An Efficient Shrunken Estimators For The Mean Of Normal Population With Known Variance

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Abstract

This article considers a shrunken estimator of Al-Hermyari and Al-Goburi (1) to estimate the mean (θ) of a normal distribution N(θ, σ^2) with known variance (σ^2), when a guess value ($\theta_\circ$) is available about the mean (θ) as an initial estimate. This estimator is shown to be more efficient than the classical estimators especially when θ is close to $\theta_\circ$. General expressions for bias and MSE of considered estimator are given, with some examples. Numerical results, comparisons and conclusions are reported.

Introduction

Let X be a random variable that has a normal distribution N(θ, σ^2) when θ is an unknown parameter and σ^2 is a known parameter, and when a prior information ($\theta_\circ$) available about the mean based on any one of the following reasons (2)

e.g.:

(i) we believe $\theta_\circ$ is close to the true value θ, or

(ii) we fear that $\theta_\circ$ may be near the true of θ,

i.e, Something bad happens if $\theta \approx \theta_\circ$, and we do not know about it.

In such a situation it is natural to start with the MLE ($\hat{\theta}$) of (θ) and modify it by moving it closer to $\theta_\circ$ so that the resulting estimator, though perhaps biased has a smaller mean squared error (MSE) than that of $\hat{\theta}$ in some interval around $\theta_\circ$.

Several authors studied shrunken estimators for the mean (θ) of a normal distribution (see, e.g. Thompson(2,3), Mehta and Srinivasan (4), Kambo, Handa and Al-Hemyari (5) and others).

In this paper we have studied a shrunken estimator for the mean (θ) of a normal distribution N(θ, σ^2) with known σ^2, which has the following form:

$$\tilde{\theta} = \begin{cases} \psi(\hat{\theta})(\hat{\theta}-\theta_0)+\theta_0 & , If\ \hat{\theta} \in R, \\ (1-\psi(\hat{\theta}))(\hat{\theta}-\theta_0)+\theta_0 & , If\ \hat{\theta} \notin R, \end{cases} \quad ....[1]$$

where $0 \le \psi(\hat{\theta}) \le 1$ is a shrinkage weight function that may be constant or a function of $\hat{\theta}$, and R be a suitable region of $\hat{\theta}$ depending on θ_0 (e.g. pre test region).

General expressions for the bias and Mean squared error of considered estimator ($\tilde{\theta}$) defined in [1] are given for any shrinkage weight function $\Psi(.)$ and for any region R. Examples of $\Psi(.)$, R and numerical results are given and compared with known estimators.

Expressions for Bias and Mean Squared Error of Estimator $\tilde{\theta}$

General expressions for bias and MSE of a shrunken estimator $\tilde{\theta}$ for any shrinkage weight function $\Psi(.)$ and any region R can be expressed as:

$$B(\tilde{\theta}/\theta,R) = \{\int_R (2\psi(\hat{\theta})-1)(\hat{\theta}-\theta_0) + \int_{-\infty}^{\infty}[(1-\psi(\hat{\theta}))(\hat{\theta}-\theta_0)+(\theta_0-\theta)]\} f(\hat{\theta}/\theta)d\hat{\theta} ....[2]$$

$$\text{MSE}(\tilde{\theta}/\theta,R) = \{\int_R [(2\psi(\hat{\theta})-1)(\hat{\theta}-\theta_0)^2 + 2(\theta_0-\theta)(2\psi(\hat{\theta})-1)(\hat{\theta}-\theta_0)] +$$

$$\int_{-\infty}^{\infty}[(1-\psi(\hat{\theta}))^2(\hat{\theta}-\theta_0)^2 + 2(\theta_0-\theta)(1-\psi(\hat{\theta}))(\hat{\theta}-\theta_0)+(\theta_0-\theta)^2]\} f(\hat{\theta}/\theta)d\hat{\theta}, ...[3]$$

where $f(\hat{\theta}/\theta)$ is a p.d. f. of $\hat{\theta}$

Shrunken Estimators $\tilde{\theta}_1$ and $\tilde{\theta}_2$

In this section we assumed two types of shrunken estimator ($\tilde{\theta}$) defined in [1] by taken two different shrinkage weight function $\Psi(.)$ to estimate the mean (θ) of a normal distribution N(θ, σ^2)

with known variance, and we assumed the pre- test region R of level of significance α, such that:

$$R=\{\hat{\theta}, T(\hat{\theta}) \in [L_{1-\alpha/2}, U_{\alpha/2}]\}, \quad ....[4]$$

where $L_{1-\alpha/2}$ and $U_{\alpha/2}$ are the lower and upper $100(\alpha/2)$ percentile points of the test statistic T used for testing $H_\circ : \theta = \theta_\circ$ against $H_1 : \theta \neq \theta_\circ$

Let us take $\psi_1(\hat{\theta}) = \sqrt{w}/c$ as a first type, where $w = n(\bar{x} - \theta_\circ)^2/\sigma^2$, $c = Z_{\alpha/2}$ and $\psi_2(\hat{\theta}) = a/b$ as the second, where a and b are natural numbers such that $(a \leq b)$.

Now, the shrunken estimators $\tilde{\theta}_1$ and $\tilde{\theta}_2$, using [1], are respectively given by:

$$\tilde{\theta}_1 = \begin{cases} \frac{\sqrt{w}}{c}(\bar{X} - \theta_\circ) + \theta_\circ & , \text{ if } \bar{X} \in R, \\ (1 - \frac{\sqrt{w}}{c})(\bar{X} - \theta_\circ) + \theta_\circ & , \text{ if } \bar{X} \notin R, \end{cases} \quad ...[5]$$

and,

$$\tilde{\theta}_2 = \begin{cases} a(\bar{X} - \theta_\circ)/b + \theta_\circ & , \text{ if } \bar{X} \in R, \\ (1 - a/b)(\bar{X} - \theta_\circ) + \theta_\circ, & \text{ if } \bar{X} \notin R, \end{cases} \quad ...[6]$$

By using equation [2], the expressions for bias of $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are respectively given as follows:-

$$B(\tilde{\theta}_1 / \theta, R) = (\sigma/\sqrt{n})\{(2/c)[J_2(\ell,u) - 2\lambda J_1(\ell,u) + \lambda^2 J_\circ(\ell,u)] - [J_1(\ell,u) - \lambda J_\circ(\ell,u)] + (1+\lambda^2)/c\}, ...[7]$$

and,

$$B(\tilde{\theta}_2 / \theta, R) = (\sigma/\sqrt{n})\{\{(2a/b - 1)[J_1(\ell,u) - \lambda J_\circ(\ell,u)] + (a/b)\lambda\}\}......[8]$$

Using equation [3], the expressions for MSE of $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are respectively given by:-

$$MSE(\tilde{\theta}_1 / \theta, R) = (\sigma^2 / n)\{(4\lambda / c)[J_2(\ell,u) - 2\lambda J_1(\ell,u) + \lambda^2 J_0(\ell,u)]$$
$$+ 2\lambda[J_1(\ell,u) - \lambda J_0(\ell,u)]$$
$$+ (2/c)[J_3(\ell,u) - 3\lambda J_2(\ell,u) + 3\lambda^2 J_1(\ell,u) - \lambda^3 J_0(\ell,u)]$$
$$+ [1 + (3 + 6\lambda^2 + \lambda^4)/c^2 - 2(3\lambda + \lambda^3)/c + (2\lambda / c^2)(1 + \lambda^2)\} \ldots\ldots [9]$$

and,

$$MSE(\tilde{\theta}_2 / \theta, R) = (\sigma^2 / n)\{(2a/b - 1)[J_2(\ell,u) - \lambda^2 J_0(\ell,u)] + (1 - a/b)^2 + (a\lambda / b)^2\} \ldots [10]$$

It is possible, for example to take b=1 and find the value of (a) by minimizing the $\text{MSE}(\tilde{\theta}_2 / \theta)$ with respect to a; therefore the value of (a) is as:

$$a = \frac{MSE(\bar{X}/\theta) - (\theta_0 - \theta)B(\bar{X}/\theta) - \int_R [(\bar{X}/\theta)^2 + (\theta_0 - \theta)^2] f(\bar{X}/\theta)d\bar{x}}{MSE(\bar{X}/\theta) - 2(\theta_0 - \theta)B(\bar{X}/\theta) + (\theta_0 - \theta)^2} \ldots\ldots [11]$$

by simple calculation,

$$a = \frac{1 - J_2(\ell,u) + \lambda^2 J_0(\ell,u)}{1 + \lambda^2} \ldots\ldots [12]$$

where:

$$\lambda = \sqrt{n}(\theta_0 - \theta)/\sigma, \quad \ell = \lambda - c, \quad u = \lambda + c, \quad z = \sqrt{n}(\bar{x} - \theta)/\sigma$$

and

$$J_r(\ell,u) = \int_{\ell}^{u} Z^r \exp(-Z^2/2)dz, r = 0,1,2, \quad \ldots\ldots [13]$$

Conclusons and Numerical Results

In this section some conclusions and numerical results concerning $\tilde{\theta}_1$ and $\tilde{\theta}_2$ will be presented as follows:

(1) $MSE(\tilde{\theta}_i / \theta, R)$ is an even function of λ, where $\lambda = \sqrt{n}(\theta_0 - \theta)/\sigma$, for i = 1,2.

(2) $\lim_{n \to \infty} MSE(\tilde{\theta}_i / \theta, R) = 0$, then $\tilde{\theta}_i$ (i=1,2) are consistent estimators.

(3) The bias, mean squared error, and relative efficiency $[R.Eff = MSE(\hat{\theta}) / MSE(\tilde{\theta}_i)]$ of estimators $\tilde{\theta}_i$ (i=1,2) with

region (R) were computed for different values of λ which is involved in these estimators, The following results based on these computations:-

(i) Numerical computations are performed by taken α =0.002, 0.01, 0.02, 0.05, 0.08 , ... and λ= 0.0 (0.1) 2.0.

(ii) In tables (1) and (2), some sample values of R. efficiency relative to $\bar{x}$ and bias ratio $\frac{\sqrt{n}}{\sigma} B(\tilde{\theta}_i / \theta)$ (shown in parenthesis) of $\tilde{\theta}_i$ (i=1,2) with λ are given for some selected values of α. It was observed that generally R. Eff ($\tilde{\theta}$ i) increases as α decreases and decreases as λ increases; Therefore, for each α, the estimators $\tilde{\theta}_i$ (i= 1,2) have higher relative efficiency when λ =0. The relative efficiency of $\tilde{\theta}_2$ increased with the increasing of b for each α and λ.

(4) The considered estimators $\tilde{\theta}$i (i= 1,2) have higher relative efficiency than the classical estimator and than the known estimators which are considered by Thompson(1), Mehta and Srinivasan(4) and Hirano(6).

Referernces

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Table (1): R. Eff ($\tilde{\theta}_1/\theta$) and B ($\tilde{\theta}_1/\theta$)

α	λ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.002	R.Ef B	7.21 (0.0)	6.21 (.05)	5.05 (.15)	4.02 (.22)	3.19 (.29)	2.55 (.35)	2.07 (.42)	1.71 (.49)	1.42 (.56)	1.19 (.62)	1.12 (.68)
0.01	R.Ef B	4.83 (0.0)	4.33 (.06)	3.70 (0.14)	3.06 (.21)	2.49 (.28)	2.02 (.35)	1.64 (.42)	1.34 (.49)	1.09 (.56)	0.89 (.64)	0.80 (.72)
0.05	R.Ef B	2.39 (0.0)	2.11 (.08)	1.76 (.17)	1.42 (.26)	1.11 (.35)	0.87 (.44)	0.68 (.54)	0.53 (.65)	0.41 (.77)	0.32 (.90)	0.25 (1.04)
0.1	R.Ef B	1.14 (0.0)	1.0 (.11)	0.82 (.22)	0.65 (.34)	0.51 (.46)	0.39 (.59)	0.31 (.74)	0.24 (.89)	0.18 (1.06)	0.14 (1.24)	0.10 (1.45)

Table (2): R.Eff ($\tilde{\theta}_1/\theta$) and B ($\tilde{\theta}_1/\theta$)

α	λ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.002	R.Ef B	30.80 (0.0)	16.21 (.09)	9.38 (.18)	6.16 (.26)	4.47 (.33)	3.48 (.37)	2.88 (.41)	2.48 (.43)	2.21 (.45)	2.01 (.46)	1.82 (.47)
0.01	R.Ef B	11.59 (0.0)	8.79 (.08)	6.16 (.26)	4.85 (.23)	3.93 (.29)	3.18 (.33)	2.74 (.36)	2.44 (.39)	2.23 (.41)	2.07 (.42)	1.70 (.43)
0.05	R.Ef B	4.83 (0.0)	4.47 (.06)	4.03 (.12)	3.60 (.17)	3.24 (.22)	2.94 (.26)	2.70 (.30)	2.50 (.34)	2.32 (.38)	2.14 (.41)	1.62 (.42)
0.1	R.Ef B	4.01 (0.0)	3.87 (.05)	3.67 (.10)	3.45 (.15)	3.22 (.20)	3.00 (.25)	2.78 (.29)	2.56 (.35)	2.34 (.37)	2.12 (.40)	1.53 (.41)

مقدرات التقلص الكفوءة لمتوسط التوزيع الطبيعي عندما يكون التباين معلوماً

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الخلاصة

في هذا البحث درس مقدر التقلص ذو المرحلة الواحدة للباحثين Al- Hemyari, and Al- Goburi [1]، لتقدير الوسط الحسابي (θ) للتوزيع الطبيعي $N(\theta, \sigma^2)$ عندما يكون التباين (σ^2) معلوماً بافتراض توافر المعلومات المسبقة (θ_0) بشكل تقديرات اولية حول الوسط الحسابي (θ). اشتقت الصيغة العامة للتحيز (bias) ومتوسط مربعات الخطأ (MSE) لهذا المقدر لمختلف دوال التقلص الموزونة $\psi(.)$ ومختلف المجالات (R)، ثم اقترحت بعض الامثلة واعطيت النتائج العددية بتناول مختلف الثوابت التي تضمنتها. قورنت المقدرات المقترحة في هذا البحث مع المقدرات الكلاسيكية والمشابهة وبينت كفاءتها.