Revista Colombiana de Estadística
Nº 25 - 26, págs. 33 - 46

MAXIMUM-LIKELIHOOD ESTIMATION OF THE 4-PARAMETER GENERALIZED WEIBULL DISTRIBUTION

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Abstract

Weibull distribution is being increasingly employed by researches in technology, medicine and other areas. Its generations to four parameters have been proposed, independently, by Stacy (1962) and Cohen (1969). In this paper, we obtain the maximum-likelihood estimators of the parameters of generalized Weibull distribution (g.w.d). The variance-covariance matrix is derived. We also consider the special cases when the threshold parameter and/or a shape parameter are known.

Key Words: Generalized Weibull distribution; Generalized gamma distribution; Maximum-Likelihood estimators; Digamma and trigamma functions; Variance-covariance matrix; Mellin transform.

1. Introduction

The Weibull distribution, first introduced in the literature by a Swedish physicist, Waloddi Weibull (1939, 1951) is being extensively used to fit a rather large class of data arising in fatigue and failure analysis, life-testing and reliability studies. Earlier, the researches confined themselves to techniques of estimation in 2 and 3 parameter distributions and properties of the estimators; for ex-

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ample, Kao (1959), Menon (1963), Cohen (1965), Dubey (1965, 1967), Harter and Moore (1965, 1967), Mann (1968), Bain and Antle (1967, 1970), Lawless (1978), Schneider and Weissfeld (1989).

In this paper, we consider the generalization of the Weibull distribution to include four parameters by Cohen (1969). The generalized Weibull distribution (g.w.d.) is very versatile, since several standard and non-standard distributions fall through as special cases, Arora (1973). For instance,

$\tau = p$: pth root Gamma (p > 0 and a +ve integer)
$\tau = 1$: Pearson Type III
$\tau = 1, \alpha = 0$: Type X or 2-parameter Exponential
$\tau = 2, \alpha = -\frac{1}{2}$: 2-parameter Half-Normal
$\tau = 2, \alpha = -\frac{1}{2}, \beta = 1, \gamma = 0$: Half-Normal
$\tau = 1, \gamma = 0, \beta = 2$: Chi-square with $2(\alpha + 1)$ d.f.

We discuss parameter estimation using the maximum likelihood technique. It is well known that the estimation equations do not yield closed form solutions for the estimators. Some form of iterative technique(s) must, therefore, be employed. The variance-covariance matrix of the estimators is derived. The cases when the location parameter is zero or is known and additionally when one of the shape parameters is known are discussed.

2. Maximum-Likelihood Estimators

We consider the 4-parameter generalized Weibull distribution (g.w.d), derived by Cohen (1969):

$$f(x;\alpha,\beta,\gamma,\tau) = \begin{cases} k(x-\gamma)^{\tau\alpha+\tau-1}\exp\left\{-(x-\gamma)^{\tau}/\beta\right\}, & x > \gamma \\ 0 ; & \text{elsewhere} \end{cases} \tag{2.1a}$$

where α, τ are the two shape parameters, β is the scale parameter and γ is the location, or threshold, parameter $\alpha > -1$, $\tau > 0$, $\beta > 0$, $-\infty < \gamma < \infty$ and

$$k = \tau\beta^{-\alpha-1}\left(\Gamma(\alpha+1)\right)^{-1} \tag{2.1b}$$

If $x_1, x_2, \ldots x_n$ are the values of a random sample of size n from the g.w.d (2.1a); the likelihood function is given by

$$L(\alpha,\beta,\gamma,\tau) = k^n \exp\left\{-\sum (x_i-\gamma)^\tau/\beta\right\} \prod (x_i-\gamma)^{\tau\alpha+\tau-1} \; ; \; x_i > \gamma \qquad (2.2)$$

and

$$\ln L(\alpha,\beta,\gamma,\tau) = n \ln\tau - n(\alpha+1)\ln\beta - n\ln(\Gamma(\alpha+1)) - \sum(x_i-\gamma)^\tau/\beta + (\tau\alpha+\tau-1)\sum \ln(x_i-\gamma). \qquad (2.3)$$

Denoting by G, H, J and K, the partial derivatives of $\ln L(\alpha,\beta,\gamma,\tau)$ with respect to α, β, γ and τ respectively, the m.l. estimators $\hat\alpha$, $\hat\beta$, $\hat\gamma$ and $\hat\tau$ must satisfy the following system of equations:

$$G \equiv -n\ln\hat\beta - n\,\psi(\hat\alpha) + \hat\tau\sum\ln(x_i-\hat\gamma) = 0 \qquad (2.4a)$$

$$H \equiv -n\hat\beta^{-1}(\hat\alpha+1) + \hat\beta^{-2}\sum(x_i-\hat\gamma)^{\hat\tau} = 0 \qquad (2.4b)$$

$$J \equiv \hat\tau\hat\beta^{-1}\sum(x_i-\hat\gamma)^{\hat\tau-1} - (\hat\tau\hat\alpha+\hat\tau-1)\sum(x_i-\hat\gamma)^{-1} = 0 \qquad (2.4c)$$

and

$$K \equiv n\hat\tau^{-1} - \hat\beta^{-1}\sum(x_i-\hat\gamma)^{\hat\tau}\ln(x_i-\hat\gamma) + (\hat\alpha+1)\sum\ln(x_i-\hat\gamma) = 0 \qquad (2.4d)$$

where $\Psi(\alpha)$ is the digamma (or psi) function defined in Abramowitz and Stegun (1964) as

$$\psi(\alpha) = \frac{\partial(\ln(\Gamma(\alpha+1)))}{\partial\alpha}. \qquad (2.4e)$$

The system of non-linear equations (2.4a) - (2.4d) does not yield explicit solutions for the m.l. estimators $\hat\alpha$, $\hat\beta$, $\hat\gamma$ and $\hat\tau$. Some iterative techniques must, therefore, be used. Harter (1967), for instance, found that a hybrid procedure using the rule of false position, the Newton-Raphson technique, and the gradient method provides best results for estimating parameters of a generalized Gamma distribution.

If the value $\hat\gamma$ satisfying the equations (2.4a) - 2.4d) is such that $\hat\gamma > \min(x_1, x_2, x_n)$ then $\hat\gamma$ is the m.l estimator for γ Otherwise, the m.l. estimator is $\hat\gamma >$

$\min(x_1, x_2, \ldots, x_n)$ and we solve (2.4a), (2.4b) and (2.4d) iteratively for $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\tau}$ with $(x_i - \gamma)$ replaced by $(x_i - \min(x_1, x_2, \ldots, x_n))$.

The problem can somewhat be simplified by noting that equation (2.4b) yields

$$\hat{\beta} = \frac{\Sigma(x_i - \hat{\gamma})^{\hat{\tau}}}{n(\hat{\alpha}+1)}. \tag{2.5}$$

which, in turn, implies solving three non-linear equations (2.6a) - (2.6c) for $\hat{\alpha}$, $\hat{\gamma}$ and $\hat{\tau}$:

$$\psi(\hat{\alpha}) - \ln(n(\alpha+1)) = \hat{\tau} n^{-1} \Sigma \ln(x_i - \hat{\gamma}) - \ln \Sigma (x_i - \hat{\gamma})^{\hat{\tau}}, \tag{2.6a}$$

$$(\hat{\alpha}+1-\hat{\tau}^{-1}) \Sigma (x_i - \hat{\gamma})^{-1} = n(\hat{\alpha}+1) \left[\Sigma (x_i - \hat{\gamma})^{\hat{\tau}-1}\right] \left[\Sigma (x_i - \hat{\gamma})^{\hat{\tau}}\right]^{-1}, \tag{2.6b}$$

$$\hat{\tau} = (\hat{\alpha}+1)^{-1} \left[\left(\Sigma (x_i - \hat{\gamma})^{\hat{\tau}} \ln(x_i - \hat{\gamma})\right)\left(\Sigma (x_i - \hat{\gamma})^{\hat{\tau}}\right)^{-1} - n^{-1} \Sigma \ln(x_i - \hat{\gamma})\right]^{-1}. \tag{2.6c}$$

The values $\hat{\alpha}$, $\hat{\gamma}$ and $\hat{\tau}$ thus obtain are used in (2.5) to obtain $\hat{\beta}$.

3. Variance-covariance matrix of the m.l. estimators

To obtain the variance-covariance matrix, we calculate the following partial derivatives of G, H, J and K:

$$\frac{\partial G}{\partial \alpha} \equiv G_\alpha = -n\,\psi'(\alpha) \tag{3.1a}$$

$$\frac{\partial G}{\partial \beta} \equiv G_\beta = -n\,\beta^{-1} = \frac{\partial H}{\partial \alpha} \equiv H_\alpha \tag{3.1b}$$

$$\frac{\partial G}{\partial \gamma} \equiv G_\gamma = -\tau \Sigma (x_i - \gamma)^{-1} = \frac{\partial J}{\partial \alpha} \equiv J_\alpha \tag{3.1c}$$

$$\frac{\partial G}{\partial \tau} = G_\tau = \sum \ln(x_i - \gamma) \qquad = \frac{\partial K}{\partial \alpha} = K_\alpha \tag{3.1d}$$

$$\frac{\partial H}{\partial \beta} = H_\beta = n\beta^{-2}(\alpha+1) - 2\beta^{-3} \sum (x_i - \gamma)^\tau \tag{3.1e}$$

$$\frac{\partial H}{\partial \gamma} = H_\gamma = -\tau\beta^{-2} \sum (x_i - \gamma)^{\tau-1} \qquad = \frac{\partial J}{\partial \beta} = J_\beta \tag{3.1f}$$

$$\frac{\partial H}{\partial \tau} = H_\tau = \beta^{-2} \sum (x_i - \gamma)^\tau \ln(x_i - \gamma) \qquad = \frac{\partial K}{\partial \beta} = K_\beta \tag{3.1g}$$

$$\frac{\partial J}{\partial \gamma} = J_\gamma = -\tau(\tau-1)\beta^{-1} \sum (x_i - \gamma)^{\tau-2} - (\tau\alpha+\tau-1) \sum (x_i - \gamma)^{-2} \tag{3.1h}$$

$$\frac{\partial J}{\partial \tau} = J_\tau = \beta^{-1} \sum (x_i - \gamma)^{\tau-1} + \tau\beta^{-1} \sum (x_i - \gamma)^{\tau-1} \ln(x_i - \gamma) - (\alpha+1) \sum (x_i - \gamma)^{-1} \tag{3.1i}$$

$$= \frac{\partial K}{\partial \gamma} = K_\gamma$$

$$\frac{\partial K}{\partial \tau} = K_\tau = -n\tau^{-2} - \beta^{-1} \sum (x_i - \gamma)^\tau \ln^2(x_i - \gamma) \tag{3.1j}$$

$\Psi'(\alpha)$ in (3.1a) is the trigamma function defined in Abramowitz and Stegun (1964) as

$$\psi'(\alpha) = \frac{\partial^2(\ln(\Gamma(\alpha+1)))}{\partial \alpha^2} = \frac{\partial \psi(\alpha)}{\partial \alpha}. \tag{3.1k}$$

We recall that under certain regularity conditions, (Kendall and Stuart, (1961)), the joint m.l. estimators tend to a multivariate normal distribution, with variance-covariance matrix whose inverse is given by

$$V_{rs}^{-1} = -E\left(\frac{\partial^2 \log L}{\partial \theta_r \partial \theta_s}\right) = E\left(\frac{\partial \log L}{\partial \theta_r} \cdot \frac{\partial \log L}{\partial \theta_s}\right). \tag{3.2}$$

Thus the inverse dispersion matrix of joint m.l. estimators of the g.w.d will be the form

$$V^{-1} = -E\begin{vmatrix} G_\alpha & G_\beta & G_\gamma & G_\tau \\ G_\beta & H_\beta & H_\gamma & H_\tau \\ G_\gamma & H_\gamma & J_\gamma & J_\tau \\ G_\tau & H_\tau & J_\tau & K_\tau \end{vmatrix} \qquad (3.3)$$

where G_α, G_β, ..., K_τ are given in (3.1a) - (3.1j) above. The necessary expected values are calculated in Appendix I and (3.3) can then be written as

$$V^{-1} = \left[V^{ij}\right]_{\substack{i=1,2,3,4 \\ j=1,2,3,4}} \qquad (3.4)$$

where

$$V^{11} = n\psi'(\alpha) \qquad (3.4a)$$

$$V^{12} = n\beta^{-1} \qquad = V^{21} \qquad (3.4b)$$

$$V^{13} = n\tau\beta^{-1/\tau}\Gamma(\alpha+1-\tau^{-1})(\Gamma(\alpha+1))^{-1} \qquad = V^{31} \qquad (3.4c)$$

$$V^{14} = n\tau^{-1}\left[\ln\beta + \psi(\alpha)\right] \qquad = V^{41} \qquad (3.4d)$$

$$V^{22} = n\beta^{-2}(\alpha+1) \qquad (3.4e)$$

$$V^{23} = n\tau\beta^{-1-1/\tau}\Gamma(\alpha+2-\tau^{-1})(\Gamma(\alpha+1))^{-1} \qquad = V^{32} \qquad (3.4f)$$

$$V^{24} = -n\beta^{-1}\tau^{-1}(\alpha+1)\left[\ln\beta + \psi(\alpha+1)\right] \qquad = V^{42} \qquad (3.4g)$$

$$V^{33} = n\beta^{-2/\tau}\Gamma(\alpha+1-2\tau^{-1})(\Gamma(\alpha+1))^{-1}(\tau^2(\alpha+1)-2\tau+1) \qquad (3.4h)$$

$$V^{34} = n\tau^{-1}\beta^{-1/\tau}\Gamma(\alpha+1-\tau^{-1})(\Gamma(\alpha+1))^{-1} - n\beta^{-1/\tau}\left[\ln\beta + \psi(\alpha+1-\tau^{-1})\right]\Gamma(\alpha+21-\tau^{-1})(\Gamma(\alpha+1))^{-1}$$

$$= V^{43} \qquad (3.4i)$$

$$V^{44} = n\tau^{-2} + n\tau^{-2}(\alpha+1)(\ln\beta)\left[\ln\beta + 2\psi(\alpha+1)\right] - n\tau^{-2}(\Gamma(\alpha+1))^{-1}\frac{\partial^2(\Gamma(\alpha+2))}{\partial\alpha^2}$$

$$(3.4j)$$

3.1 Case I: Assume γ is know

In many applications, it is possible to assume γ is zero or is know. In studying life-distributions, for example, it is often partial to assume $\gamma = 0$. Maximum likelihood estimates of α, β and τ can be found by solving iteratively (2.4a), (2.4b) and (2.4d). From (2.4b), with $\gamma = 0$, we obtain

$$\hat{\beta} = \frac{\sum x_i^{\hat{\tau}}}{n(\hat{\alpha}+1)} \, . \tag{3.5}$$

Substituting $\hat{\beta}$ in (2.4b) and (2.4d), the maximum-likelihood estimators $\hat{\alpha}$ and $\hat{\tau}$ of α and τ respectively, satisfy the equations (3.6a) and (3.6b):

$$\psi(\hat{\alpha}) - \ln(n(\hat{\alpha}+1)) = \hat{\tau} n^{-1} \sum \ln x_i - \ln \sum x_i^{\hat{\tau}}, \tag{3.6a}$$

$$\hat{\tau} = (\hat{\alpha}+1)^{-1} \left[\left(\sum x_i^{\hat{\tau}} \ln x_i \right) \left(\sum x_i^{\hat{\tau}} \right)^{-1} - n^{-1} \sum \ln x_i \right]^{-1} . \tag{3.6b}$$

Of course, if $\gamma \neq 0$, we replace each x_i in (3.5), (3.6a), and (3.6b) by $(x_i - \gamma)$.

Again, some hybrid of iterative techniques must be employed to obtain the values $\hat{\alpha}$, and $\hat{\tau}$ from the system (3.6a - 3.6b), which are subsequently substituted in (3.5) to get $\hat{\beta}$.

The asymptotic variance-covariance matrix of $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\tau}$ is the inverse of the matrix

$$n \begin{vmatrix} \psi'(\alpha) & \beta^{-1} & -\tau^{-1}\left[\ln\beta + \psi(\alpha)\right] \\ \beta^{-1} & \beta^{2}(\alpha+1) & -\beta^{-1}\tau^{-1}(\alpha+1)\left[\ln\beta+\psi(\alpha+1)\right] \\ -\tau^{-1}\left[\ln\beta+\psi(\alpha)\right] & -\beta^{-1}\tau^{-1}(\alpha+1)\left[\ln\beta+\psi(\alpha+1)\right] & \begin{array}{l} \tau^{-2}+\tau^{-2}(\alpha+1)(\ln\beta)\left[\ln\beta 2\psi(\alpha+1)\right] \\ +\tau^{-2}\left(\Gamma(\alpha+1)\right)^{-1} \dfrac{\partial^2(\Gamma(\alpha+2))}{\partial\alpha^2} \end{array} \end{vmatrix} \tag{3.7}$$

which, after some algebraic manipulations, reduces to

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} \tag{3.8}$$

where

$$a_{11} = \mathrm{Var}(\hat{\alpha}) = n^{-1}\,\Delta_{11}^{-1}\left(\Delta_{22}^{-1} + \alpha + 1\right), \tag{3.8a}$$

$$a_{12} = \mathrm{Cov}(\hat{\alpha}, \hat{\beta}) = -\,n^{-1}\,\beta\left[(1 + \Delta_{22}^{-1}\psi'(\alpha))\,\Delta_{11}^{-1} + (\ln\beta + \psi(\alpha))\,\Delta_{22}^{-1}\right], \tag{3.8b}$$

$$a_{13} = \mathrm{Cov}(\hat{\alpha}, \hat{\tau}) = -\,n^{-1}\,\tau\Delta_{22}^{-1}, \tag{3.8c}$$

$$a_{22} = \mathrm{Var}(\hat{\beta}) = n^{-1}\,\beta^2\Delta_{11}^{-1}\left[\psi'(\alpha) + \Delta_{22}^{-1}\left\{\beta^2\left(\ln\beta + \psi(\alpha)\right)\Delta_{11} + \psi'(\alpha)\right\}^2\right], \tag{3.8d}$$

$$a_{23} = \mathrm{Cov}(\hat{\beta}, \hat{\tau}) = n^{-1}\,\beta\tau\Delta_{22}^{-1}\left[\beta^2\left(\ln\beta + \psi(\alpha)\right)\Delta_{11} + \psi'(\alpha)\right], \tag{3.8e}$$

$$a_{33} = \mathrm{Var}(\hat{\tau}) = n^{-1}\,\tau^2\Delta_{22}^{-1}\,\Delta_{11}, \tag{3.8f}$$

and where

$$\Delta_{11} = \left(\psi'(\alpha)\right)(\alpha + 1) - 1, \tag{3.8g}$$

$$\Delta_{22} = \left(\psi'(\alpha)\right)^2(\alpha + 1)^2 - \psi'(\alpha) - 1. \tag{3.8h}$$

Using the approximation

$$\psi'(\alpha) \doteqdot 24\,n^{-1}\,\alpha^{-2} + \frac{1}{6}\alpha^{-3}, \tag{3.8 i}$$

we obtain, for large α,

$$\mathrm{Var}(\hat{\alpha}) \doteqdot 24\,n^{-1}\,\alpha^3, \tag{3.8 j}$$

$$\mathrm{Var}(\hat{\tau}) \doteqdot 6\,n^{-1}\,\tau^2\alpha, \tag{3.8 k}$$

and

$$\mathrm{Cov}(\hat{\alpha}, \hat{\tau}) \doteqdot -12\,n^{-1}\,\tau\alpha^2. \tag{3.8 l}$$

3.2 Case II: Assume γ and τ are known

We next examine the case when the location or threshold parameter g and one of the shape parameters, say τ, are known. Maximum likelihood estimates of α and β are the solutions of the equations (2.4a) and (2.4b).

From (2.4b), we have $\hat{\beta}$ as in (2.5). Substituting its value in (2.4a), we obtain the following equation for $\hat{\alpha}$:

$$\psi(\alpha) - \ln(n(\hat{\alpha} + 1)) = \hat{\tau} n^{-1} \sum \ln(x_i - \hat{\gamma}) - \ln \sum (x_i - \hat{\gamma})^{\hat{\tau}}. \tag{3.9}$$

The value $\hat{\alpha}$ is obtained from (3.9) and subsequently used in (2.5) to obtain $\hat{\beta}$.

The asymptotic variance-covariance matrix of $\hat{\alpha}$ and $\hat{\beta}$ is

$$n^{-1} \begin{vmatrix} \psi'(\alpha) & \beta^{-1} \\ \beta^{-1} & \beta^{-2}(\alpha+1) \end{vmatrix} = n^{-1} \Delta_{11}^{-1} \begin{vmatrix} \alpha+1 & -\beta \\ -\beta & \beta^2 \psi'(\alpha) \end{vmatrix} \tag{3.10}$$

Thus

$$\mathrm{Var}(\hat{\alpha}) \doteqdot n^{-1} \Delta_{11}^{-1} (\alpha+1), \tag{3.11a}$$

$$\mathrm{Var}(\hat{\beta}) \doteqdot n^{-1} \Delta_{11}^{-1} \beta^2 \psi'(\alpha), \tag{3.11b}$$

and

$$\mathrm{Cov}(\hat{\alpha}, \hat{\beta}) \doteqdot -n^{-1} \Delta_{11}^{-1} \beta, \tag{3.11c}$$

where Δ_{11} is defined in (3.8g).

Using the approximation (3.8i) for $\psi'(\alpha)$ for large α, we get

$$\mathrm{Var}(\hat{\alpha}) \doteqdot 2n^{-1} \alpha^2, \tag{3.12a}$$

$$\mathrm{Var}(\hat{\beta}) \doteqdot 2n^{-1} \beta^2, \tag{3.12b}$$

$$\mathrm{Cov}(\hat{\alpha}, \hat{\beta}) \doteqdot -2n^{-1} \beta\alpha. \tag{3.12c}$$

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Appendix 1

A1.1

$$E(X-\gamma)^m = k\int_\gamma^\infty (x-\gamma)^{\tau\alpha+\tau+m-1} \exp\left\{-(x-\gamma)^\tau/\beta\right\} dx$$

where k is defined in (2.1b).

We let

$$W = \beta^{-1}(X-\gamma)^\tau. \qquad \text{(A1.1a)}$$

It is then fairly straightforward to obtain

$$E(X-\gamma)^m = \beta^{m/\tau}\Gamma(\alpha + m\tau^{-1} + 1)\,(\Gamma(\alpha+1))^{-1}$$

whence

$$E\sum(X_i-\gamma)^m = n\beta^{m/\tau}\Gamma(\alpha+m\tau^{-1}+1)\,(\Gamma(\alpha+1))^{-1}. \qquad \text{(A1.1b)}$$

By letting $m = -1, -2\tau, \tau-1$ and $\tau-2$ in (A1.1b), we get some of the mathematical expectations required in (3.3).

A1.2

$$E\left(\ln(X-\gamma)\right) = k\int_\gamma^\infty (x-\gamma)^{\tau\alpha+\tau-1} \ln(x-\gamma) \exp\left\{-(x-\gamma)^\tau/\beta\right\} dx$$

where k, as before, is defined in (2.1b). Again using the transformation in (A1.1a), we get

$$E\left(\ln(X-\gamma)\right) = \tau^{-1}\ln\beta + (\tau\,\Gamma(\alpha+1))^{-1}\int_0^\infty (\ln w)\, w^\alpha e^{-w}\, dw. \qquad \text{(A1.2a)}$$

Using Mellin Transform, Erdelyi (1954), the integral in (A1.2a) can be evaluated as $\Gamma(\alpha+1)\,\psi(\alpha)$. Further algebraic manipulations give

$$E\left(\ln(X-\gamma)\right) = \tau^{-1}\,(\ln\beta + \psi(\alpha)) \qquad \text{(A1.2b)}$$

where $\psi(\alpha)$ is defined in (2.4e).

whence

$$E\left(\sum \ln (X_i - \gamma)\right) = n\tau^{-1} \, (\ln\beta + \psi(\alpha)). \tag{A1.2c}$$

A1.3

$E[(X - \gamma) \ln (X - \gamma)]$ can similarly be evaluated by making use of the transformation (A1.1a) and Mellin Transform. We have

$$E\left[(X - \gamma)^{\tau}\ln(X - \gamma)\right] = \beta\tau^{-1} \, (\alpha+1) \left[\ln\beta + \psi(\alpha + 1)\right],$$

whence

$$E\left[\sum (X_i - \gamma)^{\tau}\ln (X_i - \gamma)\right] = n\beta\tau^{-1} \, (\alpha+1)\left[\ln\beta + \psi(\alpha+1)\right]. \tag{A1.3a}$$

A1.4

Two more expectations are needed. Proceeding as above, we get

$$E\left[(X - \gamma)^{\tau-1} \ln(X - \gamma)\right] = \tau^{-1}\beta^{1-1/\tau}(\Gamma(\alpha+1))^{-1}\Gamma(\alpha+2-\tau^{-1})\left[\ln\beta + \psi(\alpha+1-\tau^{-1})\right]$$

whence

$$E\left[\sum(X_i-\gamma)^{\tau-1}\ln(X-\gamma)\right]=n\tau^{-1}\beta^{1-1/\tau}(\Gamma(\alpha+1))^{-1}\Gamma(\alpha+2-\tau^{-1})\left[\ln\beta+\psi(\alpha+1-\tau^{-1})\right]. \tag{A1.4a}$$

And

$$E\left[(X - \gamma)^{\tau} \ln^2(X - \gamma)\right] = \beta\tau^{-2}(\alpha+1)\ln^2\beta + 2\beta\tau^{-1}(\alpha+1)(\ln\beta)\psi(\alpha+1) + \beta\tau^{-2}(\Gamma(\alpha+1))^{-1}\frac{\partial^2(\Gamma(\alpha+2))}{\partial\alpha^2}$$

whence

$$E\left[\sum(X_i - \gamma)^{\tau} \ln^2(X_i - \gamma)\right] = n\beta\tau^{-2}(\alpha+1)(\ln\beta)\left[\ln\beta + 2\psi(\alpha+1)\right] + n\beta\tau^{-2}(\Gamma(\alpha+1))^{-1}\frac{\partial^2(\Gamma(\alpha+2))}{\partial\alpha^2}. \tag{A1.4b}$$