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Some Characterizations of the Exponential Distribution by Generalized Order Statistics, with Applications to Statistical Prediction Problem

Algunas caracterizaciones de la distribución exponencial mediante estadísticas de orden generalizado, con aplicaciones al problema de predicción estadística

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Abstract

Some new characterization properties of the exponential distribution based on two non-adjacent m-generalized order statistics (consequently m-dual generalized order statistics), $m \neq -1$, coming from two independent exponential distributions are derived. The result of this paper provides a beneficial strategy to predict the failure time of some survived components in a lifetime experiment by using the result of another independent lifetime experiment.

***Key words*:** characterization of distributions; Exponential distribution; generalized order statistics.

Resumen

Se derivan algunas propiedades de caracterización nuevas de la distribución exponencial basadas en dos estadísticas de orden generalizado m no adyacentes (en consecuencia, estadísticas de orden generalizado m dual), $m \neq -1$, procedentes de dos distribuciones exponenciales independientes. El resultado de este artículo proporciona una estrategia beneficiosa para predecir el tiempo de falla de algunos componentes sobrevividos en un experimento de por vida utilizando el resultado de otro experimento de por vida independiente.

***Palabras clave*:** caracterización de distribuciones; distribución Exponencial; estadísticas de orden generalizadas.

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1. Introduction

Kamps (1995) introduced the concept of generalized order statistics (GOSs) as a unified approach to a variety of models of ascendingly ordered random variables (RVs). The concept of dual GOSs, denoted by DGOSs, was introduced by Burkschat et al. (2003) as a parallel concept of GOSs to enable a common approach to descendingly ordered RVs. Burkschat et al. (2003) has shown that (cf. Theorem 3.3) there is a direct link between DGOSs and GOSs.

The subclasses $m-$GOSs and $m-$DGOSs of GOSs and DGOSs, respectively, contain many important models of ordered RVs such as ordinary order statistics (OOSs), lower and upper record values, $k-$records, sequential order statistics (SOSs) and type II censored OOSs. For any $1 \leq r \leq n$, the marginal probability density functions (PDFs) of the rth m-GOS $X(r,n;m,k)$ and m-DGOS $X^*(r,n;m,k)$, based on a continuous distribution function (DF) $F_X(x) = P(X \leq x)$ with a PDF $f_X(x)$, are given, respectively, by (cf. Kamps 1995 and Ahsanullah 2004)

$$f_{X(r,n;m,k)}(x) = \frac{C_{r-1}^{(n)}}{(r-1)!}\overline{F}_X^{\gamma_r^{(n)}-1}(x)\left[\frac{1-\overline{F}_X^{m+1}(x)}{m+1}\right]^{r-1} f_X(x),\ m \neq -1, \quad (1)$$

and

$$f_{X^*(r,n;m,k)}(x) = \frac{C_{r-1}^{(n)}}{(r-1)!}F_X^{\gamma_r^{(n)}-1}(x)\left[\frac{1-F_X^{m+1}(x)}{m+1}\right]^{r-1} f_X(x),\ m \neq -1,$$

where $\overline{F} = 1 - F$, $\gamma_r^{(n)} = k + (n-r)(m+1)$ and $C_{r-1}^{(n)} = \prod_{i=1}^{r}\gamma_i^{(n)},\ 1 \leq r \leq n$.

A characterization in statistics is a specific distributional property of a statistic that uniquely identify related parametric family of distributions. In statistical applications, the researchers usually want to verify whether the data that they are dealing with belong to a certain family of DFs. Therefore, the researchers have to rely on a characterization of the assumed distribution and check if the corresponding conditions are satisfied. Classical results in characterizations can be found in El-Adll (2018), Galambos & Kotz (1978), Nagaraja (2006), and Rao & Chanabhng (1998). Different results of characterization and its applications in terms of GOSs and DGOSs are derived by many authors. Among these authors are Arnold et al. (2008), Beutner & Kamps (2008), Castaño Martínez et al. (2012), Khan et al. (2012), Öncel et al. (2005), Samuel (2008), Shah Imtiyaz et al., (2014; 2015; 2018; 2020), Tavangar & Hashemi (2013), and Wesolowski & Ahsanullah (2004).

In this paper, we prove some new characteristic properties of the exponential DF $\exp(\alpha)$, with mean $\frac{1}{\alpha}$, $\alpha > 0$. The exponential distribution is prominent in life testing experiments and reliability problems. The exponential distribution is known for its memoryless property in the sense that the length of time the component has worked in the past does not affect its behavior in the future (the components with exponentially distributed lifetime have a constant failure rate). The result of this paper enables us to predict the time at which some survived components will have failed or to predict the mean failure time of unobserved lifetimes in a lifetime experiment by using the result of another independent lifetime experiment. Throughout this paper, "$X \overset{d}{=} Y$" means that the RVs X and Y have the same DFs and "$X \sim F$" means that the RV X has the DF F.

The rest of this paper is organized as follows. In Section 2, we reveal some characterization properties for the exponential distribution based on two non-adjacent $m-$GOSs (consequently $m-$DGOSs) from two independent exponential distributions. In Section 3, we use the results of Section 2 in an application of the prediction problem concerning the lifetime experiments.

2. Characterization Results

We assume that all the considered DFs are differentiable with respect to their arguments. Moreover, all the considered RVs are non-negative.

Theorem 1. *Let $X(r, n; m, k),\ m \neq -1$, be the rth $m-$GOS from a sample of size n drawn from a continuous DF $F_X(x)$ with PDF $f_X(x)$. Furthermore, let $Y(r, n; m, k),\ \ m \neq -1$, be the rth $m-$GOS based on a sample of size n, which is drawn from a continuous DF $F_Y(y) = P(Y \leq y)$, where Y is independent of X. Finally, let the relation*

$$X(n_1 - n_2 + r, n_1; m, k) \overset{d}{=} X(r, n_2; m, k) + \tilde{Y} \tag{2}$$

be satisfied for all $1 \leq r < n_2 < n_1$. Then, $\tilde{Y} \overset{d}{=} Y(n_1 - n_2, n_1; m, k)$ and $Y \sim \exp(\alpha)$ if and only if $X \sim \exp(\alpha),\ \alpha > 0$.

Proof. We first prove the necessity part. Let the moment generating function (MGF) of $X(r, n_2; m, k)$ be $M_{X(r,n_2;m,k)}(t)$. Then, (2) implies that

$$M_{X(n_1-n_2+r,n_1;m,k)}(t) = M_{X(r,n_2;m,k)}(t) M_{\tilde{Y}}(t). \tag{3}$$

Let us now derive the MGF of the rth $m-$GOS $X(r, n; m, k)$ based on $\exp(\alpha)$. Clearly, in view of (1), we get

$$M_{X(r,n;m,k)}(t) = \frac{\alpha C_{r-1}^{(n)}}{(m+1)^{r-1}(r-1)!} \int_0^{\infty} e^{-x(\alpha\gamma_r^{(n)} - t)} \left(1 - e^{-\alpha(m+1)x}\right)^{r-1} dx,$$

which by using the transformation $y = e^{-\alpha(m+1)x}$ takes the form

$$\begin{aligned}
M_{X(r,n;m,k)}(t) &= \frac{C_{r-1}^{(n)}}{(m+1)^r(r-1)!} \int_0^1 y^{\left(\frac{\gamma_r^{(n)}}{m+1} - \frac{t}{\alpha(m+1)}\right)-1} (1-y)^{r-1}\, dy \\
&= \frac{C_{r-1}^{(n)} \Gamma\left(\frac{\gamma_r^{(n)}}{m+1} - \frac{t}{\alpha(m+1)}\right)}{(m+1)^r \Gamma\left(\frac{\gamma_r^{(n)}}{m+1} - \frac{t}{\alpha(m+1)} + r\right)} \\
&= \prod_{i=1}^{r} \left(\frac{\frac{\gamma_i^{(n)}}{m+1}}{\frac{\gamma_r^{(n)}}{m+1} - \frac{t}{\alpha(m+1)} + r - i} \right) = \prod_{i=1}^{r} \left(1 - \frac{t}{\alpha\gamma_i^{(n)}}\right)^{-1}, \quad (4)
\end{aligned}$$

where $\Gamma(.)$ is the usual gamma function. On the other hand, in view of (3) and by using the relations $C^{(n_1)}_{n_1-n_2+r-1} = C^{(n_2)}_{r-1} C^{(n_1)}_{n_1-n_2-1}$ and $\gamma^{(n_1)}_{n_1-n_2+r} = \gamma^{(n_2)}_{r}$, we get

$$\begin{aligned}
M_{\tilde{Y}}(t) &= \frac{M_{X(n_1-n_2+r,n_1;m,k)}(t)}{M_{X(r,n_2;m,k)}(t)} = \frac{C^{(n_1)}_{n_1-n_2-1}}{(m+1)^{n_1-n_2}} \times \frac{\Gamma\left(\frac{\gamma^{(n_2)}_r}{m+1} - \frac{t}{\alpha(m+1)} + r\right)}{\Gamma\left(\frac{\gamma^{(n_2)}_r}{m+1} - \frac{t}{\alpha(m+1)} + n_1 - n_2 + r\right)} \\
&= \frac{C^{(n_1)}_{n_1-n_2-1}}{(m+1)^{n_1-n_2}} \times \frac{\Gamma\left(\frac{\gamma^{(n_1)}_{n_1-n_2}}{m+1} - \frac{t}{\alpha(m+1)}\right)}{\Gamma\left(\frac{\gamma^{(n_1)}_{n_1-n_2}}{m+1} - \frac{t}{\alpha(m+1)} + n_1 - n_2\right)} \\
&= \prod_{i=1}^{n_1-n_2} \left(\frac{\frac{\gamma^{(n_1)}_i}{m+1}}{\frac{\gamma^{(n_1)}_{n_1-n_2}}{m+1} - \frac{t}{\alpha(m+1)} + n_1 - n_2 - i} \right) = \prod_{i=1}^{n_1-n_2} \left(1 - \frac{t}{\alpha \gamma^{(n_1)}_i}\right)^{-1}, \qquad (5)
\end{aligned}$$

since, $\frac{\gamma^{(n_2)}_r}{m+1} + r = \frac{\gamma^{(n_1)}_{n_1-n_2}}{m+1}$. On comparing (5) with (4), we deduce that $M_{\tilde{Y}}(t)$ is the MGF of $Y(n_1 - n_2, n_1; m, k)$, i.e., the $(n_1 - n_2)$th m-GOS from a sample of size n_1 drawn from the DF $\exp(\alpha)$. This completes the proof of the necessity part. We now turn to prove the sufficiency part. Let the representation (2) be satisfied with $\tilde{Y} \stackrel{d}{=} Y(n_1 - n_2, n_1; m, k)$ and $Y \sim \exp(\alpha)$. Furthermore, let $X(n_1 - n_2 + r, n_1; m, k)$ and $X(r, n_2; m, k)$ in (2) be $m-$GOSs, which are based on an unknown DF $F_X(x)$ and they are independent of $Y(n_1 - n_2, n_1; m, k)$. Therefore, the convolution relation (2) implies that

$$\begin{aligned}
f_{X(n_1-n_2+r,n_1;m,k)}(x) &= \int_0^x f_{X(r,n_2;m,k)}(y) f_{Y(n_1-n_2,n_1;m,k)}(x-y)\, dy \\
&= \frac{\alpha C^{(n_1)}_{n_1-n_2-1}}{(n_1-n_2-1)!(m+1)^{n_1-n_2-1}} \int_0^x e^{-\alpha\gamma^{(n_1)}_{n_1-n_2}(x-y)} \\
&\times \left(1 - e^{-\alpha(m+1)(x-y)}\right)^{n_1-n_2-1} f_{X(r,n_2;m,k)}(y)\, dy. \qquad (6)
\end{aligned}$$

Differentiating both sides of (6) with respect to x, we get

$$\begin{aligned}
\frac{d f_{X(n_1-n_2+r,n_1;m,k)}(x)}{dx} &= \frac{\alpha^2 C^{(n_1)}_{n_1-n_2-1}}{(n_1-n_2-2)!(m+1)^{n_1-n_2-2}} \int_0^x e^{-\alpha\left(\gamma^{(n_1)}_{n_1-n_2} + (m+1)\right)(x-y)} \\
&\times \left(1 - e^{-\alpha(m+1)(x-y)}\right)^{n_1-n_2-2} f_{X(r,n_2;m,k)}(y)\, dy \\
&- \frac{\alpha^2 \gamma^{(n_1)}_{n_1-n_2} C^{(n_1)}_{n_1-n_2-1}}{(n_1-n_2-1)!(m+1)^{n_1-n_2-1}} \int_0^x e^{-\alpha\gamma^{(n_1)}_{n_1-n_2}(x-y)} \\
&\times \left(1 - e^{-\alpha(m+1)(x-y)}\right)^{n_1-n_2-1} f_{X(r,n_2;m,k)}(y)\, dy. \qquad (7)
\end{aligned}$$

On the other hand, by using the obvious relation

$$\begin{aligned}
e^{-\alpha\gamma^{(n_1)}_{n_1-n_2} z}\left(1 - e^{-\alpha(m+1)z}\right)^{n_1-n_2-1} &= e^{-\alpha\gamma^{(n_1)}_{n_1-n_2} z}\left(1 - e^{-\alpha(m+1)z}\right)^{n_1-n_2-2} \\
&- e^{-\alpha\left(\gamma^{(n_1)}_{n_1-n_2} + (m+1)\right)z}\left(1 - e^{-\alpha(m+1)z}\right)^{n_1-n_2-2},
\end{aligned}$$

and by using the representation (6), we get

$$\begin{aligned} & \frac{\alpha C^{(n_1)}_{n_1-n_2-1}}{(n_1-n_2-2)!(m+1)^{n_1-n_2-2}} \int_0^x e^{-\alpha\left(\gamma^{(n_1)}_{n_1-n_2}+(m+1)\right)(x-y)} \\ \times & \left(1-e^{-\alpha(m+1)(x-y)}\right)^{n_1-n_2-2} f_{X(r,n_2;m,k)}(y)\,dy \\ = & \frac{C^{(n_1)}_{n_1-n_2-1}}{C^{(n_1-1)}_{n_1-n_2-2}} f_{X(n_1-n_2-1,n_1-1;m,k)}(x) \\ - & (m+1)(n_1-n_2-1) f_{X(n_1-n_2,n_1;m,k)}(x). \end{aligned} \tag{8}$$

Thus, by using the relations $\gamma^{(n_1)}_{n_1-n_2}+(m+1)(n_1-n_2-1)=\gamma^{(n_1)}_1$ and

$$\frac{C^{(n_1)}_{n_1-n_2-1}}{C^{(n_1-1)}_{n_1-n_2-2}} = \frac{\prod_{i=1}^{n_1-n_2}\gamma_i^{(n_1)}}{\prod_{i=2}^{n_1-n_2}\gamma_i^{(n_1)}} = \gamma_1^{(n_1)},$$

and by combining (15) and (7), we get

$$\frac{d f_{X(n_1-n_2+r,n_1;m,k)}(x)}{dx} = \alpha\gamma_1^{(n_1)}\left[f_{X(n_1-n_2+r-1,n_1-1;m,k)}(x) - f_{X(n_1-n_2+r,n_1;m,k)}(x)\right],$$

or equivalently, by integrating from 0 to x

$$f_{X(n_1-n_2+r,n_1;m,k)}(x) = \alpha\gamma_1^{(n_1)}\left[F_{X(n_1-n_2+r-1,n_1-1;m,k)}(x) - F_{X(n_1-n_2+r,n_1;m,k)}(x)\right]. \tag{9}$$

Now, by using the relation (II) of Kamps (1995) on page 75, we get

$$\begin{aligned} & F_{X(n_1-n_2+r-1,n_1-1;m,k)}(x) - F_{X(n_1-n_2+r,n_1;m,k)}(x) \\ = & \frac{C^{(n_1-1)}_{n_1-n_2+r-2}}{(n_1-n_2+r-1)!(m+1)^{n_1-n_2+r-1}} \overline{F}_X^{\gamma^{(n_1)}_{n_1-n_2+r}-1}(x)\left[1-\overline{F}_X^{m+1}(x)\right]^{n_1-n_2+r-1}. \end{aligned} \tag{10}$$

Therefore, by combining (6), (9) and (10), we get

$$\frac{f_X(x)}{\overline{F}_X(x)} = \alpha,$$

which implies that $F_X(x) = 1-e^{-\alpha x}, x>0$. This completes the proof of the sufficiency part, as well as the proof of Theorem 1. □

Corollary 1. *Assume that the RVs X and Y are independent, as we assumed in Theorem 1. By replacing the additive relation (2) by the multiplicative relation*

$$X(n_1-n_2+r,n_1;m,k) \overset{d}{=} X(r,n_2;m,k)\times\tilde{Y}. \tag{11}$$

Then, $\tilde{Y} \overset{d}{=} Y(n_1-n_2,n_1;m,k)$ and $Y\sim Pareto(\alpha)$ (i.e., $F_Y(y)=1-y^{-\alpha},\ y>1$) if and only if $X\sim Pareto(\alpha),\ \alpha>0$.

***Proof*.** The proof immediately follows, by noting that if $X \sim Pareto(\alpha)$, then $\log X \sim \exp(\alpha)$ and

$$\log X(n_1 - n_2 + r, n_1; m, k) \stackrel{d}{=} \log X(r, n_2; m, k) + \log \tilde{Y},$$

which implies

$$X(n_1 - n_2 + r, n_1; m, k) \stackrel{d}{=} X(r, n_2; m, k) \times \tilde{Y}.$$

□

Remark 1. In (10), the product $X(r, n_2; m, k) \times \tilde{Y}$ is called random dilation of $X(r, n_2; m, k)$, cf. Beutner & Kamps (2008). Moreover, at $n_1 = n_2 + 1$, the representation (11) gives

$$X(r+1, n_2+1; m, k) \stackrel{d}{=} X(r, n_2; m, k) \times Y(1, n_2+1; m, k), \tag{12}$$

as was obtained by Beutner & Kamps (2008) for $X \sim Pareto(\alpha)$. Also, at $n_2 = n_1 - 1$, the relation (11) gives

$$X(r+1, n_1; 0, 1) \stackrel{d}{=} X(r, n_1 - 1; 0, 1) \times Y(1, n_1; 0, 1)$$

(i.e., for the OOSs model), which was obtained by Castaño Martínez et al. (2012), for $X(1, n_1; 0, 1) \sim Pareto(\alpha n_1)$. Finally, the representation (12) can be written as (for OOSs model) $X(s, n_1; 0, 1) \stackrel{d}{=} X(r, n_2; 0, 1) \times V$, $1 \le r < s, n_2 < n_1$, which was an unsolved problem due to Arnold et al. (2008).

Corollary 2. *Assume that the RVs X and Y are independent. Let $X^\star(r, n; m, k)$ and $Y^\star(r, n; m, k)$ be the rth $m-$DGOSs based on a sample of size n drawn from F_X and F_Y, respectively. By replacing the additive relation (2.1) by the multiplicative relation*

$$X^\star(n_1 - n_2 + r, n_1; m, k) \stackrel{d}{=} X^\star(r, n_2; m, k) \times Y^\star.$$

Then, $Y^\star \stackrel{d}{=} Y^\star(n_1 - n_2, n_1; m, k)$ and $Y^\star \sim power(\alpha)$, $\alpha > 0$ (i.e., $F_Y(y) = y^\alpha$, $0 < y < 1$), if and only if $X^\star \sim power(\alpha)$.

***Proof*.** The proof immediately follows from the simple relation between the GOSs and DGOSs, by noting that if $X \sim power(\alpha)$, then $-\log X \sim \exp(\alpha)$, and

$$-\log X^\star(n_1 - n_2 + r, n_1; m, k) \stackrel{d}{=} -\log X^\star(r, n_2; m, k) - \log Y^\star,$$

which implies

$$X^\star(n_1 - n_2 + r, n_1; m, k) \stackrel{d}{=} X^\star(r, n_2; m, k) \times Y^\star.$$

□

Theorem 2. *Let $X(r,n;m,k)$, $m \neq -1$, be the rth $m-GOS$ from a sample of size n drawn from a continuous DF $F_X(x)$ with PDF $f_X(x)$. Furthermore, let $Y(r,n;m,k)$, $m \neq -1$, be the rth $m-GOS$ based on a sample of size n, which is drawn from a continuous DF $F_Y(y)$, where Y is independent of X. Finally, let the relation*

$$X(n_1-n_2+r,n_1;m,k) \stackrel{d}{=} X(n_1-n_2,n_1;m,k)+\tilde{Y} \tag{13}$$

be satisfied for all $1 \leq r < n_2 < n_1$. Then, $\tilde{Y} \stackrel{d}{=} Y(r,n_2;m,k)$ and $Y \sim \exp(\alpha)$ if and only if $X \sim \exp(\alpha)$.

Proof. Clearly, the proof of the necessity part follows from Theorem 1, while the proof of the sufficiency part follows closely as the sufficiency part of Theorem 1. Namely, let the representation (13) be satisfied with $\tilde{Y} \stackrel{d}{=} Y(r;n_2,m,k)$ and $Y \sim \exp(\alpha)$. Furthermore, let $X(n_1-n_2+r,n_1;m,k)$ and $X(n_1-n_2,n_1;m,k)$ in (13) be $m-$GOSs, which are based on an unknown DF $F_X(x)$ and they are independent of $Y(r;n_2;m,k)$. Therefore, the convolution relation (13) implies that

$$\begin{aligned} f_{X(n_1-n_2+r,n_1;m,k)}(x) &= \int_0^x f_{X(n_1-n_2,n_1;m,k)}(y) f_{Y(r;n_2,m,k)}(x-y)\,dy \\ &= \frac{\alpha C_{r-1}^{(n_2)}}{(r-1)!(m+1)^{r-1}} \int_0^x e^{-\alpha\gamma_r^{(n_2)}(x-y)} \\ &\times \left(1-e^{-\alpha(m+1)(x-y)}\right)^{r-1} f_{X(n_1-n_2,n_1;m,k)}(y)\,dy. \end{aligned} \tag{14}$$

By differentiating both sides of (14) with respect to x, we get

$$\begin{aligned} \frac{d f_{X(n_1-n_2+r,n_1;m,k)}(x)}{dx} &= \frac{\alpha^2 C_{r-1}^{(n_2)}}{(r-2)!(m+1)^{r-2}} \int_0^x e^{-\alpha\left(\gamma_r^{(n_2)}+(m+1)\right)(x-y)} \\ &\times \left(1-e^{-\alpha(m+1)(x-y)}\right)^{r-2} f_{X(n_1-n_2,n_1;m,k)}(y)\,dy \\ &- \frac{\alpha^2\gamma_r^{(n_2)} C_{r-1}^{(n_2)}}{(r-1)!(m+1)^{r-1}} \int_0^x e^{-\alpha\gamma_r^{(n_2)}(x-y)} \\ &\times \left(1-e^{-\alpha(m+1)(x-y)}\right)^{r-1} f_{X(n_1-n_2,n_1;m,k)}(y)\,dy. \\ &= \alpha\gamma_r^{(n_2)}\left[f_{X(n_1-n_2+r-1,n_1;m,k)}(x) - f_{X(n_1-n_2+r,n_1;m,k)}(x)\right], \end{aligned}$$

or equivalently, by integrating from 0 to x,

$$f_{X(n_1-n_2+r,n_1,m,k)}(x) = \alpha\gamma_r^{(n_2)}\left[F_{X(n_1-n_2+r-1,n_1;m,k)}(x) - F_{X(n_1-n_2+r,n_1;m,k)}(x)\right]. \tag{15}$$

Now, by using the relation of Kamps (1995) on Page 75, we get

$$\begin{aligned} & F_{X(n_1-n_2+r-1,n_1;m,k)}(x) - F_{X(n_1-n_2+r,n_1;m,k)}(x) \\ = & \frac{C_{n_1-n_2+r-2}^{(n_1)}}{(n_1-n_2+r-1)!(m+1)^{n_1-n_2+r-1}} \overline{F}_X^{\gamma_{n_1-n_2+r}^{(n_1)}-1}(x)\left[1-\overline{F}_X^{m+1}(x)\right]^{n_1-n_2+r-1}. \end{aligned} \tag{16}$$

Therefore, by combining (14), (15) and (16), we get $f_X(x) = \alpha\overline{F}_X(x)$, which implies that $F_X(x) = 1-e^{-\alpha x}, x>0$. This completes the proof of the sufficiency part, as well as the proof of Theorem 2. □

Corollary 3. *Assume that the RVs X and Y are independent, as we assumed in Theorem 2. By replacing the additive relation (13) by the multiplicative relation*

$$X(n_1 - n_2 + r, n_1; m, k) \stackrel{d}{=} X(n_1 - n_2, n_1; m, k) \times \tilde{Y}. \tag{17}$$

Then, $\tilde{Y} \stackrel{d}{=} Y(r, n_2; m, k)$ and $Y \sim Pareto(\alpha)$ if and only if $X \sim Pareto(\alpha)$.

Proof. The proof follows exactly as the proof of Corollary 1. □

Remark 2. For OOSs model the relation (17) takes the form

$$X(n_1 - n_2 + r, n_1; 0, 1) \stackrel{d}{=} X(n_1 - n_2, n_1; 0, 1) \times Y(r, n_2; 0, 1),$$

which implies the relation $X(s, n_1; 0, 1) \stackrel{d}{=} X(r, n_1; 0, 1) \times Y(s - r, n_1 - r; 0, 1)$ that is belonging to Castaño Martínez et al. (2012).

Corollary 4. *Assume that the RVs X and Y are independent. Let $X^\star(r, n; m, k)$ and $Y^\star(r, n; m, k)$ be the rth m-DGOSs based on a sample of size n drawn from F_X and F_Y, respectively. By replacing the additive relation (13) by the multiplicative relation*

$$X^\star(n_1 - n_2 + r, n_1; m, k) \stackrel{d}{=} X^\star(n_1 - n_2, n_1; m, k) \times Y^\star.$$

Then, $Y^\star \stackrel{d}{=} Y^\star(r, n_2; m, k)$ and $Y^\star \sim power(\alpha)$ if and only if $X^\star \sim power(\alpha)$.

Proof. The proof follows as the proof of Corollary 2. □

3. Applications to the Prediction Problem

Prediction problem usually arises in life-testing experiments of medical and industrial applications. Often, in the life-testing experiments, the observations arrive in ascending order of magnitude. Consequently, in reliability theory, especially for OOSs and SOSs, $X(r, n; m, k)$ represents the life length of a $n-r+1$-out-of-n system made up of n independent life lengths (these components are identical for OOSs and non identical for SOSs). Motivation for the prediction problems arises when the experiment is terminated before its conclusion by stopping after a given time (Type I censoring) or after a given number of failures (Type II censoring). Several authors have considered prediction problems involving GOSs, see for example Aitcheson & Dunsmore (1975), Barakat et al., (2011; 2018; 2021a; 2021b; 2021c), Lawless (1971), Nagaraja (1986), Raqab (2001), and Raqab & Barakat (2018).

Theorems 1 and 2 suggest a new method for treating two prediction problems of different types. Namely, Theorem 2 treats a classical prediction problem, denoted by *Problem-1*, that predicting $X(s, n; m, k)$, $1 \leq r < s \leq n$, based on the observed $m-$GOSs $X(1, n; m, k) \leq X(2, n; m, k) \leq \cdots \leq X(r, n; m, k)$. On the other hand, Theorem 2.1 considers the prediction problem of $X(r, n; m, k)$, when the sample size of the test is enlarged from n to N, by adding some extra items $X_{n+1}, \ldots, X_N$

after observing $X(r, n; m, k)$. This problem will be noted by *Problem-2*. Clearly, the sequence $\{X(r, n; m, k)\}$ is non-increasing in n. For example, if $F_X(x)$ is continuous and for any fixed value $r < n$, the observed value of $X(r, n; 0, 1)$, denoted by $x(r, n; 0, 1)$, did not change if $\min(x_{n+1}, \ldots, x_N) > x(r, n; 0, 1)$, otherwise we get $x(r, n; 0, 1) < x(r, N; 0, 1)$. In the preceding two prediction problems, the failure times of the un-observed lifetimes in a lifetime experiment are predicted by using the result of another independent lifetime experiment. Below, more details are given.

Problem-1: Let us assume that there are two independent lifetime experiments. The first one contains n_1 items, which follow $X \sim \exp(\alpha)$. Furthermore, let us assume that s items were observed until they failed. The second experiment contains $n_2 = n_1 - s$ items, which follow $Y \sim \exp(\alpha)$. Furthermore, in the second experiment let us assume that r failure times were observed. Theorem 2.2 enables us to predict

$$X(s+1, n_1; m.k), X(s+2, n_1; m.k), \ldots, \text{and } X(s+r, n_1; m.k)$$

by $X(s, n_1; m.k) + Y(1, n_2; m.k)$, $X(s, n_1; m.k) + Y(2, n_2; m.k), \ldots,$ and $X(s, n_1;\, m.k) + Y(r, n_2; m.k)$, respectively.

Problem-2: Let us assume that there are two independent lifetime experiments. The first one contains n_2 items, which follow $X \sim \exp(\alpha)$. Furthermore, let us assume that r items were observed until they failed. The second experiment contains n_1 items, which follow $Y \sim \exp(\alpha)$, where $n_1 > n_2$. Furthermore, in the second experiment let us assume that $s = n_1 - n_2$ failure times were observed. If we decided to enlarge the number of installed items in the first experiment to n_1, Theorem 1 would enable us to predict

$$X(s+1, n_1; m.k), X(s+2, n_1; m.k), \ldots, \text{and } X(s+r, n_1; m.k)$$

by $X(1, n_2; m.k) + Y(s, n_1; m.k)$, $X(2, , n_2; m.k) + Y(s, n_1; m.k), \ldots,$ and $X(r, n_2;\, m.k) + Y(s, n_1; m.k)$, respectively.

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