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Stress-Strength Reliability Estimation of Time-Dependent Models with Fixed Stress and Phase Type Strength Distribution

Estimación de la confiabilidad de resistencia a la tensión de modelos dependientes del tiempo con estrés fijo y distribución de fuerza de tipo de fas

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Abstract

The time-dependent stress-strength reliability models deal with systems whose strength or the stress imposed on it or both are time-dependent. In this paper, we consider time-dependent stress-strength reliability model which is subjected to constant stress and it causes a change in the strength of the system over each run of the system. Assuming a continuous phase-type distribution for the initial strength and exponential distribution for the duration of each run of the system called cycle time we derived the expression for the stress-strength reliability of the system at time t. The model is further extended to the cases where cycle time distribution is Gamma and Weibull. Simulation studies are conducted to assess the variations in stress-strength reliability, $R(t)$ at different time points, corresponding to the changes in the initial strength distribution and cycle time distribution.

***Key words*:** EM algorithm; Exponential distribution; Gamma distribution; Phase type distribution; Stress-Strength reliability; Weibull distribution.

Resumen

Los modelos de confiabilidad tensión-resistencia dependientes del tiempo tratan con sistemas cuya fuerza o el estrés que se le impone o ambos dependen de tiempo. En este artículo, consideramos modelos de confiabilidad de resistencia-tensión dependientes del tiempo que está sometido a un estrés constante y provoca un cambio en la fuerza del sistema después de cada ejecución del sistema. Asumiendo una fase continua distribución de tipo para la fuerza inicial y distribución exponencial para la duración de cada

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ejecución del sistema llamado tiempo de ciclo que obtuvimos la expresión de la fiabilidad tensión-resistencia del sistema en el tiempo t. El modelo se amplía aún más a los casos en los que la distribución del tiempo de ciclo es Gamma y Weibull. Se realizan estudios de simulación para evaluar las variaciones en la confiabilidad tensión-resistencia, $R(t)$ en diferentes puntos de tiempo, correspondiente a los cambios en la distribución y el ciclo de la fuerza inicial distribución del tiempo.

***Palabras clave*:** Algoritmo EM; Distribución de tipo de fase; Distribución Gamma; Distribución exponencial; Distribución de Weibull; Fiabilidad de resistencia al estrés.

1. Introduction

Stress-strength models are of special importance in reliability theory. It deals with the evaluation of the probability that the strength of a component of a system exceeds the stress imposed on that component. Thus if X and Y represent the stress applied on the system and the strength of the system respectively, then the stress-strength reliability of the system is given by $R = P[X < Y]$. This model was introduced by Birnbaum (1956) and after that several researchers studied the estimation of R under various assumptions on the distribution of stress and strength. A detailed discussion of the computation of stress-strength reliability and various inference methods of R is given in Kotz & Pensky (2003). Assuming that stress and strength are independent random variables following different probability distributions Baklizi (2008), Kundu & Raqab (2009), Rezaei et al. (2010), Wong (2012), Huang et al. (2012), Rao, (2012; 2014), Kizilaslan & Nadar (2015), Ghitany et al. (2015), Jose et al. (2019) and Xavier & Jose (2020) have discussed the problem of estimating R for single and multi-component systems.

In a random period of time, if a system is subjected to repeated stress it will induce a change in the distribution of strength of the system over time. Stress-strength reliability models with stress or strength or both having dynamic characteristic are called dynamic or time-dependent stress-strength reliability models. Yadav (1973) and Gopalan & Venkateswarlu (1982; 1983) had studied several time-dependent models. An example of a deteriorating multi-component system is our mobile phone. Many often we may have to download files to mobile phones. Each file consumes the memory space of the phone corresponding to the size of that file. It causes a reduction in the speed of functioning of the phone. So each time we download a new file, there is an increase in the number of files piled up in the phone memory and a reduction in the functioning speed of the phone.

Here, we consider the stress-strength reliability model in which the system is subject to constant stress and it causes a change in the strength of the system over time. We assume that the strength of the system reduces by a constant over each run of the system, and is called random fixed strength and the time for completion of the run is called cycle time. The system fails whenever the stress imposed exceeds its strength. The stress-strength reliability of the dynamic system changes with respect to time. So in the case of dynamic or time-dependent stress-strength model, we are interested in the estimation of stress-strength reliability at time t,

$$R(t) = P[X(t) < Y(t)], \tag{1}$$

where $X(t)$ and $Y(t)$ represent the stress on the system and strength of the system at time t respectively.

A real-life system will always have some strength. When stress is applied to the system continuously, its strength will change and the stress-strength reliability measures the chance of further functioning of the system. For instance, suppose that an individual is a smoker. When he smokes, tobacco will burn and a toxic chemical particle called cigarette tar will be left behind. Whenever the person inhales the cigarette smoke, tar builds up inside the lungs. With the accumulation of tar, the color of the lungs will be converted to dark and eventually the tar can kill cilia in the lungs, which helps to trap pollutants. When cilia get damaged, the tar travels deeper into the lungs. From the lungs, it can move to other parts of the body. It can affect every organ in the body and can cause diseases like cancer, heart disease, etc. So when he smokes over time, there will be an increase in accumulation of tar and simultaneously a decrease in healthy cilia.

Siju & Kumar (2016; 2017) estimated time-dependent stress-strength reliability of the models with random fixed strength and fixed stress assuming Weibull distribution for initial strength and cycle time distribution as exponential. In this paper, we consider the estimation of the stress-strength reliability function $R(t)$ of the time-dependent models with cycle time distribution as exponential, Gamma or Weibull. The system is subjected to a constant stress and the distribution of the initial strength is continuous phase-type distribution.

There are several motivations for using phase-type distribution in reliability analysis. They constitute a very versatile class of distributions defined on the non-negative real numbers that lead to models which are algorithmically tractable. Hence they play an important role as a computational vehicle of much of applied probability. For a detailed discussion on phase-type distribution one may refer to Neuts (1975). Asmussen et al. (1996) introduced expectation maximization (EM) algorithm for fitting phase-type distribution. Expectation Maximization uniformization (EM unif) algorithm was developed by Bladt et al. (2011) as an improvement over EM algorithm.

Consider a continuous time Markov chain $\{X_t, t \geq 0\}$ on a discrete state space with a finite number of states $(m+1)$, of which $(m+1)^{th}$ state is absorbing and states $1, 2, \ldots, m$ are transient. Suppose that the generator matrix of the chain is

$$Q = \begin{pmatrix} B & t_B \\ \mathbf{0} & 0 \end{pmatrix}$$

where the diagonal elements of the matrix B say $B_{ii} \leq 0$; $i \leq m$ and the non diagonal elements of the matrix say $B_{ij} \geq 0$ for $i \neq j$, with $Be + t_B = \mathbf{0}$, where e is an $m \times 1$ column vector of unity and $\mathbf{0}$ is the $1 \times m$ zero matrix.

Let $\beta_i = P[X(0) = i]$ denote the initial probability. Hence the initial probability vector of $\{X(t), t \geq 0\}$ is given by (β, β_{m+1}) where $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$ and $\beta e + \beta_{m+1} = 1$.

A continuous phase-type (CPH) distribution is defined as the distribution of time to get absorbed to the absorbing state of a CTMC. If Z is a CPH random variable, then its probability density function (pdf) is defined as

$$f(z) = \beta e^{Bz} t_B; \quad z \geq 0 \tag{2}$$

where $t_B = -Be$ and we write as $Z \sim CPH(\beta, B)_m$. The set of parameters (β, B) is said to be a representation of the CPH distribution. The dimension of B is said to be the order of the representation.

Several standard statistical distributions like exponential distribution, erlang distribution, generalized erlang distribution and hyper-exponential distribution can be obtained as special cases of CPH distribution. The class of CPH distribution possesses several interesting closure properties which makes it a good choice for modelling many characteristics. Most of these results are well established in Neuts (1975).

Many authors have considered deteriorating complex multi-component manufacturing systems whose failure times, lifetimes or repair times follow phase-type distribution. Barron et al. (2004) analyzed an R-out-of-N repairable system assuming the lifetimes of components follow phase-type distribution. Barron & Yechiali (2017) dealt with deteriorating repairable Markovian system consisting of stochastically independent and identical units following discrete phase-type distribution (DPH). Eryilmaz (2018) considered CPH distribution as the distributions of stress and strength and studied stress-strength reliability for a single unit and multi-component systems. Jose et al. (2020) estimated stress-strength reliability for a single unit and multi-component systems assuming DPH as the distributions of stress and strength components.

This paper is organized as follows. In section 2, we briefly describe the stress-strength reliability of time-dependent model with cycle time following exponential distribution, constant stress and random-fixed strength. In sections 3 and 4, the model is extended to the case in which the cycle time distribution follows Gamma distribution and Weibull distribution respectively. Section 5 deals with the EM algorithm for estimating the parameters of CPH distribution. In section 6, simulation studies are conducted to assess the variations in $R(t)$ corresponding to the changes in the initial strength distribution and cycle time distribution. Section 7 deals with the concluding remarks.

2. Estimation of $R(t)$ with Cycle Time Follows Exponential Distribution

Consider a system which is allowed to run continuously within an interval of time. The initial strength of the system is assumed to follow CPH distribution and the stress on the system over the entire time interval is assumed to be fixed. Let x_0 be the stress applied on the system throughout the time interval $(0, t)$ and Y_0 be the initial strength of the system. The distribution of Y_0 is $CPH(\beta, B)_p$ with p.d.f.,

$$f_{Y_0}(y_0) = \beta e^{By_0} t_B; y_0 \geq 0. \tag{3}$$

Strength of the system is assumed to be decreasing over each run by a constant a_0. Let X_i denote the stress acting on the system during the i^{th} cycle time and Y_i be the corresponding strength of the system. Then probability of functioning of the system after n runs is given by

$$\begin{aligned} R_n &= P[(X_1 < Y_1) \cap (X_2 < Y_2) \cap \cdots \cap (X_n < Y_n)] \\ &= P[(x_0 < Y_0 - a_0) \cap (x_0 < Y_0 - 2a_0) \cap \cdots \cap (x_0 < Y_0 - na_0)] \\ &= P[(x_0 + na_0 < Y_0)] \\ &= \beta e^{B(x_0+na_0)} e_p' \end{aligned} \tag{4}$$

where $e_p = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}_{1\times p}$.

Now, assume that the cycle time has exponential distribution with rate λ, so that $N(t)$, the number of runs during an interval of time $(0, t)$ follows Poisson process. Hence the probability of n runs within $(0, t)$ is given by

$$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}. \tag{5}$$

Therefore, the value of $R(t)$ can be obtained as

$$\begin{aligned} R(t) &= \sum_{n=0}^{\infty} P_n(t) R_n \qquad (6) \\ &= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \beta e^{Bx_0 + Bna_0} e_p' \\ &= e^{-\lambda t} \beta e^{Bx_0} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{Bna_0} e_p' \\ &= e^{-\lambda t} \beta e^{Bx_0} e^{\lambda t e^{Ba_0}} e_p'. \end{aligned}$$

Now we consider the special case where the cycle time distribution is exponential with rate λ and the distribution of initial strength of the system is exponential with rate θ. Then

$$R(t) = e^{-\theta x_0 - \lambda t(1-e^{-\theta a_0})}. \tag{7}$$

Similarly, when the cycle time distribution is exponential with rate λ and the distribution of initial strength of the system is hyper-exponential with rate parameters $\theta_1, \theta_2, \ldots, \theta_s$ and mixing parameters $p_1, p_2, \ldots, p_s$ respectively, then

$$R(t) = \sum_{i=1}^{s} p_i e^{-\theta_i x_0 - \lambda t(1-e^{-\theta_i a_0})}. \tag{8}$$

If the initial strength distribution is a finite mixture of independent distributions, we have the following result.

Result 1. Consider a single component system subject to a fixed stress x_0, and initial strength distribution follows a finite mixture of continuous probability distributions $f(y_i)$, $i = 1, 2, \ldots, k$ with mixing probabilities $p_1, p_2, \ldots, p_k$. If cycle time distribution is exponential, then the stress-strength reliability function of the system is

$$R(t) = \sum_{i=1}^{s} p_i R_i(t),$$

where $R_i(t)$ is the stress-strength reliability function of the system with initial strength distribution is $f(y_i)$, fixed stress x_0, and cycle time distribution is exponential.

The graph of $R(t)$ corresponding to various values of parameters of cycle time distribution and initial strength distribution are given in Figures 1, 2 and 3. In all the case, $R(t)$ value increases with a decrease in the value of the cycle time parameter.

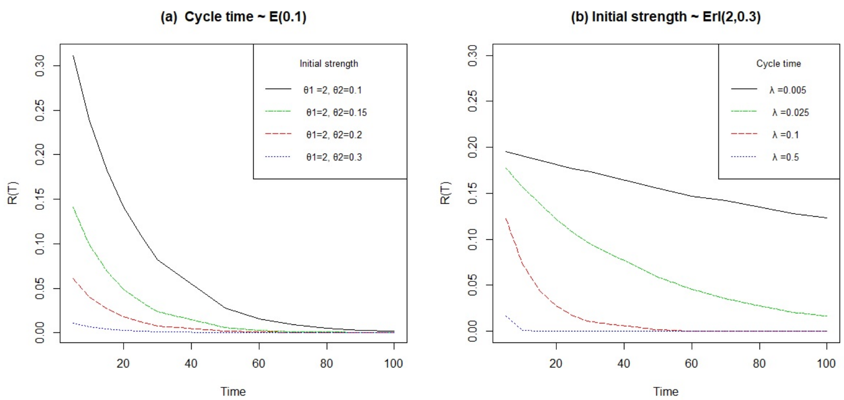

FIGURE 1: Change in $R(t)$ for systems with erlang initial strength and exponential cycle time distribution.

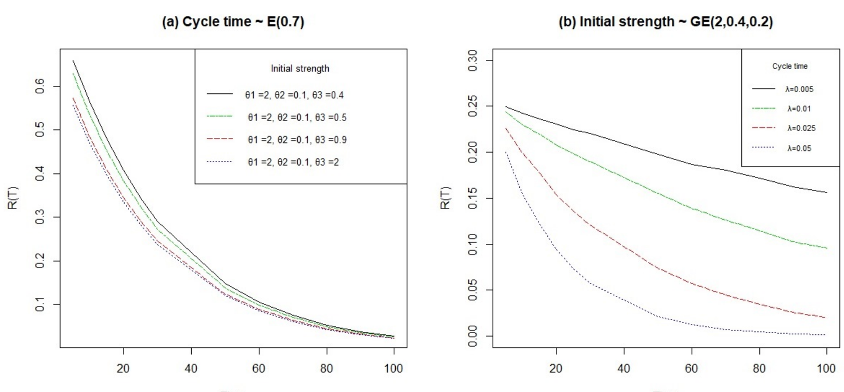

FIGURE 2: Change in $R(t)$ for systems with generalized erlang initial strength and exponential cycle time distribution.

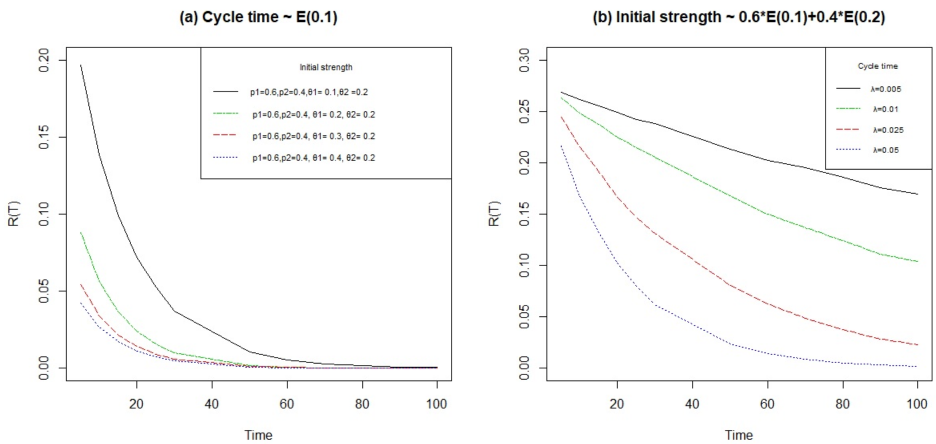

FIGURE 3: Change in $R(t)$ for systems with hyperexponential initial strength and exponential cycle time distribution.

2.1. Reliability of Series and Parallel Systems

Consider n identical systems connected in series. Let the initial strength of each system be distributed as $CPH(\beta, B)_p$. Then the corresponding stress-strength reliability can be obtained as

$$\begin{aligned}
R_s(t) &= \prod_{k=1}^{n}(R_k(t)) \\
&= (R(t))^n \\
&= (e^{-\lambda t}\beta e^{Bx_0} e^{\lambda t e^{Ba_0}} e_p')^n \\
&= e^{-\lambda t n}(\beta \otimes \cdots \otimes \beta) e^{(B\oplus\cdots\oplus B)x_0} e^{\lambda t(e^{Ba_0}\oplus\cdots\oplus e^{Ba_0})}(e_p' \otimes \cdots \otimes e_p') \\
&= e^{-\lambda t n}\beta^* e^{B^* x_0} e^{\lambda t C} e_p'^*, \qquad (9)
\end{aligned}$$

where $\beta^* = (\beta \otimes \beta \cdots \otimes \beta)$, $B^* = B \oplus B \cdots \oplus B$, $C = e^{Ba_0} \oplus e^{Ba_0} \cdots \oplus e^{Ba_0}$ and $e_p'^* = e_p' \otimes e_p' \cdots \otimes e_p'$

Now, consider n systems connected in parallel with identical initial strength.

$$\begin{aligned}
R_p(t) &= 1 - \prod_{k=1}^{n}(1 - R_k(t)) \\
&= 1 - (1 - R(t))^n \\
&= 1 - (1 - e^{-\lambda t}\beta e^{Bx_0} e^{\lambda t e^{Ba_0}} e_p')^n \qquad (10)
\end{aligned}$$

3. Estimation of $R(t)$ with Cycle Time Follows Gamma Distribution

In the previous section, we assumed that cycle time distribution is exponential. Now, we consider more general case in which we assume that the cycle times follows Gamma distribution with p.d.f.,

$$f(z) = \frac{a^k z^{k-1} e^{-az}}{(k-1)!}; z \geq 0. \tag{11}$$

Then the number of runs during the time interval $(0, t)$ has the distribution,

$$P_n(t) = e^{-at} \sum_{r=nk}^{(n+1)k-1} \frac{(at)^r}{r!}; n = 0, 1, 2, \dots \tag{12}$$

Therefore, using equation (6), the reliability of the system at time t with constant stress and random fixed strength following CPH distribution and Gamma cycle time distribution can be obtained as

$$\begin{aligned} R(t) &= \sum_{n=0}^{\infty} P_n(t) R_n \\ &= \sum_{n=0}^{\infty} e^{-at} \sum_{r=nk}^{(n+1)k-1} \frac{(at)^r}{r!} \beta e^{Bx_0 + Bna_0} e'_p \\ &= e^{-at} \beta e^{Bx_0} \sum_{n=0}^{\infty} \sum_{r=nk}^{(n+1)k-1} \frac{(at)^r}{r!} e^{Bna_0} e'_p \end{aligned} \tag{13}$$

Consider the initial strength as exponential random variable with rate θ and the distribution of cycle times as Gamma. Then we can obtain the expression for $R(t)$ as,

$$R(t) = e^{-at} \sum_{n=0}^{\infty} \sum_{r=nk}^{(n+1)k-1} \frac{(at)^r}{r!} e^{-\theta(x_0 + na_0)} \tag{14}$$

When the initial strength has hyper-exponential distribution with rate parameters $\theta_1, \theta_2, \dots, \theta_s$ and mixing parameters $p_1, p_2, \dots, p_s$ respectively, then the expression for $R(t)$ is

$$\begin{aligned} R(t) &= e^{-at} \sum_{n=0}^{\infty} \sum_{r=nk}^{(n+1)k-1} \frac{(at)^r}{r!} \sum_{i=1}^{s} p_i e^{-\theta_i(x_0 + na_0)}, \\ &= \sum_{i=1}^{s} p_i \left\{ e^{-at} \sum_{n=0}^{\infty} \sum_{r=nk}^{(n+1)k-1} \frac{(at)^r}{r!} e^{-\theta_i(x_0 + na_0)} \right\} \end{aligned} \tag{15}$$

If the initial strength distribution is a finite mixture of independent distributions, we have the following result.

Result 2. Consider a single component system subject to a fixed stress x_0, and initial strength distribution follows a finite mixture of continuous probability distributions $f(y_i)$, $i = 1, 2, \ldots, k$ with mixing probabilities $p_1, p_2, \ldots, p_k$. If cycle time distribution is Gamma, then the stress-strength reliability function of the system is

$$R(t) = \sum_{i=1}^{k} p_i R_i(t),$$

where $R_i(t)$ is the stress-strength reliability function of the system with initial strength distribution is $f(y_i)$, fixed stress x_0, and cycle time distribution is Gamma.

Figures 4, 5 and 6 represent the variation in $R(t)$ with respect to distinct parameter values of cycle time and initial strength distributions, when the cycle times are Gamma random variables. Here also the value $R(t)$ decreases with an increase in time, as in the case of exponential cycle times.

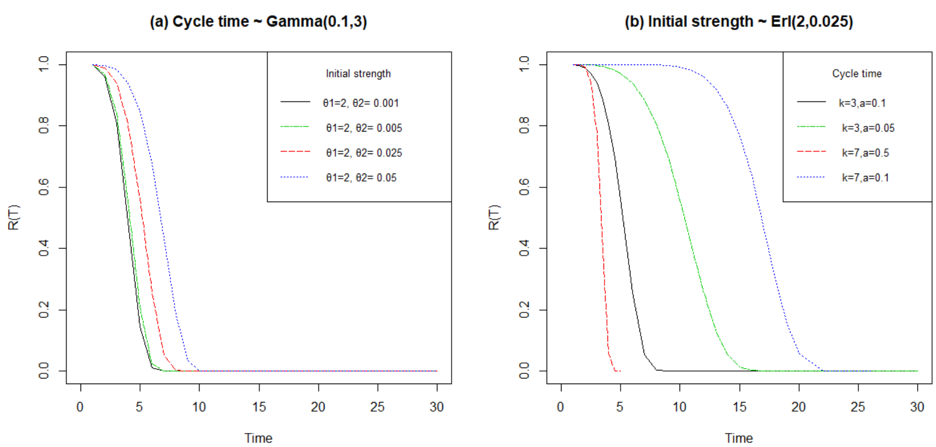

FIGURE 4: Change in $R(t)$ for systems with erlang initial strength and Gamma cycle time distribution.

4. Estimation of $R(t)$ with Cycle Time Follows Weibull Distribution

Assume that the cycle time distribution of the system is Weibull with shape parameter c. Then the p.d.f. of the cycle time distribution is given by,

$$f(t) = ct^{c-1}e^{-t^c}; \quad t \geq 0. \tag{16}$$

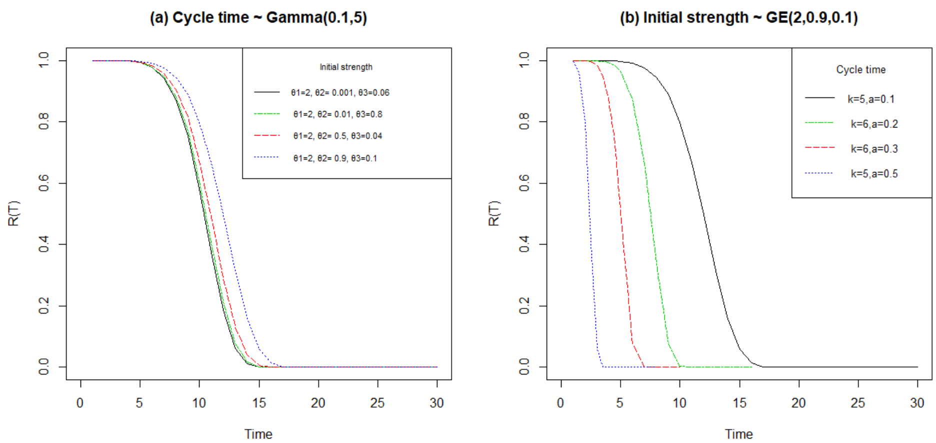

FIGURE 5: Change in $R(t)$ for systems with generalized erlang initial strength and Gamma cycle time distribution.

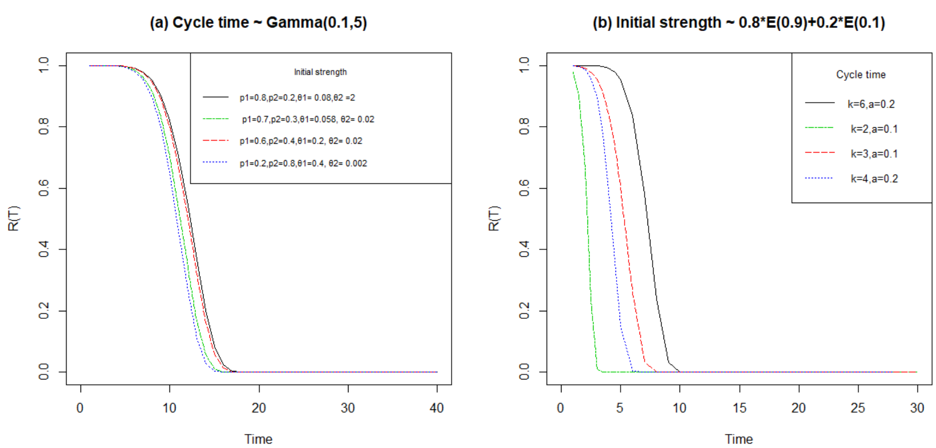

FIGURE 6: Change in $R(t)$ for systems with hyperexponential initial strength and Gamma cycle time distribution.

The corresponding c.d.f. has the form

$$F(t) = 1 - e^{-t^c}; \quad t > 0. \tag{17}$$

Then the number of runs during the time interval $(0, t)$ is a Weibull renewal process.

Let $t_1, t_2, \ldots$ denote the time at which renewal take place. $S_n = t_1 + t_2 + \cdots + t_n$ be the total time up to the instant of n^{th} renewal. Then the distribution of S_n, is given by

$$F_n(t) \quad = \int_0^t F_{n-1}(t-x)dF(x), \tag{18}$$

where $F_0(t) \equiv 1$.

Let $P_n(t)$ denote the probability of n renewals in $(0,t)$, which is defined as

$$P_n(t) = F_n(t) - F_{n+1}(t); n = 0, 1, 2, \ldots \tag{19}$$

Lomnicki (1966) obtained an infinite series expression for $P_n(t)$ in terms of the Poissonian function say $Q_n(t^c)$ as

$$P_n(t) = \sum_{i=n}^{\infty} \eta_n(i) Q_i(t^c); n = 0, 1, 2, \ldots \tag{20}$$

where

$$\eta_n(i) = \sum_{r=n}^{i} (-1)^{r+n} \begin{pmatrix} i \\ r \end{pmatrix} \frac{\mu_n(r)}{\lambda(r)}; n = 0, 1, 2, \ldots; i = n, n+1, \ldots \tag{21}$$

$$\lambda(j) = \frac{\Gamma(cj+1)}{\Gamma(j+1)};\ j = 0, 1, 2, \ldots \tag{22}$$

$$\mu_{n+1}(i) = \sum_{j=n}^{i-1} \mu_n(j)\lambda(i-j); \quad n = 0, 1, 2, \ldots; \quad i = n+1, n+2, \ldots \tag{23}$$

$$\mu_0(i) = \lambda(i); \quad i = 0, 1, 2, \ldots \tag{24}$$

The Poissonian function say $Q_n(t^c)$ satisfies

$$\frac{t^{cr}}{r!} = \sum_{i=r}^{\infty} \begin{pmatrix} i \\ r \end{pmatrix} Q_i(t^c). \tag{25}$$

$$Q_i(t) = e^{-t}\frac{t^k}{k!}. \tag{26}$$

$\eta_n(i)$ are bounded by $2^{i(n+1)}$ and hence $P_n(t)$ is absolutely convergent. Note that $\sum_{n=0}^{i} \eta_n(i) = 1$ and $\sum_{n=0}^{\infty} P_n(t) = 1$.

Therefore, using equation (6), the reliability of the system at time t with constant stress and random fixed strength following CPH distribution and Weibull cycle time distribution can be obtained as

$$R(t) = \sum_{n=0}^{\infty} \sum_{i=n}^{\infty} (\eta_n(i) Q_i(t^{\nu}) \beta e^{B(x_0+na_0)} e'_p \tag{27}$$

If the initial strength is exponential with rate θ and the cycle times is Weibull distributed with shape parameter ν, then

$$R(t) = \sum_{n=0}^{\infty} \sum_{i=n}^{\infty} (\eta_n(i) Q_i(t^{\nu}) e^{-\theta(x_0+na_0)}$$

When the initial strength has hyper-exponential distribution with rate parameters $\theta_1, \theta_2, \ldots, \theta_s$ and mixing parameters $p_1, p_2, \ldots, p_s$ respectively, then the expression for $R(t)$ is

$$\begin{aligned} R(t) &= \sum_{n=0}^{\infty}\sum_{i=n}^{\infty}(\eta_n(i)Q_i(t^\nu)\sum_{i=1}^{s}p_i e^{-\theta_i(x_0+na_0)} \\ &= \sum_{i=1}^{s}p_i\left\{\sum_{n=0}^{\infty}\sum_{i=n}^{\infty}(\eta_n(i)Q_i(t^\nu)e^{-\theta_i(x_0+na_0)}\right\} \end{aligned} \tag{28}$$

If the initial strength distribution is a finite mixture of independent distributions, we have the following result.

Result 3. Consider a single component system subject to a fixed stress x_0, and initial strength distribution follows a finite mixture of continuous probability distributions $f(y_i)$, $i = 1, 2, \ldots, k$ with mixing probabilities $p_1, p_2, \ldots, p_k$. If cycle time distribution is Weibull, then the stress-strength reliability function of the system is

$$R(t) = \sum_{i=1}^{k} p_i R_i(t),$$

where $R_i(t)$ is the stress-strength reliability function of the system with initial strength distribution $f(y_i)$, fixed stress x_0, and cycle time distribution is Weibull.

Figure 7 represents the variation in $R(t)$ with respect to distinct parameter values of cycle time and initial strength distributions, when the cycle times are Weibull random variables. From Figure 7, we can see that the value of $R(t)$ decreases with increase in strength parameter values as well as cycle time parameter values.

5. ML Estimation of Parameters of CPH Distribution

We have used EM uniformization algorithm for estimating the parameters of initial strength distribution. While dealing with phase-type random variables, we will only have the absorption times, and the entire underlying structure of the Markov process is not available. Asmussen et al. (1996) have presented a procedure for fitting CPH distributions via the EM algorithm. Using uniformization method, Bladt et al. (2011) developed an alternative method to compute the E-step in the EM algorithm, and is called as EM unif algorithm.

Let $\{y_1, y_2, \ldots, y_M\}$ be a sample of size M from $CPH(\beta, B)$, then $\{y_1, y_2, \ldots, y_M\}$ denote the time until absorption. Assume that we have complete observation of a Markov jump process with p states, $\{X(t),\ t \geq 0\}$. Consider y $\in \{y_1, y_2, \ldots, y_M\}$ and suppose that k jumps takes place before absorption. Let $x_0, x_1, \ldots, x_m$ be the sequence of states visited and the time taken between each of the jumps be $t_0, t_1, \ldots, t_m$, ie $t_0 + t_1 + \cdots + t_m = y$.

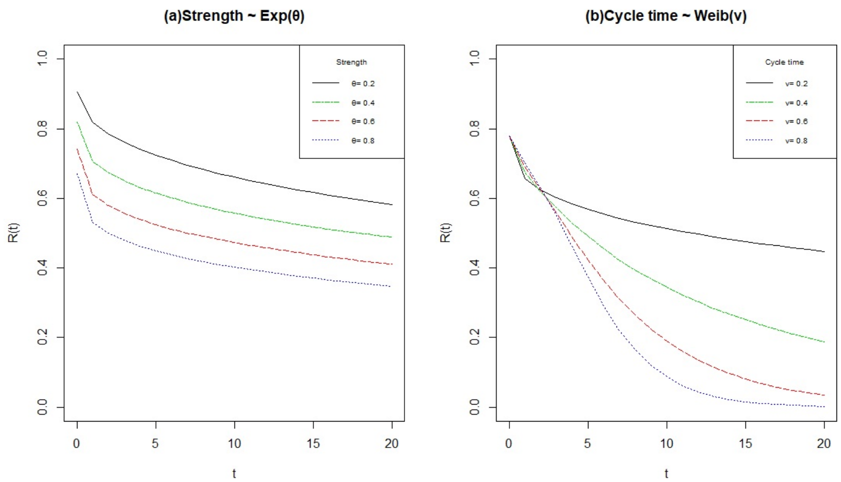

FIGURE 7: Change in $R(t)$ for systems with exponential initial strength and Weibull cycle time distribution.

We have to find the MLE of $\Gamma = (\beta, B, t)$ of $CPH(\alpha, A)$ using the observed data $\{y_1, y_2, \ldots, y_M\}$. Let $u^k = (x_0^k, x_1^k, \ldots, x_m^k)$ denote the trajectory of the underlying Markov jump processes(MJPs) $\{X(t), t \geq 0\}$ corresponding to the observation y_k and $u = \{u^k; k = 1, 2, \ldots, M\}$ gives the full data set for the M absorption times.

Let B_k represent the number of processes that start from state k, N_k be the number of processes which exit from state k to the absorbing state, N_{kl} be the number of jumps from k^{th} state to l^{th} state among all processes, and Z_k be the total time spent in the state k before absorption for all processes. Let B_k^m, Z_k^m, N_k^m and N_{kl}^m be the corresponding statistics for the m^{th} observation. The likelihood function of the complete data is

$$L = \prod_{k=1}^{p} \beta_k^{B_k} \prod_{k=1}^{p} \prod_{l \neq k}^{p} t_{kl}^{N_{kl}} e^{-t_{kl} Z_k} \prod_{k=1}^{p} t_k^{N_k} e^{-t_k Z_k} \tag{29}$$

The log-likelihood function for the complete data is given by

$$\begin{aligned} l_f(\Gamma; x) = & \sum_{k=1}^{p} B_k \log(\beta_k) + \sum_{k=1}^{p} \sum_{l \neq k}^{p} N_{kl} \log(t_{kl}) - \sum_{k=1}^{p} \sum_{l \neq k}^{p} t_{kl} Z_k \\ & + \sum_{k=1}^{p} N_k log(t_k) - \sum_{k=1}^{p} t_k Z_k \end{aligned} \tag{30}$$

Asmussen et al. (1996) considered

$$E_\Gamma \left(B_k \mid Y_m = y_m\right) = \sum_{m=1}^{M} E_\Gamma \left(B_k^m \mid Y_m = y_m\right) = \sum_{m=1}^{M} \frac{\beta_k b_k \left(y_m \mid \Gamma\right)}{\beta b \left(y_m \mid \Gamma\right)} \tag{31}$$

$$E_{\Gamma}\left(Z_k \mid Y_m = y_m\right) = \sum_{m=1}^{M} E_{\Gamma}\left(Z_k^m \mid Y_m = y_m\right) = \sum_{m=1}^{M} \frac{c_k\left(y_m, k \mid \Gamma\right)}{\beta b\left(y_m \mid \Gamma\right)} \tag{32}$$

$$E_{\Gamma}\left(B_k \mid Y_m = y_m\right) = \sum_{m=1}^{M} E_{\Gamma}\left(B_k^m \mid Y_m = \; y_m\right) = \sum_{m=1}^{M} \frac{t_k a_k\left(y_m \mid \Gamma\right)}{\beta b\left(y_m \mid \Gamma\right)} \tag{33}$$

$$E_{\Gamma}\left(B_k \mid Y_m = y_m\right) = \sum_{m=1}^{M} E_{\Gamma}\left(B_k^m \mid Y_m = \; y_m\right) = \sum_{m=1}^{M} \frac{t_{kl} c_l\left(y_{m,k} \mid \Gamma\right)}{\beta b\left(y_m \mid \Gamma\right)} \tag{34}$$

where $a(y|\Gamma) = \beta e^{By}$, $b(y|\Gamma) = e^{By} t_B$ and $c(y, k|\Gamma) = \int_0^y \beta e^{Bw} e_k e^{(B(y-w))} t_B dw$, $k = 1, 2, \ldots, p$ and e_k is the $p \times 1$ column vector with k^{th} element as unity and all other elements as zeros.

Bladt et al. (2011) replaced the constant unit of time between two transitions by independent exponential random variables with the same parameter and interprets a continuous time Markov process as a discrete time Markov chain. The EM unif algorithm is given below.

Algorithm

Step1.Set $\Gamma_0 = (\beta_0, B_0, t_{B_0})$, initial value of the parameters.

Step 2.(E-step): Calculate the function $r(\Gamma) = E_{\Gamma_0}(l_f(\Gamma; x) \mid Y = y)$

Step 3.(M-step): $\Gamma_0 = argmax_{\Gamma} r(\Gamma)$

Step 4. Goto step 2.

The E-step and M-step are repeated until convergence.

6. Numerical Analysis

In this section, we carry out Monte Carlo simulation to illustrate the estimation of the value of $R(t)$ of the time-dependent system with fixed stress, random cycle times and random fixed initial strength distributions. Here we consider three examples. In the first example consider the system with exponential cycle time distribution so that the number of cycles have Poisson process, in the second example we consider the system with Gamma cycle time distribution so that the number of cycles have Gamma renewal process and in the third example we consider the system with Weibull cycle time distribution so that the number of cycles have Weibull renewal process.

In each case we have simulated 10,000 observations each on cycle time and initial strength. The parameters of the models are estimated by ML method using the simulated data sets. The entire procedure is repeated 2,000 times and obtained the bootstrap lower confidence interval (LCI) of $R(t)$, at different time points, by using the estimates of $R(t)$ corresponding to first 1,000 samples. The remaining 1,000 simulated samples are used for computing the coverage probability (CP).

Example 1. We take cycle time distributions as exponential and initial strength distribution as generalized erlang. Let $X_1, X_2, \ldots, X_k$ are independent exponential random variables with parameters λ_i, $i = 1, 2, \ldots, k$ respectively. Now consider the sum $Z = \sum_{i=1}^{k} X_i$, the phase-type representation of Z is given by $Z \sim PH(\beta, B)$, where $\beta = (1, 0, 0, \ldots, 0)$ and

$$B = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & . & . & . & 0 & 0 \\ 0 & -\lambda_2 & \lambda_2 & . & . & . & 0 & 0 \\ . & & & & & & & \\ . & & & & & & & \\ 0 & 0 & 0 & . & . & . & -\lambda_{k-1} & \lambda_{k-1} \\ 0 & 0 & 0 & . & . & . & 0 & -\lambda_k \end{bmatrix}$$

The distribution of Z is called k-generalized erlang distribution.

Assume that the cycle time distributions as exponential with parameter $\lambda = 0.1$ and initial strength distribution as generalized erlang with parameters 2, 0.1 and 0.5. The value of the stress is taken as $x_0 = 10$ and the strength reducing constant as $a_0 = 6$. We have simulated 10,000 observations from exp(λ) and GE(2,0.1,0.5).

Using the simulated data, estimated the cycle time distribution parameter and parameters of the initial strength distribution. The estimated values are given in Table 1. The graph of the histograms corresponding to the simulated data on cycle times, number of runs during the time interval $(0, t)$ and initial strength along with the actual distribution are given in Figure 8.

Table 1: Estimated values of parameters (with cycle time follows exponential distribution).

	λ	β	B	t_B
Actual value	0.1	$\begin{pmatrix} 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -0.1 & 0.1 \\ 0 & -0.5 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0.5 \end{pmatrix}$
MLE	0.1001	$\begin{pmatrix} 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -0.1 & 0.1011 \\ 0 & -0.4999 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0.4996 \end{pmatrix}$

Table 2: Estimates of $R(t)$ (Cycle time follows exponential distribution).

Parameters		$t = 10$	$t = 50$	$t = 100$
$\lambda = 0.1$	$R(t)$	0.4367	0.3598	0.2822
$k = 2$	$\widehat{R}(t)$	0.4385	0.3607	0.2829
$\theta_1 = 0.1$	RMSE	0.0086	0.0068	0.0056
$\theta_2 = 0.5$	90 % LCI	(0.4277 ,1)	(0.3522,1)	(0.2762,1)
	CP	0.836	0.875	0.8967
	95% LCI	(0.4257,1)	(0.3500,1)	(0.2742,1)
	CP	0.902	0.935	0.914
	99% LCI	(0.4213,1)	(0.3459,1)	(0.2714,1)
		CP 0.923	0.969	0.934

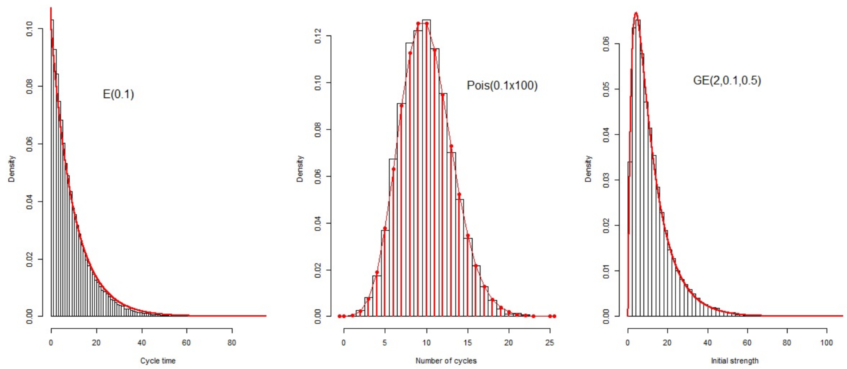

FIGURE 8: Histogram of simulated data with superimposed density curves.

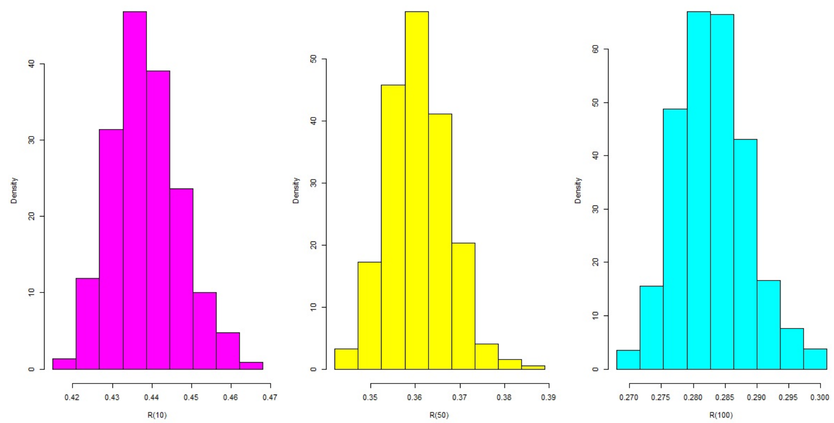

FIGURE 9: Histogram of sampling distribution of $R(t)$ for $\lambda = 0.1$; $\theta_1 = 2, \theta_2 = 0.1$, $\theta_3 = 0.5$.

The bootstrap lower confidence interval (LCI) of $R(t)$ at time points $t = 10$, 50, 100 and the corresponding coverage probabilities are given in Table 2. The histograms corresponding to the sampling distribution of $R(t)$ at $t = 10$, $t = 50$ and $t = 100$ are given in Figure 9.

Example 2. In this example we took the cycle time distribution as Gamma. So that the number of runs during the time interval $(0, t)$ follows a renewal process. Consider the parameters of cycle time distribution as $k = 1$ and $a = 0.1$. We took the initial strength distribution as exponential with parameter $\theta = 0.01$.

If Z is an exponential random variable with parameter θ having the pdf

$$f(x) = \theta e^{-\theta x};\ x > 0, \quad \theta > 0. \tag{35}$$

The phase-type representation of Z is given by $\beta = (1)$ and $B = [\theta]$.

The value of stress is fixed as $x_0 = 2$ and the strength reducing constant as $a_0 = 0.28$. We have estimated the parameters of cycle time distribution, initial strength distribution and $R(t)$ using the simulated data. The estimated values are given in Table 3. The graph of the histograms corresponding to the simulated data on cycle times, number of runs during the time interval $(0, t)$ and initial strength along with the actual distribution are given in Figure 10.

TABLE 3: Estimated values of parameters (Cycle time follows Gamma distribution).

	a	k	β	B	t_B
Actual value	0.1	1	$\begin{pmatrix} 1 \end{pmatrix}$	$\begin{pmatrix} -0.01 \end{pmatrix}$	$\begin{pmatrix} 0.01 \end{pmatrix}$
MLE	0.0976	0.9792	$\begin{pmatrix} 1 \end{pmatrix}$	$\begin{pmatrix} -0.01 \end{pmatrix}$	$\begin{pmatrix} 0.0099 \end{pmatrix}$

TABLE 4: Estimates of $R(t)$ (Cycle time follows Gamma distribution).

Parameters		$t = 2$	$t = 4$	$t = 5$
$a = 0:1$	$R(t)$	0.9423	0.7607	0.6316
$k = 1$	$\widehat{R}(t)$	0.9424	0.7633	0.6365
$\theta = 0.01$	RMSE	0.0359	0.0064	0.0234
	90% LCI	(0.9419,1)	(0.7601,1)	(0.6308,1)
	CP	0.8375	0.8675	0.815
	95% LCI	(0.9411,1)	(0.7579,1)	(0.6293,1)
	CP	0.8396	0.925	0.874
	99% LCI	(0.9398,1)	(0.7541,1)	(0.6153,1)
	CP	0.9075	0.9375	0.945

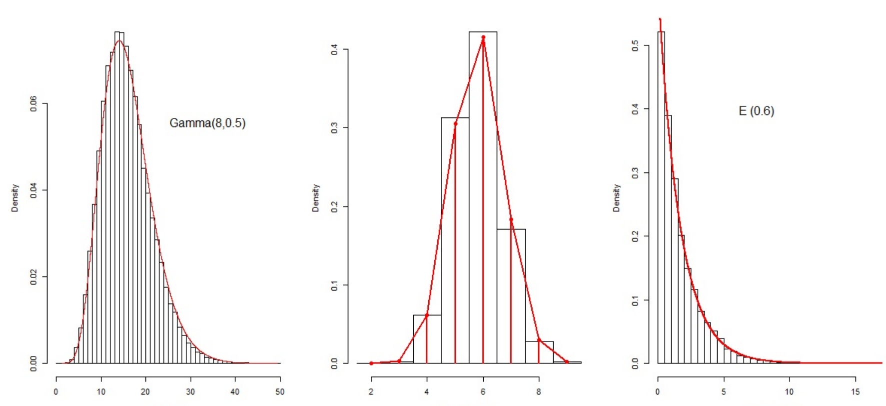

FIGURE 10: Histogram of simulated data with superimposed density curves.

The bootstrap lower confidence interval (LCI) of $R(t)$ at time points $t = 2, 4, 5$ and the corresponding coverage probabilities are given in Table 4. The histograms of the sampling distribution of $R(t)$ at $t = 2$, $t = 4$ and $t = 5$ are given in Figure 11.

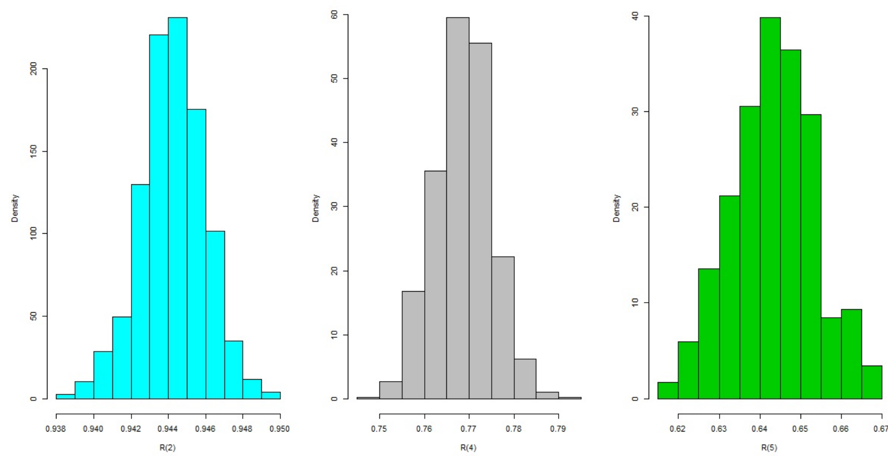

FIGURE 11: Histogram of sampling distribution of $R(t)$ for $a = 0.1$, $k = 1$ and $\theta = 0.01$.

Example 3. Here we assume that the distribution of cycle times as Weibull with shape parameter value $c = 1.5$ and the distribution of initial strength as hyper-exponential with parameter $p_1 = 0.2$, $p_2 = 0.8$, $\theta_1 = 0.3$, $\theta_2 = 0.5$. The value of the stress is taken as $x_0 = 0.005$ and the strength reducing constant as $a_0 = 0.001$.

Let $X_1, X_2, \ldots, X_k$ are k independent exponential random variables with parameters $\lambda_1, \lambda_2, \ldots, \lambda_k$ respectively. Assume that Z takes the value of x_i with probability p_i. Then Z_i can be expressed as a mixture of X_i's with k components and is called hyper - exponential distribution. Then $Z \sim PH(\beta, B)$, where

$$\beta = (p_1, p_2, \ldots, p_k)$$

and

$$B = \begin{bmatrix} -\lambda_1 & 0 & . & . & . & 0 \\ 0 & -\lambda_2 & . & . & . & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & . & . & . & -\lambda_k \end{bmatrix}$$

The probability density function of the hyper exponential distribution is

$$f(x) = \sum_{i=1}^{k} p_i \lambda_i e^{-\lambda_i x}; x \geq 0 \tag{36}$$

TABLE 5: Estimated values of parameters (Cycle time follows Weibull distribution).

	c	β	B	t_B
True value	1.5	$\begin{pmatrix} 0.2 & 0.8 \end{pmatrix}$	$\begin{pmatrix} -0.3 & 0 \\ 0 & -0.5 \end{pmatrix}$	$\begin{pmatrix} 0.3 \\ 0.5 \end{pmatrix}$
MLE	1.5631	$\begin{pmatrix} 0.2009 & 0.7991 \end{pmatrix}$	$\begin{pmatrix} -0.2972 & 0 \\ 0 & -0.4950 \end{pmatrix}$	$\begin{pmatrix} 0.2972 \\ 0.4950 \end{pmatrix}$

Using the simulated data, we have estimated the cycle time distribution parameter and parameters of the initial strength distribution. The estimated values are given in Table 5.

The graph of the histograms corresponding to the simulated data on cycle times, number of runs during the time interval $(0, t)$ and initial strength along with the actual distribution are given in Figure 12. The bootstrap lower confidence interval (LCI) of $R(t)$ at time points $t = 2, 5, 10$ and the corresponding coverage probabilities are given in Table 6. The histograms corresponding to the sampling distribution of $R(t)$ at $t = 2$, $t = 5$ and $t = 10$ are given in Figure 13.

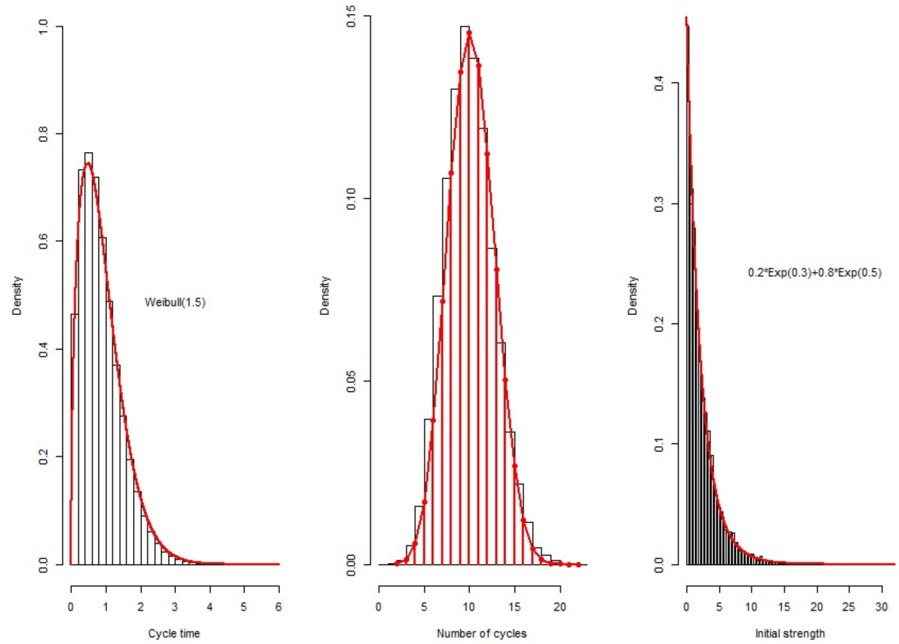

FIGURE 12: Histogram of simulated data with superimposed density curves.

TABLE 6: Estimates of $R(t)$ (Cycle time follows Weibull distribution).

Parameters		$t = 2$	$t = 5$	$t = 10$
$c = 1.5$	$R(t)$	0.9941	0.5794	0.0024
$p_1 = 0.2$	$\widehat{R}(t)$	0.9945	0.5796	0.0027
$p_2 = 0.8$	RMSE	0.0003	0.0001	0.0002
$\theta_1 = 0.3$	90% LCI	(0.9936,1)	(0.5787,1)	(0.0019,1)
$\theta_2 = 0.5$	CP	0.8931	0.8311	0.8411
	95% LCI	(0.9924,1)	(0.5764,1)	(0.0015,1)
	CP	0.8984	0.9080	0.9343
	99% LCI	(0.9918,1)	(0.5759,1)	(0.0009,1)
	CP	0.9423	0.9135	0.9498

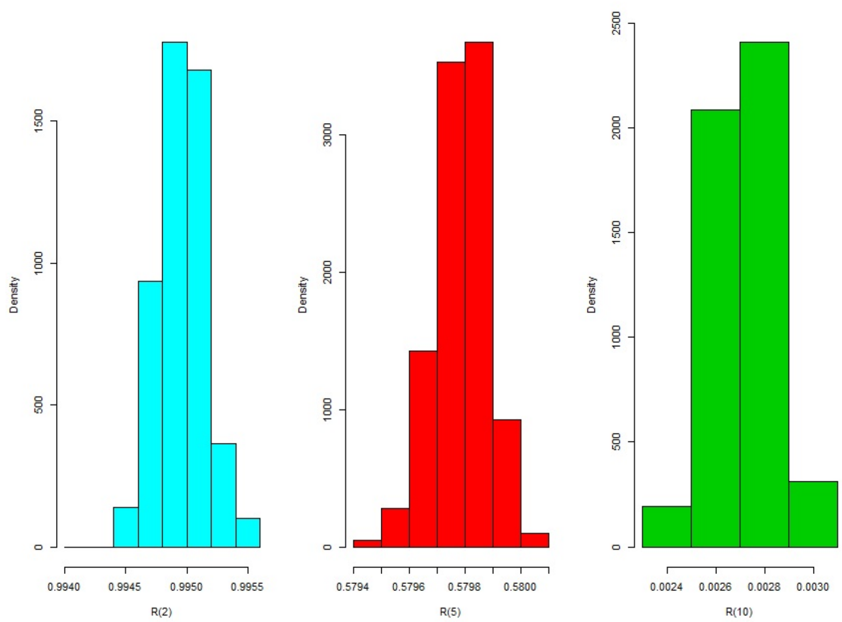

FIGURE 13: Histogram of sampling distribution of $R(t)$ for $\nu = 1.5$; $\theta = 0.5$.

7. Conclusion

In this paper, we considered time dependent stress-strength reliability model subjected to constant stress. The stress causes a change in the strength of the system during each run of the system. $R(t)$ is estimated under the assumption that the initial strength has CPH distribution and the cycle times have either exponential, Gamma or Weibull distribution. We have obtained the expression of $R(t)$ for identical systems connected in series as well as parallel configuration when the cycle times follow an exponential distribution. Variation in $R(t)$ with respect to time is studied for various cases of systems with cycle times following exponential, Gamma and Weibull distribution. ML estimates, bootstrap lower confidence intervals and corresponding coverage probability are also obtained for $R(t)$ at different points of time.

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