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# A single queue with mutually replacing $m$ servers

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ABSTRACT.
Customers arriving in a Poisson stream are served one by one exponentially by the $m$ servers $S_1, S_2, \ldots, S_m$. A server who has just completed a service either continues the next service or is replaced by another server. The replacement of server at each service completion is governed by a probabilistic criterion of availability of servers. Transient solutions in terms of Laplace transforms of the probability generating functions are obtained and among some special cases the earlier known results of Madan (1990) for the case of 2 servers are deduced. The steady state solutions and the mean queue lengths have been explicitly for some particular cases.

**Key words** Poisson stream, exponential service, availability criterion, Laplace transform, probability generating service, sequential service, steady state, mean queue length.

## 1. Introduction

Madan (1990) has studied a 2-server queue with correlated availability of servers. The idea was motivated by some papers dealing with correlated arrivals, correlated departures and some other similar situations. (c.f. Chaudhry (1965), Mohan (1955), Mirari (1969)). in this paper, we generalize the problem to $m$-servers. The mathematical model is briefly described by the following assumptions:

(1) Arrivals occur one by one in a Poisson stream with mean arrival rate $\lambda(>0)$.
(2) The system has $m$ servers designated as $S_1, S_2, \ldots, S_m$ and only one of them serves the customers at a time. The service is provided on a first

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come, first served basis and the service times of $S_j$ are exponentially distributed with mean $1/\mu_j, j = 1, 2, \ldots, m$.

(3) Whenever an empty system starts with a first arriving customer, the servers $S_1, S_2, \ldots, S_m$ are available with respective probabilities $\pi_j = 1, 2, \ldots, m$, where $\sum_{j=1}^{m} \pi_j = 1$.

(4) Subsequently, however, at the completion of each service, a server who has just completed a service either continues the next service or is replaced by another server. The availability criterion of server is determined by the conditional probability $p_{ij}$ = the probability that the server $S_j$ has just completed a service. Obviously, when $j = i$, it means that the server $S_i$ continues with the next service. Thus the $m \times m$ availability matrix is given by

$$\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}$$

(5) It has been assumed that the replacement of servers is instantaneous.

## 2. Equations governing the system

Define $P_n^{(j)}(t)(n \geq 0)$ as the probability that at time $t$ there are $n$ customers in the queue excluding the one being served by the jth servers $j = 1, 2, \ldots, m$ and let $Q(t)$ be the probability that at time $t$ the queue length is zero and none of the $m$ servers is providing service. The following set of forward equations govern the system for $j = 1, 2, \ldots, m$:

$$\frac{d}{dt} P_n^{(j)}(t) + (\lambda + \mu_j) P_n^{(j)}(t) = \lambda P_{n-1}^{(j)}(t) + \sum_{k=1}^{m} p_{kj} \mu_k P_{n+1}^{(k)}(t) \qquad (n > 0) \quad (1)$$

$$\frac{d}{dt} P_0^{(j)}(t) + (\lambda + \mu_j) P_0^{(j)}(t) = \lambda \pi_j Q(t) + \sum_{k=1}^{m} p_{kj} \mu_k P_0^{(k)}(t) \quad (2)$$

$$\frac{d}{dt} Q(t) + \lambda Q(t) = \sum_{k=1}^{m} \mu_k P_0^{(k)}(t) \quad (3)$$

We assume that initially there are no customers either waiting in the queue or being served so that the initial condition is

$$Q(0) = 1 \quad (4)$$

Taking Laplace transform of equation (1) to (3) and using equation (4), we have

$$(s+\lambda+\mu_j)\bar{P}_n^{(j)}(s) = \lambda\bar{P}_{n-1}^{(j)}(s) + \sum_{k=1}^{m} p_{kj}\mu_k\bar{P}_{n+1}^{(k)}(s) \qquad (n>0) \tag{5}$$

$$(s+\lambda+\mu_j)\bar{P}_0^{(j)}(s) = \lambda\pi_j\bar{Q}(s) + \sum_{k=1}^{m} p_{kj}\mu_k\bar{P}_0^{(k)}(s) \tag{6}$$

$$(s+\lambda)\bar{Q}(s) = 1 + \sum_{k=1}^{m} \mu_k\bar{P}_0^{(k)}(s) \tag{7}$$

## 3. The time-dependent solution

Let $\bar{P}^{(j)}(s,z) = \sum_{n=0}^{\infty}\bar{P}_n^{(j)}(s)z^n, |z| \le 1$ define the probability generating functions of queue length under various states in terms of their Laplace transforms. Multiplying equations (5) and (6) by suitable powers of $z$ and simplifying, we obtain

$$(s+\lambda+\mu_j-\lambda z)z\bar{P}^{(j)}(s,z) = \lambda z\pi_j\bar{Q}(s) + \sum_{k=1}^{m} p_{kj}\mu_k\bar{P}^{(k)}(s,z)$$

$$-\sum_{k=1}^{m} p_{kj}\mu_k\bar{P}^{(k)}(s), \quad j=1,2,\dots,m \tag{8}$$

We solve the system of equations given by (8) for $\bar{P}^{(j)}(s,z), j=1,2,\dots,m$ and have

$$\bar{P}^{(j)}(s,z) = \frac{N^{(j)}}{\triangle} \tag{9}$$

Where $\triangle$ is the determinant of the $m\times m$ matrix

$$\begin{bmatrix} G_1 & -p_{21}\mu_2 & \cdots & -p_{m1}\mu_m \\ -p_{12}\mu_1 & G_2 & \cdots & -p_{m2}\mu_m \\ \cdots & \cdots & \cdots & \cdots \\ -p_{1m}\mu_1 & -p_{2m}\mu_2 & \cdots & G_m \end{bmatrix}$$

and

$$G_j = (s+\lambda+\mu_j-\lambda z)z - p_{jj}\mu_j$$

$$H_j = \lambda z\pi_j\bar{Q}(s) - \sum_{k=1}^{m} p_{kj}\mu_k\bar{P}_0^{(k)}(s)$$

and $N^{(j)}$ is the $m \times m$ determinant which is obtained from the determinant $\triangle$ by replacing the $j$ th column of $\triangle$ by the column vector $(H_1, H_2, \ldots, H_m)'$.

We note that due to the vector $(H_1, H_2, \ldots, H_m)'$, appearing in $N^{(j)}$, the numerator of each of the equations given by (9) involves $m+1$ unknowns, namely, $\bar{Q}(s)$ and $\bar{P}_0^{(k)}(s), k = 1, 2, \ldots, m$. We proceed to determine these unknowns. It is easy to see that each $G_j = (s + \lambda + \mu_j - \lambda z)z - p_{jj}\mu_j$ has only one zero inside the unit circle $|z| = 1$ for $j = 1, 2, \ldots, m$ and, for that matter, due to the product term $G_1 G_2 \ldots G_m$ appearing in $\triangle$, the denominator of the right hand side of (9) has $m$ zeros inside $|z| = 1$. These zeros give rise to $m$ equations in $m+1$ unknows mentioned above. In addition, equation (6) also involves the same $m+1$ unknowns. Thus there are in all $m+1$ equations in $m+1$ unknowns which are sufficient to determine the $m+1$ unknowns. Hence all the probability generating functions $\bar{P}^{(j)}(s, z)$, for $j = 1, 2, \ldots, m$, can be completely determined.

## 4. Some particular cases

**Case 1** (sequential service)

If we let $p_{12} = p_{23} = p_{34} = \ldots = p_{m-1,m} = p_{m1} = 1$ and all other p's are zero, this essentially means that the servers are providing sequential service. In this case, the corresponding results will be given by (9) where now, we will have

$$G_j = (s + \lambda + \mu_j - \lambda z), \quad j = 1, 2, \ldots, m$$
$$H_j = \lambda z \pi_j \bar{Q}(s) - \mu_{j-1} \bar{P}_0^{(j-1)}(s), \ j = 2, 3, \ldots, m$$
$$H_1 = \lambda z \pi_1 \bar{Q}(s) - \mu_m \bar{P}_0^{(m)}(s)$$

**Case 2** (each server completes his cycle)

Let $p_{ii} = 1$ for $i = 1, 2, \ldots, m$ and $p_{ij} = 0$ for $i \neq j$ which means that $S_1, S_2, \ldots S_m$ whosoever starts service continues serving until the queue becomes empty again. In this case,

$$G_j = (s + \lambda + \mu_j - \lambda z)z - \mu_j$$
$$H_j = \lambda z \pi_j \bar{Q}(s) - \mu_j \bar{P}_0^{(j)}(s)$$

and then (9) yields

$$\bar{P}^{(j)}(s, z) = \frac{\lambda z \pi_j \bar{Q}(s) - \mu_j \bar{P}_0^{(j)}(s)}{(s + \lambda + \mu_j - \lambda z)z - \mu_j}, \qquad j = 1, 2, \ldots, m \tag{10}$$

The denominator of the right hand side of equation (10) has one zero inside the unit circle $|z| = 1$. Let $z = z_j$ be this zero. This zero gives

$$\lambda z_j \pi_j \bar{Q}(s) - \mu_j \bar{P}_0^{(j)}(s) = 0 \text{ for } j = 1, 2, \ldots, m \tag{11}$$

Using equation (11) in (7), we have

$$\bar{Q}(s) = \frac{1}{s+\lambda-\lambda\sum_{j=1}^{m}\pi_j z_j} \tag{12}$$

and hence

$$\bar{P}_0^{(j)} = \frac{\lambda z_j \pi_j}{\mu_j\{s+\lambda-\lambda\sum_{j=1}^{m}\pi_j z_j\}}, \quad j=1,2,\dots,m \tag{13}$$

**Steady state**

Let $P_n^{(j)}, j=1,2,\dots,m$ and $Q$ be the respective steady state probabilities corresponding to $P_n^{(j)}(t)$ and $Q(t)$ and, for that matter, let $P^{(j)}(z)$ be the steady state probability generating functions corresponding to $\bar{P}^{(j)}(s,z)$. Then the steady state solution can be obtained by using the well-known Tauberian property $\lim_{s\to 0} sf(s) = \lim_{t\to\infty} f(t)$.

We proceed to derive the steady state results only for the particular case 2 as follows: Applying the above Tauberian property, equation (10) yields

$$P^{(j)}(z) = \frac{\lambda z \pi_j Q - \mu_j P_0^{(j)}}{(\lambda+\mu_j-\lambda z)z-\mu_j}, j=1,2,\dots,m \tag{14}$$

Now, $z=1$ is a zero of the denominator of the right hand side of (14). Therefore, its denominator must vanish for this zero, giving

$$P_0^{(j)} = (\frac{\lambda}{\mu_j})\pi_j Q, \quad j=1,2,\dots,m \tag{15}$$

Using (15), equation (14) can be written as

$$P^{(j)}(z) = \frac{(z-1)\lambda\pi_j Q}{(\lambda+\mu_j-\lambda z)z-\mu_i}, \quad j=1,2,\dots,m \tag{16}$$

For $z=1$, equation (16) is indeterminate of the (0/0) form. Therefore, using L'Hopital's rule, we have

$$P^{(j)}(1) = \frac{\lambda\pi_j Q}{\mu_j-\lambda}, \quad j=1,2,\dots,m \tag{17}$$

Using (17) in the normalizing condition $\sum_{j=1}^{m} P^{(j)}(1) + Q = 1$, we obtain

$$Q = \frac{1}{1+\lambda\sum_{j=1}^{m}(\frac{\pi_j}{\mu_j-\lambda})} \tag{18}$$

Which is the probability that the system is empty and none of the servers is providing service.

Using the value of $Q$ from (18) in equation (16) we get

$$P^{(j)}(z) = \left[\frac{\lambda(z-1)\pi_j}{(\lambda+\mu_j-\lambda z)z-\mu_j}\right]\left[1+\lambda\sum_{j=1}^{m}\frac{\pi_j}{\mu_j-\lambda}\right]^{-1}, \quad j=1,2,\ldots,m \tag{19}$$

Factoring $(\lambda+\mu_j-\lambda z)z-\mu_j$ as $(z-1)(\mu_j-\lambda z)$ and canceling out the factor $(z-1)$, we can write equation (19) as

$$P^{(j)}(z) = \frac{\lambda\pi_j}{(\mu_j-\lambda z)}\left[1+\lambda\sum_{j=1}^{m}\frac{\pi_j}{\mu_j-\lambda}\right]^{-1}, \quad j=1,2,\ldots,m \tag{20}$$

which can again be re-written as

$$P^{(j)}(z) = \frac{\lambda\pi_j}{\mu_j}\left[1+\lambda\sum_{j=1}^{m}\frac{\pi_j}{\mu_j-\lambda}\right]^{-1}[1-\frac{\lambda z}{\mu_j}]^{-1}, \quad j=1,2,\ldots,m \tag{21}$$

Expanding the last factor of the right hand side of equation (21) and picking up the coefficient of the nth power of $z$, and simplifying, we have

$$P_n^{(j)} = \pi_j\left[1+\lambda\sum_{j=1}^{m}\frac{\pi_j}{\mu_j-\lambda}\right]^{-1}\left(\frac{\lambda}{\mu_j}\right)^{n+1}, \quad n\ge 0, j=1,2,\ldots,m \tag{22}$$

## The mean queue length

Let $p_n = \sum_{j=1}^{m} p_n^{(j)}$ denote the steady state probability that the queue length is $n \ge 0$, irrespective of whosoever server is providing service. Then, the mean queue length, $L_q$ is given by

$$L_q = \sum_{n=0}^{\infty} np_n = \left[\sum_{n=0}^{\infty}\sum_{j=1}^{m} n\{\pi_j[1+\lambda\sum_{j=1}^{m}\frac{\pi_j}{\mu_j-\lambda}]^{-1}\left(\frac{\lambda}{\mu_j}\right)^{n+1}\}\right] \tag{23}$$

Carrying out the summations and simplifying, we have the mean queue length as

$$L_q = \left[1+\lambda\sum_{j=1}^{m}\frac{\pi_j}{\mu_j-\lambda}\right]^{-1}\left[\sum_{j=1}^{m}\pi_j\left(\frac{\lambda}{\mu_j-\lambda}\right)^2\right] \tag{24}$$

**Case 3** (the case of two servers)

If the system has two servers, then $m = 2$ and, for that matter, $\pi_j = 0 = \bar{P}^{(j)}(s,z)$, for $j \geq 3$. Also, $p_{ij} = 0$ for $i, j \geq 3$ so that now we have a $2 \times 2$ selection matrix given by $\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$. Then in this case the equations in matrix form will be

$$\begin{bmatrix} G_1 & -p_{21}\mu_2 \\ -p_{12}\mu_1 & G_2 \end{bmatrix} \begin{bmatrix} \bar{P}^{(1)}(s,z) \\ \bar{P}^{(2)}(s,z) \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \tag{25}$$

where

$$G_1 = (s + \lambda + \mu_1 - \lambda z)z - p_{11}\mu_1$$

$$G_2 = (s + \lambda + \mu_2 - \lambda z)z - p_{22}\mu_2$$

$$H_1 = \lambda z \pi_1 \bar{Q}(s) - p_{11}\mu_1 \bar{P}_0^{(1)}(s) - p_{21}\mu_2 \bar{P}^{(2)}(s)$$

$$H_2 = \lambda z \pi_2 \bar{Q}(s) - p_{12}\mu_1 \bar{P}_0^{(1)}(s) - p_{22}\mu_2 \bar{P}^{(2)}(s)$$

Solving (25) simultaneously, we have

$$\bar{P}^{(1)}(s,z) = \frac{H_1[(s + \lambda + \mu_2 - \lambda z)z - p_{22}\mu_2] + p_{21}\mu_2 H_2}{[(s + \lambda + \mu_1 - \lambda z)z - p_{11}\mu_1][(s + \lambda + \mu_2 - \lambda z)z - p_{22}\mu_2]} \tag{26}$$

$$\bar{P}^{(2)}(s,z) = \frac{H_2[(s + \lambda + \mu_1 - \lambda z)z - p_{11}\mu_1] + p_{12}\mu_1 H_1}{[(s + \lambda + \mu_1 - \lambda z)z - p_{11}\mu_1][(s + \lambda + \mu_2 - \lambda z)z - p_{22}\mu_2]} \tag{27}$$

Results in equations (26) and (27) agree with known results. (see Madan (1990))

Under the conditions of particular case 2 discussed above, we have $p_{11} = 1 = p_{22}$ and $p_{12} = 0 = p_{21}$ for $j = 2$. Consequently, the steady state results corresponding to the equations (16), (18), (22) and (24) can be derived as follows:

$$P^{(j)}(z) = \frac{(z-1)\lambda \pi_j Q}{(\lambda + \mu_j - \lambda z)z - \mu_j}, j = 1, 2. \tag{28}$$

$$Q = \frac{(\mu_1 - \lambda)(\mu_2 - \lambda)}{\lambda\pi_1(\mu_2 - \lambda) + \lambda\pi_2(\mu_1 - \lambda) + (\mu_1 - \lambda)(\mu_2 - \lambda)} \tag{29}$$

$$P_n^{(j)} = [\frac{(\mu_1 - \lambda)(\mu_2 - \lambda)}{\lambda\pi_1(\mu_2 - \lambda) + \lambda\pi_2(\mu_1 - \lambda) + (\mu_1 - \lambda)(\mu_2 - \lambda)}]\pi_j(\frac{\lambda}{\mu_j})^{n+1}$$

$$n \geq 0. \quad j = 1, 2. \tag{30}$$

$$L_q = Q\left[\pi_1(\frac{\lambda}{\mu_1 - \lambda})^2 + \pi_2\left(\frac{\lambda}{\mu_2 - \lambda}\right)^2\right] \tag{31}$$

where $Q$ is given in equation (29).

Again, all the above results given in equations (28), (29), (30) and (31) agree with results of Madan (1990), except for notations.

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