ⨁-s-extending.

Proof

Let A be a direct summand fully invariant of ⨁-s-extending module M and let L be a closed submodule in A, so L is closed in M. But M is⨁-s-extending. Then L has a weakly supplement W which is direct summand in M, so M=W⊕W', M=W+L and W∩L≪M. Now A=A∩M=A∩(W+L) by (modular law) A=L+(A∩W), but A=(A∩ W) ⊕ $(A \cap W')$. So A∩ W is direct summand in A and L∩ $(A \cap W)$= W∩L≪M, since A is direct summand in M. Hence A∩ W is weakly supplement of L in N. Thus A is ⊕-s-extending.

Recall that, if for each closed submodules F, G and S in a module M such that F∩(G+S)= F∩G+F∩S, then a module M is said to be local distributive [10].

A direct sum of ⊕-s-extending module need not necessary ⊕-s-extending. In fact, for example. M=Z⊕Z_2 as Z-module is not ⊕-s-extending (since it is not weakly supplement extending module, while Z and Z_2 are ⊕-s-extending. We give the condition so we get the proposition:

Proposition (18)

Let M=M_1⊕ M_2 where $M_1$and M_2 are ⊕-s-extending such that M is local distributive module. Then M is ⊕-s-extending module.

Proof

Let F be a closed submodule in M. To prove F∩M_i is closed in M_i, since M is local distributive module, then we have F=((F∩M_1) ⊕ (F∩M_2)). Hence F∩M_1 is closed in M_1 and F∩M_2 is closed in M_2. But M_1 and M_2 are ⊕-s-extending module. Then there exists a weakly supplement submodule G_1 of M_1, G_2 of M_2 and G_1, G_2 are direct summand such that G_1+(F∩M_1)= M_1 and $G_2 + (F \cap M_2) = M_2$, $G_1 \cap (F \cap M_1)$ = (G_1∩F) ≪ M_1 and G_2∩(F∩M_2)=(G_2∩F)≪ M_2. Now M= M_1⊕M_2 = (G_1+(F∩M_1)) ⊕ (G_2+(F∩M_2))=(G_1 ⊕G_2)+F. Then M = (G_1⊕ G_2) + F and (G_1⊕G_2)∩F = (G_1∩F)⊕(G_2∩F)≪(F∩M_1) ⊕(F∩M_2) ≪ M_1⊕ M_2 ≪ M. We have G_1 ⊕G_2 is direct summand of M. Then M is ⊕-s-extending module.

Recall that, a module M is distributive if for any submodule F, G and S in M such that F∩(G+S)= F∩G+F∩S. Following [10]. Every distributive module is local distributive. Thus, we have this corollary:

Corollary (19)

Let M=M_1⊕ M_2 where $M_1$and M_2 are ⊕-s-extending module such that M is a distributive module. Then M is ⊕-s-extending.

2. Conclusions

We proved the following results:

1. A module M is ⊕-s-extending if and only if each submodule in M is essential in submodule has a weakly supplement which is direct summand.
2. A module M is ⊕-s-extending if and only if each closed submodule in M has a (weakly) supplement which is direct summand.
3. A module M is ⊕-s-extending if and only if M is closed ⊕-supplemented.
4. We obtain the following result: Extending modules $\rightleftarrows$ ⊕-s-extending modules $\rightleftarrows$ weakly supplement extending modules.

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