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Results on the Fractional Cumulative Residual Entropy of Coherent Systems

Resultados en la entropia residual acumulativa fraccional de sistemas coherentes

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Abstract

Recently, Xiong et al. (2019) introduced an alternative measure of uncertainty known as the fractional cumulative residual entropy (FCRE). In this paper, first, we study some general properties of FCRE and its dynamic version. We also consider a version of fractional cumulative paired entropy for a random lifetime. Then we apply the FCRE measure for the coherent system lifetimes with identically distributed components.

***Key words*:** Fractional cumulative residual entropy; Paired entropy; Coherent systems.

Resumen

Recientemente, Xiong et al. (2019) introdujeron una medida alternativa de incertidumbre conocida como entropia residual acumulativa fraccionada (FCRE). En este articulo, primero, estudiamos algunas propiedades generales de FCRE y su version dynami. También consideramos una version de entropia pareada acumulativa fraccionaria para una vida aleatoria. Luego, aplicamos la medida FCRE para la vida util del sistema coherente con componentes distribuidos de manera idéntica.

***Palabras clave*:** Entropia residual acumulativa fraccionada; Entropia pareada; Sistemas coherentes.

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1. Introduction

The behavior of engineering systems often requires use of concepts of entropy and its generalizations. Let X denotes the lifetime of a system with probability density function (pdf) f and distribution function F, respectively. Shannon (1948) introduced a measure of uncertainty associated with f as

$$H(X) = -\int_0^{+\infty} f(x)\log f(x)dx.$$

The quantity $H(X)$ is often named differential entropy. Recently, new measures of information have been proposed in the literature. Replacing the pdf by the survival function $\bar{F} = 1 - F$ in Shannon entropy, the cumulative residual entropy (CRE) was defined by Rao et al. (2004) as follows:

$$\mathcal{E}(X) = \int_0^{+\infty} \bar{F}(x)\Lambda(x)dx = \mathbb{E}\left(\frac{\Lambda(X)}{\lambda(X)}\right),$$

where $\Lambda(.) = -\log \bar{F}(.)$ and $\lambda(.) = \frac{f(.)}{\bar{F}(.)}$ is the failure rate of F. Asadi & Zohrevand (2007) also considered a dynamic version of the CRE as

$$\mathcal{E}(X;t) = \mathcal{E}(X_t) = \int_t^{\infty} \left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)[\Lambda(x) - \Lambda(t)]dx, \quad t \geq 0,$$

where $X_t = (X - t|X \geq t)$ is the residual lifetime. Clearly, X_t denotes the system lifetime conditioned to the survival of the system at time t. Some interesting results and extensions regarding CRE have been studied by Psarrakos & Navarro (2013), Psarrakos & Toomaj (2017) and Navarro & Psarrakos (2017). Di Crescenzo & Longobardi (2009) proposed another information measure analogue to $\mathcal{E}(X)$, called it cumulative entropy (CE) and is defined as

$$\mathcal{CE}(X) = \int_0^{+\infty} F(x)\tilde{\Lambda}(x)dx,$$

where $\tilde{\Lambda}(x) = -\log F(x)$. Longobardi (2014) obtained more results of CE and stochastic orders. Tahmasebi et al. (2020) studied on a shift-dependent measure of CE and its applications in blind image quality assessment. Ubriaco (2009) defined a new entropy based on fractional calculus as follows:

$$H_p(X) = \int_0^{+\infty} f(x)[-\log f(x)]^p dx, \quad 0 < p \leq 1.$$

The fractional entropy is concave, positive and non-additive. From a physical sense, it also satisfies Lesche and thermodynamic stability. Recently, Xiong et al. (2019) defined fractional cumulative residual entropy (FCRE) as

$$\mathcal{E}_p(X) = \int_0^{+\infty} \bar{F}(x)[\Lambda(x)]^p dx = \int_0^1 \frac{\varphi_p(u)}{f(\bar{F}^{-1}(u))}du, \quad 0 < p \leq 1, \tag{1}$$

where $\varphi_p(u) = u[-\log u]^p$ and $\bar{F}^{-1}(u) = \sup\{x : \bar{F}(x) \geq u\}$ is known as the quantile function of $\bar{F}$. Note that $\varphi_p(0) = \varphi_p(1) = 0$. It is clear that $\mathcal{E}_1(X) = \mathcal{E}(X)$. Klein et al. (2016) proposed the $\phi - entropy$ based on cdf F and survival function $\bar{F}$ as

$$\mathcal{CPE}(X) = \int_{\mathbb{R}} \phi(F(x)) + \phi(\bar{F}(x))dx,$$

where ϕ is the entropy generating function defined on $[0, 1]$ with $\phi(0) = \phi(1) = 0$.

The study on information properties of coherent systems is a relevant subject in reliability and survival theories. A system is said to be coherent if it does not have any irrelevant components and its structure function is monotone. Recently, Toomaj et al. (2017) studied on the CRE of coherent systems when the component lifetimes are identically distributed. Toomaj et al. (2018) obtained some results on information properties of coherent systems.Da Costa Bueno & Balakrishnan (2020) considered a cumulative residual inaccuracy measure for coherent systems. Moreover, Calì et al. (2020) obtained some results on the generalized cumulative entropy in coherent systems. Rahimi et al. (2020) studied on extended cumulative entropy based on kth lower record values for the coherent systems lifetime.

In the present paper we study general properties of $\mathcal{E}_p(X)$ and obtain some results of $\mathcal{E}_p(T)$ for a coherent system with lifetime T.

This paper is organized as follows: In Section 2, we present some general properties of FCRE and its dynamic version, we also propose a version of fractional cumulative paired entropy for a random lifetime based on the $\phi - entropy$. In Section 3, we study the FCRE measure for the coherent systems lifetime with identically distributed components.

2. General properties of FCRE

Before proceeding to give the results of this section, we overview some preliminary concepts on stochastic orderings between random variables. For more details of these concepts one can see Shaked & Shanthikumar (2007).

Definition 1. Suppose that X and Y are the non-negative random variables with cdfs F and G, respectively, then

1. X is smaller than Y in the hazard rate order, denoted by $X \leq_{hr} Y$, if $\lambda_X(x) \geq \lambda_Y(x)$ for all x, where $\lambda_X(x)$ and $\lambda_Y(x)$ are the failure rate functions X and Y, respectively.

2. X is smaller than Y in the dispersive order, denoted by $X \leq_{disp} Y$, if $f(F^{-1}(u)) \geq g(G^{-1}(u))$ for all $u \in (0, 1)$, where F^{-1} and G^{-1} are right continuous inverses of F and G, respectively.

3. X is said to have decreasing failure rate (DFR) if $\lambda_X(x) = \frac{f(x)}{\bar{F}(x)}$ is decreasing in x.

4. X is smaller than Y in the convex transform order, denoted by $X \leq_c Y$, if $G^{-1}F(x)$ is a convex function on the support of X.

5. X is smaller Y in the increasing convex order, denoted by $X \leq_{icx} Y$, if $\mathbb{E}(\phi(X)] \leq \mathbb{E}(\phi(X)]$ for all increasing convex functions ϕ such that the expectations exist.

6. X is smaller than Y in the star order , denoted by $X \leq_* Y$, if $\frac{G^{-1}F(x)}{x}$ is increasing in $x \geq 0$.

7. X is smaller than Y in the supper additive order, denoted by $X \leq_{su} Y$, if $G^{-1}F(t+u) \geq G^{-1}F(t) + G^{-1}F(u)$ for $t \geq 0, u \geq 0$.

8. X is said to have increasing(decreasing) failure rate average (IFRA(DFRA)) if $\frac{\lambda(x)}{x}$ is increasing (decreasing) function in $x > 0$.

Here we aim to present some general results of FCRE and FCPE. We will focus on certain lower and upper bounds and stochastic orderings properties of these measures, the proof of which follows on the same lines are given by Psarrakos & Toomaj (2017) and Navarro & Psarrakos (2017).

Proposition 1. *Let X be an absolutely continuous non-negative random variable with $\mathcal{E}_p(X) < \infty$. Then*

$$\mathcal{E}_p(X) = \mathbb{E}[h_p(X)], \quad 0 < p \leq 1,$$

where $h_p(x) = \int_0^x [\Lambda(z)]^p dz$.

Proposition 2. *If X denotes an absolutely continuous nonnegative random variable with finite $\mu = \mathbb{E}(X)$. Then*

$$\mathcal{E}_p(X) \geq \tilde{h}(\mu), \quad 0 < p \leq 1,$$

where $\tilde{h}(\mu) = \int_0^\mu [\Lambda(z)]^p dz$.

Proposition 3. *Let X and Y be two nonnegative random variables such that $X \leq_{icx} Y$, then $\mathcal{E}_p(X) \leq \mathcal{E}_p(Y)$.*

Proposition 4. *Let X and Y be two nonnegative random variables with finite positive means and such that $X \leq_{icx} Y$, then $\frac{\mathcal{E}_p(X)}{\mathbb{E}(X)} \leq \frac{\mathcal{E}_p(Y)}{\mathbb{E}(Y)}$.*

Proposition 5. *If X is IFRA(DFRA), then for $0 < p \leq 1$ we have*

$$\mathcal{E}_p(X) \leq (\geq) \mathbb{E}\left(\frac{X}{[\Lambda(X)]^{1-p}}\right).$$

Proposition 6. *Let X and Y be two random variable with cdfs F_X and G_Y, respectively. If $X \leq_{st} Y$, then*

$$I_p(X,Y) \leq \mathcal{E}_p(X) \leq I_p(X,Y), \ \ 0 < p \leq 1,$$

where $I_p(X,Y) = \int_0^{+\infty} \bar{F}_X(u)[-\log \bar{G}_Y(u)]^p du$.

The dynamic version of the FCRE of the residual lifetime $X_t = (X - t \mid X > t)$ is given by

$$\mathcal{E}_p(X;t) = \mathcal{E}_p(X_t) = \int_t^{\infty} \frac{\bar{F}(x)}{\bar{F}(t)}[\Lambda(x) - \Lambda(t)]^p dx, \;\; 0 < p \leq 1, \; t \geq 0. \tag{2}$$

Note that $\mathcal{E}_p(X;0) = \mathcal{E}_p(X)$. This function is called dynamic fractional cumulative residual entropy (DFCRE).

Proposition 7. *Let X be a random variable with cdf F and $Y = aX + b$, where $a > 0$ and $b \geq 0$. Then*

$$\mathcal{E}_p(Y;t) = a\mathcal{E}_p(X; \frac{t-b}{a}), \quad 0 < p \leq 1, \quad t \geq 0.$$

Proposition 8. *Let X be a non-negative random variable with absolutely cdf $F(x)$, then an alternative expression of the DFCRE is*

$$\mathcal{E}_p(\tilde{X}_t) = \int_t^{\infty} \frac{\bar{F}(x)}{\bar{F}(t)} \left([\Lambda(x) - \Lambda(t)]^p + 1\right) dx = \mathcal{E}_p(X_t) + m(t).$$

Proposition 9. *Let X be a random variable with survival function $\bar{F}(.)$. Then*

$$\mathcal{E}_p(X;t) = \mathbb{E}[h_p(X;t) \mid X > t], \; 0 < p \leq 1, \; t \geq 0,$$

where $h_p(X;t) = \int_t^x [\Lambda(z) - \Lambda(t)]^p dz, \;\; z > t$.

Proposition 10. *Let X be the random lifetime of a system with cdf F_X , then we have*

$$\mathcal{E}_p(X_t) \leq [\mathcal{E}(X_t)]^p, \;\; 0 < p \leq 1.$$

Proof. Since $\frac{\bar{F}(x)}{\bar{F}(t)} \leq \left[\frac{\bar{F}(x)}{\bar{F}(t)}\right]^p$ for $0 < p \leq 1$, from (2) we have

$$\mathcal{E}_p(X_t) = \int_t^{\infty} \frac{\bar{F}(x)}{\bar{F}(t)}[\Lambda(x) - \Lambda(t)]^p dx \leq \int_t^{\infty} \left[\frac{\bar{F}(x)}{\bar{F}(t)}[\Lambda(x) - \Lambda(t)]\right]^p dx.$$

By noting that $g(x) = x^p$, for all $0 < p \leq 1$, is a concave function of x, Jensen's inequality gives

$$\mathcal{E}_p(X_t) \leq \left[\int_t^{\infty} \frac{\bar{F}(x)}{\bar{F}(t)}[\Lambda(x) - \Lambda(t)] dx\right]^p.$$

Hence, the proof is completed.

Definition 2. Let X be a random variable with cdf F. Then the fractional cumulative paired entropy (FCPE) is defined as

$$\begin{aligned} \mathcal{FCPE}_p(X) &= \int_0^{+\infty} \bar{F}(x)[\Lambda(x)]^p dx + \int_0^{+\infty} F(x)[\tilde{\Lambda}(x)]^p dx \\ &= \mathcal{E}_p(X) + \mathcal{CE}_p(X), \;\; 0 < p \leq 1, \end{aligned}$$

where $\mathcal{E}_p(X) = \int_0^{+\infty} F(x)[\tilde{\Lambda}(x)]^p dx$ is the fractional cumulative past entropy (FCPE).

Note that $\mathcal{FCPE}_p(X)$ is useful to describe information in dynamic reliability systems when uncertainty is related to the future, and to the past, respectively. Hence, in the following we study some results of this measure. The results are similar to various results given in Xiong et al. (2019) and hence we omit their proofs.

Proposition 11. *Let F be the cdf of the random variable X. Then*

$$\mathcal{FCPE}_p(aX+b) = |a|\mathcal{FCPE}_p(X), \ \ 0 < p \leq 1.$$

Proposition 12. *Let X and Y be two random variable with cdfs F_X and G_Y, respectively. If $X \leq_{disp} Y$, then*

$$\mathcal{FCPE}_p(X) \leq \mathcal{FCPE}_p(Y), \ \ 0 < p \leq 1.$$

Proposition 13. *If X and Y are two absolutely continuous independent random variables, then*

$$\mathcal{FCPE}_p(X+Y) \geq \max\{\mathcal{FCPE}_p(X), \mathcal{FCPE}_p(Y)\}, \ \ 0 < p \leq 1.$$

Analogous to empirical FCRE given in Xiong et al. (2019) , we provide an estimator of FCPE using empirical approach.

Definition 3. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a lifetime distribution with absolutely continuous cdf $F(x)$. Then the empirical FCPE is defined as

$$\widehat{\mathcal{FCPE}}_{p,n}(X) = \sum_{k=1}^{n-1} U_{k+1}(1-\frac{k}{n})[-\log(1-\frac{k}{n})]^p + \sum_{k=1}^{n-1} U_{k+1}\frac{k}{n}[-\log(\frac{k}{n})]^p,$$

where $U_{k+1} = X_{(k+1)} - X_{(k)}$, $U_1 = X_1$ and $X_{(k)}$ represent the order statistic of X_k.

Proposition 14. *Let X be a random variable with $\mathbb{E}(|X|^p) < \infty$ for some $p < 1$, then the empirical FCPE converges to the FCPE of X, i.e., as $n \to \infty$*

$$\widehat{\mathcal{FCPE}}_{p,n}(X) \to \mathcal{FCPE}_p(X) \quad \textit{almost surely}.$$

Example 1. Let us consider the data set from Murthy et al. (2004), concerning the failure times of 84 mechanical components displayed in Table 1.

TABLE 1: The failure times of 84 mechanical components.

0.040	1.866	2.385	3.443	0.301	1.876	2.481	3.467	0.309	1.899	2.610	3.478	0.557
1.911	2.625	3.578	0.943	1.912	2.632	3.595	1.070	1.914	32.646	3.699	1.124	1.981
2.661	3.779	1.248	2.010	2.688	3.924	1.281	2.038	2.823	4.035	1.281	2.085	2.890
4.121	1.303	2.089	2.902	4.167	1.432	2.097	2.934	4.240	1.480	2.135	2.962	4.255
1.505	2.154	2.964	4.278	1.506	2.190	3.000	4.305	1.568	2.194	3.103	4.376	1.615
2.223	3.114	4.449	1.619	2.224	3.117	4.485	1.652	2.229	3.166	4.570	1.652	2.300
3.344	4.602	1.757	2.324	3.376	4.663							

The mean and standard deviation of this data set are 2.55 and 1.11, respectively. Since the coefficient of skewness is 0.1 which indicates that the data is right skewed. We propose a two-parameter Weibull distribution with cdf $F(x) = 1 - \exp(-ax^b)$. Based on the $p - value = 0.9687$ of the Kolmogorov-Smirnov test, we can conclude that Weibull distribution with parameters $\hat{a} = 0.082$ and $\hat{b} = 2.37$ can be fitted to this data set. Figure 1 shows the function $\widehat{\mathcal{FCPE}}_{p,n}(X)$ for $0 < p \leq 1$. It decreases in empirical measure of FCPE for different values of p. Hence, the FCPE measure is particularly suitable to measure variability in data distributions that are skewed to the right, such as those concerning the failure times of 84 mechanical components. This is confirmed by Figure 2 where the density is positively skewed.

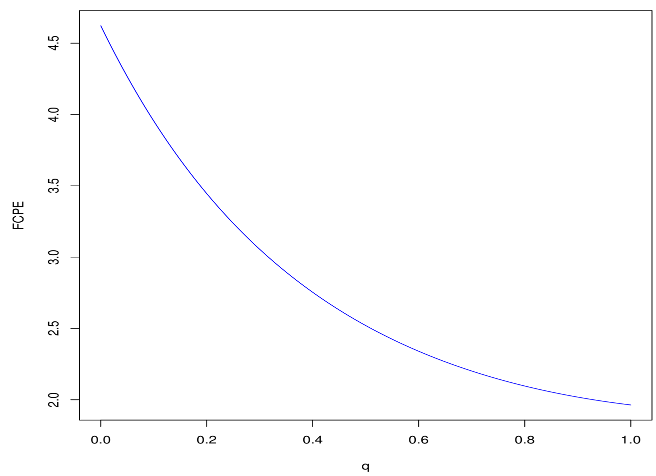

FIGURE 1: Plot of $\widehat{\mathcal{FCPE}}_{p,n}(X)$ for $0 < p \leq 1$.

3. FCRE of Coherent Systems

Let T denote the lifetime of a coherent system consisting of m independent and identically distributed (i.i.d.) components with the common distribution F_X, then its survival function $\bar{F}_T$ can be written as

$$\bar{F}_T(t) = \tilde{q}(\bar{F}_X(t)),$$

where $\tilde{q} : [0, 1] \to [0, 1]$ is a distortion function and depends on the structure of a system and the survival copula of the component lifetime. The function $\tilde{q}$ is a continuous increasing function such that $\tilde{q}(0) = 0$ and $\tilde{q}(1) = 1$. For more details on coherent systems see Burkschat & Navarro (2018) and Navarro et al. (2013). The distortion function $\tilde{q}$ depends on both the structure of the system and the survival copula of the random vector $(X_1, \dots, X_n)$. However, it does not depend

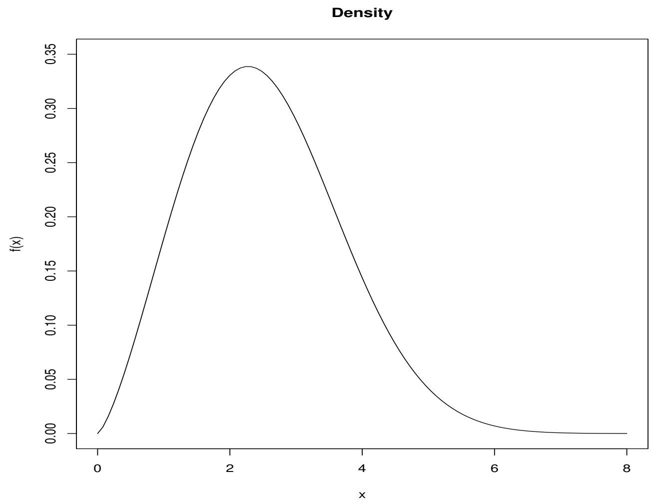

FIGURE 2: Plot of Weibull density with parameters $\hat{a}$=0.082 and $\hat{b} = 2.37$.

on $\bar{F}$. For example if we consider a 2-out-of-3 system with i.i.d. components, then we have $\tilde{q}(u) = 3u^2 - 2u^3$. Also, for a parallel system with i.i.d. components and lifetime $T = \max((X_1, \ldots, X_m)$ we have $\tilde{q}(u) = 1 - (1-u)^m$. Hence, the FCRE of the random lifetime T is obtained as follows:

$$\begin{aligned} \mathcal{E}_p(T) &= \int_0^{+\infty} \bar{F}_T(x)[-\log \bar{F}_T(x)]^p dx \\ &= \int_0^{+\infty} \varphi_p(\bar{F}_T(x))dx \\ &= \int_0^{+\infty} \varphi_p(\tilde{q}(\bar{F}_X(x)))dx \\ &= \int_0^{1} \frac{\varphi_p(\tilde{q}(u))}{f_X(\bar{F}_X^{-1}(u))}du, \quad 0 < p \leq 1. \end{aligned} \tag{3}$$

For example, for a parallel system with $m = 5$ i.i.d. component lifetimes of uniform distribution in $(0, 1)$, we have

$$\mathcal{E}_{0.5}(T) = 0.1841 < \mathcal{E}_{0.5}(X) = 0.3133.$$

Hence, the information measure $\mathcal{E}_{0.5}(T)$ is smaller in the case of a complex system (a parallel system) with respect to the parent distribution. As an application of equation (3), we have the following example

Example 2. (i). We consider two coherent systems with lifetimes $T_1 = \max\{X_1, \min\{X_2, X_3, X_4\}\}$ and $T_2 = \max\{\min\{X_1, X_2\}, \min\{X_3, X_4\}\}$ and i.i.d. components having the common exponential with mean θ. From (3) we obtain

$$\mathcal{E}_{0.5}(T_1) = (0.8618)\theta, \ \mathcal{E}_{0.75}(T_1) = (0.8844)\theta, \ \ \mathcal{E}_{0.5}(T_1) < \mathcal{E}_{0.75}(T_1).$$

Similarly we obtain

$$\mathcal{E}_{0.5}(T_2) = (0.5358)\theta,\ \mathcal{E}_{0.75}(T_2) = (0.5291)\theta.$$

It is clear that $\mathcal{E}_p(T_2) \leq \mathcal{E}_p(T_1)$ for $p = 0.5, 0.75$.

(ii). If the system have dependent identical exponential components with an exchangeable survival copula $\tilde{C}$, then we have

$$\mathcal{E}_p(T_1) = \theta \int_0^1 \frac{\varphi_p(\tilde{q}_1(u))}{u} du, \ \ 0 < p \leq 1,$$

where $\tilde{q}_1(u) = \tilde{C}(u, 1, 1, 1) + \tilde{C}(u, u, u, 1) - \tilde{C}(u, u, u, u)$. If the component lifetimes are dependent with the Farlie-Gumbel-Morgenstern (FGM) survival copula as

$$\tilde{C}(u_1, u_2, u_3, u_4) = u_1 u_2 u_3 u_4 [1 + \alpha(1 - u_1)(1 - u_2)(1 - u_3)(1 - u_4)], \ \ -1 \leq \alpha \leq 1.$$

Then, for $\alpha = 0.5$ we obtain $\tilde{q}_1(u) = u + u^3 - u^4[1 + \frac{(1-u)^4}{2}]$ and

$$\mathcal{E}_{0.5}(T_1) = (0.8614)\theta, \ \ \mathcal{E}_{0.75}(T_1) = (0.8845)\theta.$$

Finally, if a system with lifetime T_2 have dependent identical exponential components with FGM survival copula, then

$$\mathcal{E}_p(T_2) = \theta \int_0^1 \frac{\varphi_p(\tilde{q}_2(u))}{u} du, \ \ 0 < p \leq 1,$$

where for $\alpha = 0.5$ we have $\tilde{q}_2(u) = 2u^2 - u^4[1 + \frac{(1-u)^4}{2}]$. So, for $p = 0.5, 0.75$ we obtain

$$\mathcal{E}_{0.5}(T_2) = (0.5349)\theta, \ \ \mathcal{E}_{0.75}(T_2) = (0.5285)\theta.$$

Numerically, we see that for a constant α, $\mathcal{E}_p(T_2)$ decreases when p increases.

In the following we study some results of $\mathcal{E}_p(T)$. The results are similar to various results given in Toomaj et al. (2017) and hence we omit their proofs.

Proposition 15. *Let T be the lifetime of coherent system with i.i.d. components having common distribution $F(.)$ with distortion function $\tilde{q}$. If $\varphi_p(\tilde{q}(u)) \geq (\leq)\varphi_p(u)$ for all $u \in [0, 1]$ and $0 < p \leq 1$, then we have*

$$\mathcal{E}_p(T) \geq (\leq)\mathcal{E}_p(X).$$

Example 3. Let $T = \min\{X_1, X_2, \dots, X_n\}$ be the lifetime of series system with i.i.d. components having common distribution $F(.)$.

(i) If $F(x) = x, \ \ 0 < x < 1$. Then for $0 < p \leq 1$ we have

$$\mathcal{E}_p(T) = \left(\frac{n}{n+1}\right)^p \frac{\Gamma(p+1)}{n+1} < \mathcal{E}_p(X) = \frac{\Gamma(p+1)}{2^{p+1}}.$$

(ii) If $F(x) = 1 - \left(\frac{\beta}{x}\right)^{\alpha}$, $x > \beta$. Then for $\alpha > 1$ and $0 < p \leq 1$ we have

$$\mathcal{E}_p(T) = \left(\frac{n\alpha}{n\alpha - 1}\right)^p \frac{\beta\Gamma(p+1)}{n\alpha - 1} < \mathcal{E}_p(X) = \left(\frac{\alpha}{\alpha - 1}\right)^p \frac{\beta\Gamma(p+1)}{\alpha - 1}.$$

(iii)If $F(x) = 1 - \exp\left(-(\lambda x)^q\right)$, $x > 0$. Then for $q > 0$ and $0 < p \leq 1$ we have

$$\mathcal{E}_p(T) = \frac{\Gamma(p + \frac{1}{q})}{\lambda q n^{\frac{1}{q}}} < \mathcal{E}_p(X) = \frac{\Gamma(p + \frac{1}{q})}{\lambda q}.$$

Proposition 16. *Assume that the components have cdf F_X and pdf f_X and support S. Let T be the lifetime of a coherent system with i.i.d. components and with distortion function $\tilde{q}$.*

(i). If $f(x) \leq M$ for all $x \in S$, then

$$\mathcal{E}_p(T) \geq \frac{1}{M} \int_0^1 \varphi_p(\tilde{q}(u))du, \quad 0 < p \leq 1.$$

(ii). If $f(x) \geq L > 0$ for all $x \in S$, then

$$\mathcal{E}_p(T) \leq \frac{1}{L} \int_0^1 \varphi_p(\tilde{q}(u))du, \quad 0 < p \leq 1.$$

Example 4. (i). Let $T = max\{X_1, \min(X_2, X_3)\}$ be the lifetime of coherent system with i.i.d. components have an exponential distribution with mean $\mu > 0$, then $M = \frac{1}{\mu}$ and

$$\mathcal{E}_{0.5}(T) \geq \mu(0.2820).$$

(ii). Let $T = max\{\min(X_1, X_2, X_3), \min(X_2, X_3, X_4)\}$ be the lifetime of coherent system with i.i.d. components have a Pareto type II distribution with cdf $\bar{F}(x) = \left(\frac{\beta}{\beta + x}\right)^{\alpha}$, $x > 0$, then $M = \alpha\beta^{\alpha}$ and

$$\mathcal{E}_{0.5}(T) \geq \frac{0.2140}{\alpha\beta^{\alpha}}.$$

Proposition 17. *Suppose that T is the lifetime of a coherent system with i.i.d. components and with distortion function $\tilde{q}$. Let $\varphi_p(u) = u[-\log(u)]^p$. Then*

$$B_{1,p}\mathcal{E}_p(X_1) \leq \mathcal{E}_p(T) \leq B_{2,p}\mathcal{E}_p(X_1), \quad 0 < p \leq 1,$$

where $B_{1,p} = \inf_{u \in (0,1)} \left(\frac{\varphi_p(\tilde{q}(u))}{\varphi_p(u)}\right)$ and $B_{2,p} = \sup_{u \in (0,1)} \left(\frac{\varphi_p(\tilde{q}(u))}{\varphi_p(u)}\right)$.

***Proof*.** Proof is similar to the proof of Proposition 1. of Toomaj et al. (2017) and hence it is omitted. □

For example, if the distortion function for a 3-out-of-4 system with i.i.d. components is $\tilde{q}(u) = 6u^2 - 8u^3 + 3u^4$, then for $\alpha = 0.1, 0.3$ we obtain

$$0.1676 \leq \mathcal{E}_{0.1}(T) \leq (1.29434)\mathcal{E}_{0.1}(X_1),$$

and

$$0.0047 \leq \mathcal{E}_{0.3}(T) \leq (1.1811)\mathcal{E}_{0.3}(X_1).$$

In Table 2, the distortion functions for all coherent structures with 1-4 components are provided by Navarro et al. (2013). In Table 3, we obtain the values of $\mathcal{E}_p(T)$ for these systems when the components have a uniform distribution in $(0, 1)$.

Table 2: Distortion function $\tilde{q}$ for coherent system with 1-4 i.i.d. components.

N	T	$\tilde{q}(u)$
1	X_1	u
2	$X_{1:2} = \min(X_1, X_2)$	u^2
3	$X_{2:2} = \max((X_1, X_2)$	$2u - u^2$
4	$X_{1:3} = \min(X_1, X_2, X_3)$	u^3
5	$\min(X_1, \max((X_2, X_3))$	$2u^2 - u^3$
6	$X_{2:3}(2 - out - of - 3)$	$3u^2 - 2u^3$
7	$\max((X_1, \min(X_2, X_3))$	$u + u^2 - u^3$
8	$X_{3:3} = \max((X_1, X_2, X_3)$	$3u - 3u^2 + u^3$
9	$X_{1:4} = \min(X_1, X_2, X_3, X_4)$	u^4
10	$\max((\min(X_1, X_2, X_3), \min(X_2, X_3, X_4))$	$2u^3 - u^4$
11	$\min(X_{2:3}, X_4)$	$3u^3 - 2u^4$
12	$\min(X_1, \max((X_2, X_3), \max((X_2, X_4))$	$u^2 + u^3 - u^4$
13	$\min(X_1, \max((X_2, X_3, X_4))$	$3u^2 - 3u^3 + u^4$
14	$X_{2:4}$	$4u^3 - 3u^4$
15	$\max((\min(X_1, X_2), \min(X_1, X_3, X_4), \min(X_2, X_3, X_4))$	$u^2 + 2u^3 - 2u^4$
16	$\max((\min(X_1, X_2), \min(X_3, X_4))$	$2u^2 - u^4$
17	$\max((\min(X_1, X_2), \min(X_1, X_3), \min(X_2, X_3, X_4))$	$2u^2 - u^4$
18	$\max((\min(X_1, X_2), \min(X_2, X_3), \min(X_3, X_4))$	$3u^2 - 2u^3$
19	$\min(\max((X_1, X_2), \max((X_2, X_3), \max((X_3, X_4))$	$3u^2 - 2u^3$
20	$\min(\max((X_1, X_2), \max((X_1, X_3), \max((X_2, X_3, X_4))$	$4u^2 - 4u^3 + u^4$
21	$\min(\max((X_1, X_2), \max((X_3, X_4))$	$4u^2 - 4u^3 + u^4$
22	$\min(\max((X_1, X_2), \max((X_1, X_3, X_4), \max((X_2, X_3, X_4))$	$5u^2 - 6u^3 + 2u^4$
23	$X_{3:4}(3 - out - of - 4)$	$6u^2 - 8u^3 + 3u^4$
24	$\max((X_1, \min(X_2, X_3, X_4))$	$u + u^3 - u^4$
25	$\max((X_1, \min(X_2, X_3), \min(X_2, X_4))$	$u + 2u^2 - 3u^3 + u^4$
26	$\max((X_{2:3}, X_4)$	$u + 3u^2 - 5u^3 + 2u^4$
27	$\max((\min(X_1, X_2, X_3), \min(X_2, X_3, X_4))$	$2u - 2u^3 + u^4$
28	$X_{4:4} = \max((X_1, X_2, X_3, X_4)$	$4u - 6u^2 + 4u^3 - u^4$

Table 3: $\mathcal{E}_p(T)$ and bounds for $\mathcal{E}_p(T)$ obtained from the coherent systems given in Table 1 with i.i.d. components having a U(0,1).

N	p	$\mathcal{E}_p(T)$	$B_{1,p}$	$B_{2,p}$	$B_{2,p}\mathcal{E}_p(X_1)$
	0.1	0.4438	1	1	0.4438
1	0.2	0.3996	1	1	0.3996
	0.3	0.3644	1	1	0.3644
	0.1	0.3045	0	1.0717	0.4756
2	0.2	0.2822	0	1.1486	0.4589
	0.3	0.2648	0	1.2310	0.4485
	0.1	0.5289	0.3819	1.985	0.8809
3	0.2	0.4353	0.1459	1.970	0.7872
	0.3	0.3685	0.0557	1.9555	0.7125
	0.1	0.2310	0	1.1159	0.4952
4	0.2	0.2167	0	1.2455	0.4977
	0.3	0.2058	0	1.3902	0.5065
	0.1	0.3692	0.00014	1.0022	0.4447
5	0.2	0.3327	0.00015	1.0084	0.4029
	0.3	0.3042	0.00016	1.0178	0.3708
	0.1	0.4127	0.0002	1.0787	0.4787
6	0.2	0.3528	0.0002	1.0487	0.4190
	0.3	0.3098	0.0002	1.0283	0.3747
	0.1	0.4769	0.4094	1.2082	0.5361
7	0.2	0.4037	0.1676	1.1761	0.4699
	0.3	0.3510	0.0686	1.1504	0.4192
	0.1	0.5429	0.1458	2.9636	1.3152
8	0.2	0.4192	0.0212	2.9279	1.1699
	0.3	0.3385	0.0031	2.8926	1.0540
9	0.1	0.1860	0	1.1484	0.5096
	0.2	0.1756	0	1.3192	0.5271
	0.3	0.1678	0	1.5154	0.5522
	0.1	0.2728	0	1.0717	0.4756
10	0.2	0.2519	0	1.1486	0.4589
	0.3	0.2358	0	1.2310	0.4485
	0.1	0.3101	0	1.0062	0.4465
11	0.2	0.2798	0	1.0217	0.4082
	0.3	0.2565	0	1.0441	0.3804
	0.1	0.3402	0	1.0043	0.4457
12	0.2	0.3073	0	1.0154	0.4057
	0.3	0.2818	0	1.0316	0.3759
	0.1	0.3972	0.0002	1.0001	0.4438
13	0.2	0.3561	0.0002	1.0009	0.3999
	0.3	0.3237	0.0002	1.0029	0.3654
	0.1	0.3361	0	1.0158	0.4508
14	0.2	0.2922	0	1.0020	0.4003
	0.3	0.2608	0	1.0005	0.3645
	0.1	0.3634	0	1.0269	0.4557
15	0.2	0.3152	0	1.0082	0.4028
	0.3	0.2807	0	1.0007	0.3646
	0.1	0.3890	0.0001	1.0459	0.4641
16,17	0.2	0.3356	0.0001	1.0219	0.4083
	0.3	0.2972	0.0001	1.0082	0.3673

Continued

Table 3. *Continued*

N	p	$\mathcal{E}_p(T)$	$B_{1,p}$	$B_{2,p}$	$B_{2,p}\mathcal{E}_p(X_1)$
	0.1	0.4127	0.0002	1.0787	0.4787
18,19	0.2	0.3528	0.0002	1.0487	0.4190
	0.3	0.3098	0.0002	1.0283	0.3747
	0.1	0.4338	0.0002	1.1319	0.5023
20,21	0.2	0.3660	0.0002	1.0940	0.4371
	0.3	0.3178	0.0003	1.0656	0.3883
	0.1	0.4506	0.0003	1.2059	0.5351
22	0.2	0.3734	0.1459	1.1567	0.4622
	0.3	0.3195	0.0557	1.1186	0.4076
	0.1	0.4558	0.1676	1.2943	0.5744
23	0.2	0.3677	0.0280	1.2307	0.4917
	0.3	0.3093	0.0047	1.1811	0.4303
	0.1	0.4577	1	1.1100	0.4926
24	0.2	0.3935	1	1.0840	0.4331
	0.3	0.3470	0.0774	1.0652	0.3881
	0.1	0.4914	0.3819	1.3270	0.5889
25	0.2	0.4072	0.1459	1.2810	0.5118
	0.3	0.3479	0.0557	1.2434	0.4530
	0.1	0.4915	0.1628	1.4476	0.6424
26	0.2	0.3947	0.0265	1.3845	0.5532
	0.3	0.3304	0.0043	1.3332	0.4858
	0.1	0.5218	0.1563	1.9851	0.8809
27	0.2	0.4133	0.0244	1.9703	0.7873
	0.3	0.3413	0.0038	1.9556	0.7126
	0.1	0.5342	0	3.9378	1.7475
28	0.2	0.3926	0	3.8771	1.5492
	0.3	0.3064	0	3.8172	1.3909

In the following proposition, we compare the FCRE of two systems with distinct lifetimes.

Proposition 18. *Suppose that T_1 and T_2 are the lifetimes of two coherent systems with i.i.d. components and with distortion functions $\tilde{q}_1$ and $\tilde{q}_2$, respectively. Let $\varphi_p(u) = u[-\log(u)]^p$. Then*

$$D_{1,p}\mathcal{E}_p(T_1) \leq \mathcal{E}_p(T_2) \leq D_{2,p}\mathcal{E}_p(T_1), \quad 0 < p \leq 1,$$

where $D_{1,p} = \inf_{u\in(0,1)} \left(\frac{\varphi_p(\tilde{q}_2(u))}{\varphi_p(\tilde{q}_1(u))}\right)$ and $D_{2,p} = \sup_{u\in(0,1)} \left(\frac{\varphi_p(\tilde{q}_2(u))}{\varphi_p(\tilde{q}_1(u))}\right)$.

It is clear that if $D_{2,p} \leq 1$, then $\mathcal{E}_p(T_2) \leq \mathcal{E}_p(T_1)$. Now, let us consider two coherent systems with i.i.d components. Suppose that $T_1 = X_{1:2} = \min(X_1, X_2)$ is the lifetime of a 2- components parallel system with $\tilde{q}_1(u) = u^2$ and T_2 is the lifetime of a 2-out-of-3 system with $\tilde{q}_2(u) = 3u^2 - 2u^3$, then for $p = 0.2, 0.3$ from the previous proposition we obtain

$$(0.1582)\mathcal{E}_{0.2}(T_1) \leq \mathcal{E}_{0.2}(T_2) \leq (2.964)\mathcal{E}_{0.2}(T_1)$$

and

$$(0.062)\mathcal{E}_{0.3}(T_1) \leq \mathcal{E}_{0.3}(T_2) \leq (2.947)\mathcal{E}_{0.3}(T_1)$$

In the following example, we consider a series system with dependent and identically distributed (d.i.d) components and obtain the bounds of $\mathcal{E}_p(T)$.

Example 5. If $T = \min(X_1, X_2, X_3)$ is the lifetime of the series system with d.i.d. components having a FGM survival copula as

$$\tilde{C}(u_1, u_2, u_3) = u_1 u_2 u_3[1 + \alpha(1 - u_1)(1 - u_2)(1 - u_3)], \quad 0 \leq u_1, u_2, u_3 \leq 1, \quad -1 \leq \alpha \leq 1.$$

Then $\tilde{q}(u) = u^3[1 + \alpha(1 - u)^3]$. So, from Proposition 17, we obtain

$$0 \leq \mathcal{E}_{0.5}(T) \leq (1.73)\mathcal{E}_{0.5}(X_1).$$

Also, if $L \leq f(x) \leq M$, then from Proposition 16 we obtain

$$\frac{0.19}{M} \leq \mathcal{E}_{0.5}(T) \leq \frac{0.19}{L}.$$

Example 6. Suppose that $T = \min(X_1, X_2, X_3)$ is the lifetime of series system with d.i.d. components. If the component lifetimes are dependent with the Clayton-Oakes survival copula as

$$\tilde{C}(u_1, u_2) = \frac{u_1 u_2}{u_1 + u_2 - u_1 u_2}, \quad 0 \leq u_1, u_2 \leq 1.$$

Then $\tilde{q}(u) = \frac{u}{2-u}$. Hence, from Proposition 17, we obtain

$$(0.5)\mathcal{E}_{0.1}(X_1) \leq \mathcal{E}_{0.1}(T) \leq (1.07)\mathcal{E}_{0.1}(X_1).$$

Also, if $L \leq f(x) \leq M$, then from Proposition 16 we obtain

$$\frac{0.35}{M} \leq \mathcal{E}_{0.1}(T) \leq \frac{0.35}{L}.$$

Proposition 19. *Let T_1 and T_2 be the lifetimes of two coherent systems with the same structure and with i.i.d. components having the same copula and common distributions F and G, respectively. If $X \leq_{disp} Y$, then for any $0 < p \leq 1$ we have*

$$\mathcal{E}_p(T_1) \leq \mathcal{E}_p(T_2).$$

***Proof*.** Since both systems have a common distortion function $\tilde{q}$ and the same structure, then the proof follows from equation (3) and the assumption on the dispersive order. □

Corollary 1. *Under the assumptions of Proposition 19, if $X \leq_{hr} Y$ and X or Y is DFR, then $\mathcal{E}_p(T_1) \leq \mathcal{E}_p(T_2)$.*

Corollary 2. *Under the assumptions of Proposition 19, if $X \leq_{su} Y(X \leq_{*} Y$ or $X \leq_c Y)$, then $\mathcal{E}_p(T_1) \leq \mathcal{E}_p(T_2)$.*

Theorem 1. *Let T_1 and T_2 be the lifetimes of two coherent systems with the same structure and with i.i.d. components having common distributions F and G, respectively. If $\mathcal{E}_p(X) \leq \mathcal{E}_p(Y)$ and*

$$\inf_{u \in A_1} \left[\frac{\varphi_p(\tilde{q}(u))}{\varphi_p(u)} \right] \geq \sup_{u \in A_2} \left[\frac{\varphi_p(\tilde{q}(u))}{\varphi_p(u)} \right],$$

for $A_1 = \{u \in [0,1] : f(\bar{F}^{-1}(u)) > g(\bar{G}^{-1}(u))\}$ and $A_2 = \{u \in [0,1] : f(\bar{F}^{-1}(u)) \leq g(\bar{G}^{-1}(u))\}$, then $\mathcal{E}_p(T_1) \leq \mathcal{E}_p(T_2)$.

***Proof*.** Since $\mathcal{E}_p(X) \leq \mathcal{E}_p(Y)$, we have from (1) that

$$\mathcal{E}_p(Y) - \mathcal{E}_p(X) = \int_0^1 \Delta(u) du \geq 0, \ \ 0 < p \leq 1,$$

where $\Delta(u) = \frac{\varphi_p(u)}{g_Y(\bar{G}_Y^{-1}(u))} - \frac{\varphi_p(u)}{f_X(\bar{F}_X^{-1}(u))}$. It follows from (3) that

$$\begin{aligned}
\mathcal{E}_p(T_2) - \mathcal{E}_p(T_1) &= \int_0^1 \frac{\varphi_p(\tilde{q}(u))}{\varphi_p(u)} \Delta(u) du \\
&= \int_{A_1} \frac{\varphi_p(\tilde{q}(u))}{\varphi_p(u)} \Delta(u) du + \int_{A_2} \frac{\varphi_p(\tilde{q}(u))}{\varphi_p(u)} \Delta(u) du \\
&\geq \inf_{u \in A_1} \frac{\varphi_p(\tilde{q}(u))}{\varphi_p(u)} \int_{A_1} \Delta(u) du + \sup_{u \in A_2} \frac{\varphi_p(\tilde{q}(u))}{\varphi_p(u)} \int_{A_2} \Delta(u) du \\
&\geq \sup_{u \in A_2} \frac{\varphi_p(\tilde{q}(u))}{\varphi_p(u)} \int_{A_2} \Delta(u) du \\
&\geq 0.
\end{aligned}$$

So, the proof is completed. □

Corollary 3. *Under the assumptions of Theorem 1, if $\tilde{q}$ is strictly increasing in $(0,1)$, then $\mathcal{E}_p(X) \leq_{disp} \mathcal{E}_p(Y)$ if and only if $\mathcal{E}_p(T_1) \leq_{disp} \mathcal{E}_p(T_2)$.*

Proof. *The proof follows from Theorem 2.9 of Navarro et al. (2013).*

Corollary 4. *Suppose that T is the lifetime of a coherent system with i.i.d. components and with distortion function $\tilde{q}$, then the FCPE of the random lifetime T is obtained as*

$$\mathcal{FCPE}_p(T) = \mathcal{E}_p(T) + \mathcal{CE}_p(T), \ \ 0 < p \leq 1,$$

where

$$\mathcal{CE}_p(T) = \int_0^1 \frac{\varphi_p(\tilde{q}(1-u))}{f_X(F_X^{-1}(1-u))} du,$$

is the FCPE of T and $F^{-1}(1-u) = \sup\{x : F(x) \leq 1-u\}$.

Proof. *The proof follows from Definition 2 and equation (3).*

4. Conclusions

In this work we first obtained some results of FCRE and fractional cumulative paired entropy. Also, a dynamic version of the FCRE is considered. We studied this measure of uncertainty for the coherent system's lifetime consisting of dependent and identically distributed components. We obtained upper and lower bounds of $\mathcal{E}_p(T)$ based on a representation of distortion function of the component survival function. Finally, we can obtain some results of $\mathcal{FCPE}_p(T)$ in coherent systems for future researches.

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References

Asadi, M. & Zohrevand, Y. (2007), 'On the dynamic cumulative residual entropy', *Journal of Statistical Planning and Inference* **137**(6), 1931–1941.

Burkschat, M. & Navarro, J. (2018), 'Stochastic comparisons of systems based on sequential order statistics via properties of distorted distributions', *Probability in the Engineering and Informational Sciences* **32**(2), 246–274.

Calì, C., Longobardi, M. & Navarro, J. (2020), 'Properties for generalized cumulative past measures of information', *Probability in the Engineering and Informational Sciences* **34**(1), 92–111.

Da Costa Bueno, V. & Balakrishnan, N. (2020), 'A cumulative residual inaccuracy measure for coherent systems at component level and under nonhomogeneous poisson processes ', *Probability in the Engineering and Informational Sciences* pp. 1–26.

Di Crescenzo, A. & Longobardi, M. (2009), 'On cumulative entropies', *Journal of Statistical Planning and Inference* **139**(12), 4072–4087.

Klein, I., Mangold, B. & Doll, M. (2016), 'Cumulative paired φ-entropy', *Entropy* **18**(7), 248.

Longobardi, M. (2014), 'Cumulative measures of information and stochastic orders', *Ricerche di Matematica* **63**(1), 209–223.

Murthy, D. P., Xie, M. & Jiang, R. (2004), *Weibull Models*, Vol. 505, John Wiley & Sons.

Navarro, J., del Águila, Y., Sordo, M. A. & Suárez-Llorens, A. (2013), 'Stochastic ordering properties for systems with dependent identically distributed components', *Applied Stochastic Models in Business and Industry* **29**(3), 264–278.

Navarro, J. & Psarrakos, G. (2017), 'Characterizations based on generalized cumulative residual entropy functions', *Communications in Statistics-Theory and Methods* **46**(3), 1247–1260.

Psarrakos, G. & Navarro, j. (2013), 'Generalized cumulative residual entropy and record values', *Metrika* **76**(5), 623–640.

Psarrakos, G. & Toomaj, A. (2017), 'On the generalized cumulative residual entropy with applications in actuarial science', *Journal of Computational and Applied Mathematics* **309**, 186–199.

Rahimi, S., Tahmasebi, S. & Lak, F. (2020), 'Extended cumulative entropy based on k th lower record values for the coherent systems lifetime', *Journal of Inequalities and Applications* **2020**(1), 1–22.

Rao, M., Chen, Y., Vemuri, B. C. & Wang, F. (2004), 'Cumulative residual entropy: a new measure of information', *IEEE transactions on Information Theory* **50**(6), 1220–1228.

Shaked, M. & Shanthikumar, J. G. (2007), *Stochastic Orders*, Springer, New York.

Shannon, C. (1948), 'A mathematical theory of communication', *Bell System Technical Journal* **27**, 379–432.

Tahmasebi, S., Keshavarz, A., Longobardi, M. & Mohammadi, R. (2020), 'A shift-dependent measure of extended cumulative entropy and iIts applications in blind image quality ', *Symmetry* **12**(2), 316.

Toomaj, A., Di Crescenzo, A. & Doostparast, M. (2018), 'Some results on information properties of coherent systems', *Applied Stochastic Models in Business and Industry* **34**(2), 128–143.

Toomaj, A., Sunoj, S. M. & Navarro, J. (2017), 'Some properties of the cumulative residual entropy of coherent and mixed systems', *Journal of Applied Probability* **54**(2), 379.

Ubriaco, M. R. (2009), 'Entropies based on fractional calculus', *Physics Letters A* **373**(30), 2516–2519.

Xiong, H., Shang, P. & Zhang, Y. (2019), 'Fractional cumulative residual entropy', *Communications in Nonlinear Science and Numerical Simulation* **78**, 104879.